ARITHMETIC DIAGONAL CYCLES ON UNITARY SHIMURA VARIETIES

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Abstract. We define variants of PEL type of the Shimura varieties that appear in the context of the Arithmetic Gan–Gross–Prasad conjecture. We formulate for them a version of the AGGP conjecture. We also construct (global and semi-global) integral models of these Shimura varieties and formulate for them conjectures on arithmetic intersection numbers. We prove some of these conjectures in low dimension.

Contents

1. Introduction

The theorem of Gross and Zagier [16] relates the Néron–Tate heights of Heegner points on modular curves to special values of derivatives of certain L-functions. Ever since the appearance of [16], the problem of generalizing this fundamental result to higher dimension has attracted considerable attention. The generalization that is most relevant to the present paper is the *Arithmetic Gan–Gross–Prasad conjecture* (AGGP conjecture) [12, §27]. This conjectural generalization concerns Shimura varieties attached to orthogonal groups of signature $(2, n-2)$, and to unitary groups of signature $(1, n-1)$ (note that modular curves are closely related to Shimura varieties associated to orthogonal groups of signature $(2, 1)$ and to unitary groups of signature $(1, 1)$). In [12, §27], algebraic cycles of codimension one on such Shimura varieties are defined by exploiting embeddings of Shimura varieties attached to orthogonal groups of signature $(2, n-3)$, resp. to unitary groups of signature $(1, n-2)$. By taking the graphs of these embeddings, one obtains cycles in codimension just above half the (odd) dimension of the ambient variety.

For any algebraic variety X smooth and proper of odd dimension over a number field, Beilinson and Bloch have defined a height pairing on the *rational* Chow group $\mathrm{Ch}(X)_{\mathbb{Q},0}$ of cohomologically trivial cycles of codimension just above half the dimension. Their definition makes use of some widely open unsolved conjectures on algebraic cycles and the existence of regular proper integral models of X. By suitably replacing in the case at hand the graph cycle by a cohomologically trivial avatar, one obtains a linear form on $\mathrm{Ch}(X)_{\mathbb{Q},0}$, where now X is the product of the two Shimura varieties in question. The AGGP conjecture relates a special value of the derivative of an L-function to the non-triviality of the restriction of this linear form to a Hecke eigenspace in $\mathrm{Ch}(X)_{\mathbb{Q},0}$. It is stated in a very succinct way in [12], for orthogonal groups and for unitary groups. In the present paper, we restrict ourselves to *unitary* groups, and one of our aims is

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to give in this case more details on (a variant of) this conjecture. Our version here is also an improvement of the version of the conjecture in [55, 56]. One new feature of our version is that we use that in the case of unitary groups the *standard sign conjecture* is satisfied (more precisely, we use Ramakrishnan's Chebotarev theorem [41] to improve on the theorem of Morel–Suh [38] in this case). This allows us to construct "Hecke–Kunneth" projectors that project the total cohomology of our Shimura variety to the even-degree part.

As indicated above, the AGGP conjecture is based on conjectures of Beilinson and Bloch which seem out of reach at present. As a consequence, the conjecture in [12] has not been proved in a single case of higher dimension.[1] A variant of the AGGP conjecture, inspired by the *relative trace formula* of Jacquet-Rallis, has been proposed by the third author [55]. More precisely, this variant relates the height pairing with distributions that appear in the relative trace formula. This variant leads to local conjectures (on intersection numbers on Rapoport–Zink spaces), namely the *Arithmetic Fundamental Lemma* conjecture and the *Arithmetic Transfer* conjecture, cf. [55, 44]—and these have been proved in various cases [55, 43, 44, 35, 36, 46]. The second aim of the present paper is to formulate a global conjecture whose proof in various cases is a realistic goal. In the present paper, basing ourselves on our local papers [55, 44], we prove this conjecture for unitary groups of size $n \leq 3$.

To formulate this conjecture, we define variants of the Shimura varieties appearing in [12] and [55] which are of PEL type, i.e. are related to moduli problems of abelian varieties with polarizations, endomorphisms, and level structures. In fact, we even define integral models of these Shimura varieties, in a global version and a semi-global version. The construction of such models is the third aim of the present paper. Once these models are defined, we replace the Bloch–Beilinson pairing on the cohomologically trivial Chow group by the Gillet–Soulé pairing on the arithmetic Chow group of the (global or semi-global) integral model.

Now that we formulated the three main goals of this paper, let us be more specific.

Let F be a CM number field, with maximal totally real subfield F_0. We fix a CM type Φ of F and a distinguished element $\varphi_0 \in \Phi$. Let $n \geq 2$ and let $r\colon \mathrm{Hom}(F, \mathbb{C}) \to \{0, 1, n-1, n\}$, $\varphi \mapsto r_\varphi$, be the function defined by

$$r_\varphi := \begin{cases} 1, & \varphi = \varphi_0; \\ 0, & \varphi \in \Phi \smallsetminus \{\varphi_0\}; \\ n - r_{\overline{\varphi}}, & \varphi \notin \Phi. \end{cases}$$

Associated to these data, there is the field $E \subset \overline{\mathbb{Q}}$ which is the composite of the reflex field of r and the reflex field of Φ. Then E contains F via φ_0. We denote by $Z^{\mathbb{Q}}$ the torus

$$Z^{\mathbb{Q}} := \bigl\{\, z \in \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_m) \bigm| \mathrm{Nm}_{F/F_0}(z) \in \mathbb{G}_m \,\bigr\}.$$

We also fix an F/F_0-hermitian vector space W of dimension n with signature

$$\mathrm{sig}(W_\varphi) = (r_\varphi, r_{\overline{\varphi}}), \quad \varphi \in \Phi.$$

Let G be the unitary group of W, considered as an algebraic group over $\mathbb{Q}$.[2] Associated to (G, r) is the Shimura variety of [12]. In the present paper, we instead consider the Shimura variety associated to $\widetilde{G} := Z^{\mathbb{Q}} \times G$. We are able to formulate a PEL moduli problem $M_{K_{\widetilde{G}}}(\widetilde{G})$ of abelian varieties with additional structure (endomorphisms and polarization) which defines a model over E of the Shimura variety

$$\mathrm{Sh}_{K_{\widetilde{G}}}(\widetilde{G}) = M_{K_{\widetilde{G}}}(\widetilde{G}) \otimes_E \mathbb{C}. \tag{1.1}$$

(In fact, we demand that $K_{\widetilde{G}} = K_{Z^{\mathbb{Q}}} \times K_G$, where $K_{Z^{\mathbb{Q}}}$ is the unique maximal compact subgroup of $Z^{\mathbb{Q}}(\mathbb{A}_f)$ and where K_G is an open compact subgroup of $G(\mathbb{A}_f)$.) The group differs from the group of unitary similitudes $\mathrm{GU}(W)$ by a central isogeny. The Shimura variety corresponding to the latter group is considered by Kottwitz [28], and he formulates a PEL moduli problem over the reflex field of r which *almost* defines a model for it—but not quite, because of the possible failure of the Hasse principle for $\mathrm{GU}(W)$. This Shimura variety is also considered by

[1]However, we point out that [52] proves certain variants of this conjecture in a higher-dimensional case for *orthogonal groups* of type $\mathrm{SO}(3) \times \mathrm{SO}(4)$.

[2]This notation differs from the main body of the paper, where G denotes the unitary group of W over F_0.

Harris–Taylor [17]. In the setup of [17], we have $E = F$ and both their Shimura variety and ours are defined over F; however, ours offers a number of technical advantages over theirs.[3] The definition of our moduli problem is based on a sign invariant $\mathrm{inv}_v^r(A_0, A) \in \{\pm 1\}$ for every non-archimedean place v of F_0 which is non-split in F. Here $(A_0, \iota_0, \lambda_0)$ is a polarized abelian variety of dimension $d = [F_0 : \mathbb{Q}]$ with complex multiplication of CM type $\overline{\Phi}$ of F and (A, ι, λ) is a polarized abelian variety of dimension nd with complex multiplication of generalized CM type r of F. This sign invariant is similar to the one in [31, 32], but much simpler. This simplicity is another reflection of the advantage of our Shimura varieties over those considered by Kottwitz [28].

This sign invariant also allows us to define *global integral models* of $M_{K_{\widetilde{G}}}(\widetilde{G})$ over $\operatorname{Spec} O_E$ (at least when F/F_0 is not everywhere unramified) and *semi-global integral models* over $\operatorname{Spec} O_{E,(\nu)}$, where ν is a fixed non-archimedean place of E, of residue characteristic p. These integral models generalize those in [7] when $F_0 = \mathbb{Q}$ and when K_G is the stabilizer of a self-dual lattice in W. Here we allow K_G to be the stabilizer of certain vertex lattices. To achieve flatness, we sometimes have to impose conditions on the Lie algebras of the abelian varieties in play that are known in a similar context from our earlier local papers [43, 44] (the *Pappas wedge condition*, the *spin condition* and its refinement, the *Eisenstein conditions*). However, in contrast to Kottwitz, we do not need any unramifiedness conditions.

Once the model $M_{K_{\widetilde{G}}}(\widetilde{G})$ and its global or semi-global model are defined, we can also create a restriction situation in analogy with [12]. Namely, fixing a *totally negative* vector $u \in W$ (satisfying additional integrality conditions for the global, resp. semi-global integral situation), we define $W^\flat$ to be the orthogonal complement of u. Then $W^\flat$ satisfies the same conditions as W, with n replaced by $n-1$. We obtain a finite unramified morphism

$$M_{K_{\widetilde{H}}}(\widetilde{H}) \longrightarrow M_{K_{\widetilde{G}}}(\widetilde{G}), \tag{1.2}$$

resp. their global, resp. semi-global integral versions. Here $\widetilde{H} = Z^{\mathbb{Q}} \times H$, where $H = \mathrm{U}(W^\flat)$, considered as an algebraic group over $\mathbb{Q}$. Using the graph of the above morphism, we obtain an element in the *rational Chow group*,

$$z_{K_{\widetilde{HG}}} \in \mathrm{Ch}^{n-1}\big(M_{K_{\widetilde{HG}}}(\widetilde{HG})\big)_{\mathbb{Q}}.$$

Here $M_{K_{\widetilde{HG}}}(\widetilde{HG})$ is the model defined as above for the Shimura variety for the group $Z^{\mathbb{Q}} \times H \times G$. Using the Hecke–Kunneth projector, we construct a cohomologically trivial variant $z_{K_{\widetilde{HG}},0} \in \mathrm{Ch}^{n-1}(M_{K_{\widetilde{HG}}}(\widetilde{HG}))_{0,\mathbb{C}}$ of this element, which, via the Beilinson–Bloch pairing, in turn defines a linear form

$$\ell_{K_{\widetilde{HG}}} \colon \mathrm{Ch}^{n-1}\big(M_{K_{\widetilde{HG}}}(\widetilde{HG})\big)_{\mathbb{C},0} \longrightarrow \mathbb{C}.$$

Our variant of the AGGP conjecture is expressed in terms of this linear form.

We similarly define, under certain hypotheses, elements in the rational Chow group, resp. rational arithmetic Chow group, of the (global, or semi-global) integral model $\mathcal{M}_{K_{\widetilde{HG}}}(\widetilde{HG})$. Let us consider the Gillet–Soulé intersection product pairing on the rational arithmetic Chow group of $\mathcal{M}_{K_{\widetilde{HG}}}(\widetilde{HG})$. We have a conjecture on the value of the intersection product of $z_{K_{\widetilde{HG}}}$ and its image under a Hecke correspondence. Let us state the semi-global version, since this is the one for which we can produce concrete evidence. We also have a global version.

Conjecture 1.1 (Semi-global conjecture)**.** *Fix a non-split place v_0 of F_0 over the place $p \leq \infty$ of $\mathbb{Q}$. Let $f = \otimes_\ell f_\ell \in \mathscr{H}^p_{K_{\widetilde{HG}}}$ ($\mathscr{H}_{K_{\widetilde{HG}}}$ if p is archimedean) be a completely decomposed element of the finite Hecke algebra of $\widetilde{HG}$, and let $f' = \otimes_v f'_v \in \mathscr{H}(G'(\mathbb{A}_{F_0}))$ be a Gaussian test function in the Hecke algebra of $G' = \mathrm{Res}_{F/F_0}(\mathrm{GL}_{n-1} \times \mathrm{GL}_n)$ such that $\otimes_{v<\infty} f'_v$ is a smooth transfer of f. Assume that for some place λ prime to p, the function f has regular support at λ in the sense of Definition 8.4 and that f' has regular support at λ in the sense of Definition 7.4.*

(i) Assume that v_0 is non-archimedean of hyperspecial type, cf. Section 4.1, and that $f'_{v_0} = \mathbf{1}_{G'(O_{F_0,v_0})}$. Then

$$\mathrm{Int}_{v_0}(f) = -\partial J_{v_0}(f').$$

[3]Kottwitz [28] does not need any assumptions on the signature of W; neither do we.

(ii) Assume that v_0 is archimedean, or non-archimedean of AT type, cf. Section 4.4. Then

$$\mathrm{Int}_{v_0}(f) = -\partial J_{v_0}(f') - J(f'_{\mathrm{corr}}[v_0]),$$

where $f'_{\mathrm{corr}}[v_0] = \otimes_v f'_{\mathrm{corr},v}$, with $f'_{\mathrm{corr},v} = f'_v$ for $v \neq v_0$, is a correction function. Furthermore, f' may be chosen such that $f'_{\mathrm{corr}}[v_0]$ is zero.

We refer to the body of the text for an explanation of the terms used (cf. Conjecture 8.13). Here it suffices to remark that $\mathrm{Int}_{v_0}(f)$ is the normalized sum of the local contributions of all places ν of E over v_0 to the Gillet–Soulé intersection product of the diagonal subscheme $\mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H})$ of $\mathcal{M}_{K_{\widetilde{HG}}}(\widetilde{HG})$ and its translate by the Hecke correspondence $R(f)$ associated to f. By J, resp. ∂J_{v_0}, we denote the distributions that arise in the twisted trace formula approach to the AGGP conjecture, cf. [55]. The regularity assumption on f guarantees that for non-archimedean v_0 the intersection of the two cycles has support in characteristic p, and the regularity assumption on f' guarantees that the distributions J and ∂J_{v_0} localize.

In the global context, when the Hecke correspondence $R(f)$ satisfies a suitable regularity assumption, the intersection product localizes, i.e., is a finite sum of contributions, one from each place ν of E. We group together the local contributions from all places ν which induce a given place v_0 of F_0. From this point of view, Conjecture 1.1(i) predicts the contribution of the good places, and Conjecture 1.1(ii) the contributions from the archimedean places and certain bad places.

The conjecture is accessible in certain cases. We prove the following theorem.

Theorem 1.2. *Let v_0 be a non-archimedean place of F_0 that is non-split in F. Then Conjecture 1.1 above holds true for $n \leq 3$.*

The main input is our work in the local case (intersection product on Rapoport–Zink spaces): the proof of the AFL conjecture in the hyperspecial case for $n \leq 3$ by one of us [55] and the proof of the AT conjecture for $n \leq 3$ in [43, 44]. The passage from the local statement to a global statement is modeled on the similar passage in [30] (which also inspired the similar passage in [55]). In fact, this similarity is not only formal. Indeed, the definition of *Kudla–Rapoport divisors* uses in an essential way that the Shimura variety for $\mathrm{GU}(W)$ is replaced by the Shimura variety for $\widetilde{G}$ (in fact, in [30], one uses $Z^{\mathbb{Q}} \times \mathrm{GU}(W)$; as remarked in [7], the definition can be realized on the Shimura variety for $\widetilde{G}$). In this way, there is a direct connection between the intersection problem occurring in Conjecture 1.1 and the intersection problem in [30].

We also have a theorem for split places (hence non-archimedean), cf. Proposition 8.12.

Theorem 1.3. *Let v_0 be a place of F_0 that is split in F. Let f and f' be as in Conjecture 8.8. Then*

$$\mathrm{Int}_{v_0}(f) = \partial J_{v_0}(f') = 0.$$

The significance of this theorem is that in the global context, again under a regularity assumption on the Hecke correspondence $R(f)$, the contribution of the places v_0 which split in F is trivial.

Let us now put the results of this paper in perspective. The construction of integral models of the Shimura varieties for $\widetilde{G}$ answers a question of B. Gross; however, only partially. Indeed, we have to pay a price by having to replace the field F, over which Gross's Shimura varieties have a model, by the field E, over which our Shimura varieties have a model; and E may be strictly larger than F. This also causes us to modify our adaptation of the AGGP conjecture. It would be interesting to understand whether the Kisin–Pappas construction of integral models of Shimura varieties of abelian type [25] yields a solution of Gross's question which can be used to give a variant of the AGGP conjecture which avoids having to replace F by a bigger field.

Our current knowledge of AT conjectures forces on us to be very specific when imposing level structures in our moduli problems. It seems realistic to hope that more cases of AT conjectures than in [43, 44] can be formulated, and this would allow more flexibility for the level structures. We hope to return to this point.

How realistic is it to hope that the conjectures on the arithmetic intersection pairing can be proved, in cases that go beyond those treated in this paper? The stumbling block seems to be that in higher dimension it is difficult to avoid degenerate intersections—and degenerate intersections seem to be a challenge to currently available techniques, already in the local situation (intersection on Rapoport–Zink spaces). It might be fruitful to search for intersection problems derived from those considered here which avoid these apparently very difficult problems, in the spirit of B. Howard's papers concerning the Kudla–Rapoport divisor intersection problem, cf. [18, 19].

As is apparent, automorphic L-functions do not appear explicitly in the statements above, contrary to what happens in the AGGP conjecture. L-functions are involved implicitly because the distributions J and ∂J_{v_0} are related to them (cf. Section 7). However, more analytic work is involved to make this relation more explicit. One of us (W.Z.) hopes to return to this point and explain this issue in more detail.

We finally give an overview of the layout of the paper. In Section 2, we introduce the groups in play and define the concept of matching in this context. In Section 3, we introduce the various Shimura varieties mentioned above, and the corresponding moduli interpretation over E, and the relation between the moduli variety for $W^\flat$ and for W. In Section 4, we define the semi-global integral models of these moduli schemes and the morphisms between them. We do this in several contexts: for *hyperspecial level*, for *split level*, for *Drinfeld level*, and for *AT parahoric level.* These levels reflect the possibility of applying the AFL conjecture, resp. the AT conjecture, resp. the vanishing theorem Theorem 1.3 in the split level. The AT parahoric level is complicated by the fact that the morphisms between the semi-global integral models are in some cases not defined in the naive way. In Section 5 we give the global integral models, first without level structure and then with Drinfeld level structure. Section 6 is devoted to giving our version of the AGGP conjecture. Section 7 is preparatory for the last section. Here we explain the distributions arising in the context of the relative trace formula and their relation to L-functions. In Section 8 everything comes finally together. First, we formulate our conjecture on the arithmetic intersection numbers. We do this in the global case without level structure, in the global case with Drinfeld level structure, and in the semi-global case. Second, we give the proofs of the semi-global versions in cases of small dimension, cf. Theorem 1.2. There are two appendices. In Appendix A we define the sign invariant that is used in the formulation of the moduli schemes. In Appendix B we check that in the case of *banal signature* the relevant local models are trivial in a precise sense.

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Notation. Except in Section 2, F denotes a CM number field and F_0 denotes its (maximal) totally real subfield of index 2 (in Section 2 F/F_0 can be any quadratic extension of number fields). We denote by $a \mapsto \overline{a}$ the nontrivial automorphism of F/F_0. We fix a presentation $F = F_0(\sqrt{\Delta})$ for some totally negative element $\Delta \in F_0$, and we let Φ be the CM type for F determined by $\sqrt{\Delta}$,

$$\Phi := \{\, \varphi\colon F \to \mathbb{C} \mid \varphi(\sqrt{\Delta}) \in \mathbb{R}_{>0} \cdot \sqrt{-1}\,\}. \tag{1.3}$$

Note that by weak approximation, every CM type for F arises in this way for some F/F_0-traceless element $\sqrt{\Delta} \in F^\times$.

We use the symbols v and v_0 to denote places of F_0, and w and w_0 to denote places of F. We write $F_{0,v}$ for the v-adic completion of F_0, and we set $F_v := F \otimes_{F_0} F_{0,v}$; thus F_v is isomorphic to $F_{0,v} \times F_{0,v}$ or to a quadratic field extension of $F_{0,v}$ according as v is split or non-split in F. We often identify the CM type Φ (or more precisely, the restrictions of its elements to F_0) with the archimedean places of F_0. When v is a finite place, we write $\mathfrak{p}_v$ for the maximal ideal in O_{F_0} at v, we write ϖ_v for a uniformizer in $F_{0,v}$, and we write π_v for a uniformizer in F_v (when v splits in F this means an ordered pair of uniformizers on the right-hand side of the isomorphism $F_v \cong F_{0,v} \times F_{0,v}$). We write $O_{F_{0,(v)}}$ for the localization of O_{F_0} at the maximal ideal $\mathfrak{p}_v$, and

$O_{F_0,v} \subset F_{0,v}$ for its $\mathfrak{p}_v$-adic completion. We use analogous notation for other fields in place of F_0 and other finite places in place of v. In particular, we will often consider the v-adic completion $O_{F,v} = O_F \otimes_{O_{F_0}} O_{F_0,v}$ of O_F.

We write $\mathbb{A}$, $\mathbb{A}_{F_0}$, and $\mathbb{A}_F$ for the adele rings of $\mathbb{Q}$, F_0, and F, respectively. We systematically use a subscript f for the ring of finite adeles, and a superscript p for the adeles away from the prime number p.

We take all hermitian forms to be linear in the first variable and conjugate-linear in the second, and we assume that they are nondegenerate unless we say otherwise. For k any field, A an étale k-algebra of degree 2, and W a finite free A-module equipped with an A/k-hermitian form, we write $\det W \in k^\times/\operatorname{Nm}_{A/k} A^\times$ for the class of $\det J$, where J is any hermitian matrix (relative to the choice of an A-basis for W) representing the form. Note that this value group is trivial when $A \simeq k \times k$. We also write $-W$ for the same A-module as W, but with the hermitian form multiplied by -1. Of course W and $-W$ have the same unitary groups. When W is an F/F_0-hermitian space of dimension n and v is a place of F_0, we write W_v for the induced $F_v/F_{0,v}$-hermitian space $W \otimes_{F_0} F_{0,v}$, and we define

$$\operatorname{inv}_v(W_v) := (-1)^{n(n-1)/2} \det W_v \in F_{0,v}^\times/\operatorname{Nm} F_v^\times. \tag{1.4}$$

We say that W_v is *split* at a finite place v if $\operatorname{inv}_v(W_v) = 1$; under our normalization, the antidiagonal unit matrix always defines a split hermitian form. When v is an archimedean place, the form on W_v is isometric to $\operatorname{diag}(1^{(r)}, (-1)^{(s)})$ for some $r + s = n$, and we write $\operatorname{sig}_v(W_v) := (r, s)$ (the *signature*). In the local setting, isometry classes of n-dimensional $F_v/F_{0,v}$-hermitian spaces are classified by inv_v when v is a finite place, and by sig_v when v is an archimedean place. By the Hasse principle, two global hermitian spaces are isometric if and only if they are isometric at every place v, i.e. they have the same invariants at each finite place and the same signatures at each archimedean place. Given a global space W as above, the product formula for the norm residue symbol for the extension F/F_0 gives

$$\prod_v \operatorname{inv}_v(W_v) = 1, \tag{1.5}$$

where v ranges through the places of F_0, and where we identify $F_{0,v}^\times/\operatorname{Nm} F_v^\times \subset \{\pm 1\}$. Conversely, Landherr's theorem asserts that a collection $(W_v)_v$ of $F_v/F_{0,v}$-hermitian spaces arises as the set of local completions of a (unique, by the Hasse principle) global F/F_0-hermitian space exactly when $\operatorname{inv}_v(W_v) = 1$ for all but finitely many v and the product formula (1.5) holds. Given an embedding $\varphi\colon F \to \mathbb{C}$, we write $W_\varphi := W \otimes_{F,\varphi} \mathbb{C}$ for the induced hermitian space over $\mathbb{C}$.

For an abelian scheme A over a locally noetherian scheme S on which the prime number p is invertible, we write $T_p(A)$ for the p-adic Tate module of A and $V_p(A) := T_p(A) \otimes \mathbb{Q}$ for the rational p-adic Tate module. When S is a $\mathbb{Z}_{(p)}$-scheme, we write $\widehat{V}^p(A)$ for the rational prime-to-p Tate module of A. When S is a scheme in characteristic zero, we write $\widehat{V}(A)$ for the full rational Tate module of A.

We use a superscript $\circ$ to denote the operation $- \otimes_{\mathbb{Z}} \mathbb{Q}$ on groups of homomorphisms of abelian schemes, so that for example $\operatorname{Hom}^\circ(A, A') := \operatorname{Hom}(A, A') \otimes_{\mathbb{Z}} \mathbb{Q}$.

Given modules M and N over a ring R, we write $M \subset^r N$ to indicate that M is an R-submodule of N of finite colength r. Typically R will be $O_{F,v}$ for v a finite place of F_0. When Λ is an O_F-lattice in an F/F_0-hermitian space, we denote the dual lattice with respect to the hermitian form by $\Lambda^\vee$. We use the same notation when Λ is an $O_{F,v}$-lattice in an $F_v/F_{0,v}$-hermitian space, and we call Λ a *vertex lattice of type r* if $\Lambda \subset^r \Lambda^\vee \subset \pi_v^{-1}\Lambda$. Note that this terminology differs slightly from e.g. [29, 45]. A *vertex lattice* is a vertex lattice of type r for some r. Let us single out the following special cases. A *self-dual* lattice is, of course, a vertex lattice of type 0. An *almost self-dual* lattice is a vertex lattice of type 1. At the other extreme, a vertex lattice Λ is *π_v-modular* if $\Lambda^\vee = \pi_v^{-1}\Lambda$, and *almost π_v-modular* if $\Lambda \subset \Lambda^\vee \subset^1 \pi_v^{-1}\Lambda$.

Given a discretely valued field L, we denote the completion of a maximal unramified extension of it by $\breve{L}$.

We write 1_n for the $n \times n$ identity matrix. We use a subscript S to denote base change to a scheme (or other object) S, and when $S = \operatorname{Spec} A$, we often use a subscript A instead.

2. Group-theoretic setup

In this section we introduce the groups and linear-algebraic objects which will be in play throughout the paper. Let F/F_0 be a quadratic extension of number fields.

2.1. Similitude groups and variants.

We begin by introducing algebraic groups over F_0,

$$\begin{aligned} G' &:= \operatorname{Res}_{F/F_0}(\mathrm{GL}_{n-1} \times \mathrm{GL}_n), \\ H_1' &:= \operatorname{Res}_{F/F_0} \mathrm{GL}_{n-1}, \\ H_2' &:= \mathrm{GL}_{n-1} \times \mathrm{GL}_n, \\ H_{1,2}' &:= H_1' \times H_2'. \end{aligned}$$

Next let W be an F/F_0-hermitian space of dimension $n \geq 2$. We fix a non-isotropic vector $u \in W$, which we call the *special vector*. We denote by $W^\flat$ the orthogonal complement of u in W. We define another four algebraic groups over F_0,

$$\begin{aligned} G &:= \mathrm{U}(W), \\ H &:= \mathrm{U}(W^\flat), \\ G_W &:= H \times G, \\ H_W &:= H \times H. \end{aligned}$$

We further define the following algebraic groups over $\mathbb{Q}$. We systematically use the symbol c to denote the similitude factor of a point on a unitary similitude group.

$$\begin{aligned} Z^{\mathbb{Q}} &:= \bigl\{\, z \in \operatorname{Res}_{F/\mathbb{Q}} \mathbb{G}_m \bigm| \operatorname{Nm}_{F/F_0}(z) \in \mathbb{G}_m \,\bigr\}, \\ H^{\mathbb{Q}} &:= \bigl\{\, h \in \operatorname{Res}_{F_0/\mathbb{Q}} \mathrm{GU}(W^\flat) \bigm| c(h) \in \mathbb{G}_m \,\bigr\}, \\ G^{\mathbb{Q}} &:= \bigl\{\, g \in \operatorname{Res}_{F_0/\mathbb{Q}} \mathrm{GU}(W) \bigm| c(g) \in \mathbb{G}_m \,\bigr\}, \\ \widetilde{H} &:= Z^{\mathbb{Q}} \times_{\mathbb{G}_m} H^{\mathbb{Q}} = \bigl\{\, (z,h) \in Z^{\mathbb{Q}} \times H^{\mathbb{Q}} \bigm| \operatorname{Nm}_{F/F_0}(z) = c(h) \,\bigr\}, \\ \widetilde{G} &:= Z^{\mathbb{Q}} \times_{\mathbb{G}_m} G^{\mathbb{Q}} = \bigl\{\, (z,g) \in Z^{\mathbb{Q}} \times G^{\mathbb{Q}} \bigm| \operatorname{Nm}_{F/F_0}(z) = c(g) \,\bigr\}, \\ \widetilde{HG} &:= \widetilde{H} \times_{Z^{\mathbb{Q}}} \widetilde{G} = Z^{\mathbb{Q}} \times_{\mathbb{G}_m} H^{\mathbb{Q}} \times_{\mathbb{G}_m} G^{\mathbb{Q}} \\ &= \bigl\{\, (z,h,g) \in Z^{\mathbb{Q}} \times H^{\mathbb{Q}} \times G^{\mathbb{Q}} \bigm| \operatorname{Nm}_{F/F_0}(z) = c(h) = c(g) \,\bigr\}. \end{aligned}$$

Note that $Z^{\mathbb{Q}}$ is naturally a central subgroup of $H^{\mathbb{Q}}$ and $G^{\mathbb{Q}}$, and these inclusions give rise to product decompositions

$$\begin{array}{c} \begin{array}{rcl} \widetilde{H} & \xrightarrow{\ \sim\ } & Z^{\mathbb{Q}} \times \operatorname{Res}_{F_0/\mathbb{Q}} H \\ (z,h) & \longmapsto & (z, z^{-1}h) \end{array}, \qquad \begin{array}{rcl} \widetilde{G} & \xrightarrow{\ \sim\ } & Z^{\mathbb{Q}} \times \operatorname{Res}_{F_0/\mathbb{Q}} G \\ (z,g) & \longmapsto & (z, z^{-1}g) \end{array}, \\ \begin{array}{rcl} \widetilde{HG} & \xrightarrow{\ \sim\ } & Z^{\mathbb{Q}} \times \operatorname{Res}_{F_0/\mathbb{Q}}(H \times G) \\ (z,h,g) & \longmapsto & (z, z^{-1}h, z^{-1}g) \end{array} \end{array} \tag{2.1}$$

We also record that the decomposition $W = W^\flat \oplus Fu$ gives rise to natural closed embeddings of algebraic groups,

$$\begin{array}{rcl} \widetilde{H} & \hookrightarrow & \widetilde{G} \\ (z,h) & \longmapsto & \bigl(z, \operatorname{diag}(h,z)\bigr) \end{array} \quad \text{and} \quad \begin{array}{rcl} \widetilde{H} & \hookrightarrow & \widetilde{HG} \\ (z,h) & \longmapsto & \bigl(z, h, \operatorname{diag}(h,z)\bigr) \end{array}. \tag{2.2}$$

2.2. Orbit matching.

The following lemma is obvious.

Lemma 2.1. *The natural projections in (2.1) induce isomorphisms*

$$\widetilde{G}/\widetilde{H} \xrightarrow{\ \sim\ } \operatorname{Res}_{F_0/\mathbb{Q}} G \big/ \operatorname{Res}_{F_0/\mathbb{Q}} H$$

and

$$\widetilde{H}\backslash\widetilde{HG}/\widetilde{H} \xrightarrow{\ \sim\ } \operatorname{Res}_{F_0/\mathbb{Q}} H \big\backslash \operatorname{Res}_{F_0/\mathbb{Q}} G_W \big/ \operatorname{Res}_{F_0/\mathbb{Q}} H. \qquad \square$$

The homogeneous version of the *matching relation* is a natural injection of orbit spaces of *regular semisimple* elements,

$$H(F_{0,v})\backslash G_W(F_{0,v})_{\mathrm{rs}}/H(F_{0,v}) \hookrightarrow H_1'(F_{0,v})\backslash G'(F_{0,v})_{\mathrm{rs}}/H_2'(F_{0,v})$$

for any place v of F_0. In the case when v is a non-archimedean place not split in F, this is explained in [43, §2]. The definition extends to the archimedean places. If v is split in F, we define matching as in [57, §2]. Briefly speaking, we identify $H(F_{0,v})$ with $\mathrm{GL}_{n-1}(F_{0,v})$ and $G(F_{0,v})$ with $\mathrm{GL}_n(F_{0,v})$. This gives a natural way of matching regular semisimple elements. Using the above lemma, we obtain an injection for every prime number p,

$$\widetilde{H}(\mathbb{Q}_p)\backslash \widetilde{HG}(\mathbb{Q}_p)_{\mathrm{rs}}/\widetilde{H}(\mathbb{Q}_p) \hookrightarrow \prod_{v|p} H_1'(F_{0,v})\backslash G'(F_{0,v})_{\mathrm{rs}}/H_2'(F_{0,v}). \tag{2.3}$$

3. The Shimura varieties

For the rest of the paper we take F to be a CM field over $\mathbb{Q}$ and F_0 to be its totally real subfield of index 2. We recall from the Introduction that we fix a totally imaginary element $\sqrt{\Delta} \in F^\times$, and we denote by Φ the induced CM type for F given in (1.3).

3.1. **The Shimura data.** In this subsection we define Shimura data for some of the groups introduced in Section 2. We assume that the hermitian space W has the following signatures at the archimedean places of F_0: for a distinguished element $\varphi_0 \in \Phi$, the signature of W_{φ_0} is $(1, n-1)$, and for all other $\varphi \in \Phi$ the signature of W_φ is $(0, n)$. We also assume that the special vector u is totally negative, i.e. that $(u,u)_\varphi < 0$ for all φ.

We first define Shimura data $(G^{\mathbb{Q}}, \{h_{G^{\mathbb{Q}}}\})$ and $(H^{\mathbb{Q}}, \{h_{H^{\mathbb{Q}}}\})$; comp. [39, §1.1]. Using the canonical inclusions $G^{\mathbb{Q}}_{\mathbb{R}} \subset \prod_{\varphi\in\Phi} \mathrm{GU}(W_\varphi)$ and $H^{\mathbb{Q}}_{\mathbb{R}} \subset \prod_{\varphi\in\Phi} \mathrm{GU}(W_\varphi^\flat)$, it suffices to define the components $h_{G^{\mathbb{Q}},\varphi}$ of $h_{G^{\mathbb{Q}}}$ and $h_{H^{\mathbb{Q}},\varphi}$ of $h_{H^{\mathbb{Q}}}$. We define matrices

$$J_\varphi := \begin{cases} \mathrm{diag}\big(1, (-1)^{(n-1)}\big), & \varphi = \varphi_0; \\ \mathrm{diag}(-1,-1,\dots,-1), & \varphi \in \Phi \smallsetminus \{\varphi_0\}, \end{cases}$$

and we define the matrix $J_\varphi^\flat$ by removing a -1 from J_φ. We may then choose bases $W_\varphi \simeq \mathbb{C}^n$ and $W_\varphi^\flat \simeq \mathbb{C}^{n-1}$ such that $u \otimes 1 \in W_\varphi$ identifies with a multiple of $(0^{(n-1)}, 1)$, such that the inclusion $W_\varphi^\flat \subset W_\varphi$ is compatible with the inclusion $\mathbb{C}^{n-1} \subset \mathbb{C}^{n-1} \oplus \mathbb{C} = \mathbb{C}^n$, and such that the hermitian forms on W_φ and $W_\varphi^\flat$ have respective normal forms

$$(x,y)_\varphi = {}^t x J_\varphi \overline{y} \quad \text{and} \quad (x^\flat, y^\flat)_\varphi = {}^t x^\flat J_\varphi^\flat \overline{y}^\flat.$$

We then define the component maps

$$h_{G^{\mathbb{Q}},\varphi} \colon \mathbb{C}^\times \longrightarrow \mathrm{GU}(W_\varphi)(\mathbb{R}) \quad \text{and} \quad h_{H^{\mathbb{Q}},\varphi} \colon \mathbb{C}^\times \longrightarrow \mathrm{GU}(W_\varphi^\flat)(\mathbb{R})$$

to be induced by the respective $\mathbb{R}$-algebra homomorphisms

$$\begin{array}{ccc} \mathbb{C} \longrightarrow \mathrm{End}(W_\varphi) & & \mathbb{C} \longrightarrow \mathrm{End}(W_\varphi^\flat) \\ & \text{and} & \\ \sqrt{-1} \longmapsto \sqrt{-1} J_\varphi & & \sqrt{-1} \longmapsto \sqrt{-1} J_\varphi^\flat \end{array} .$$

By definition of Φ, the form $x, y \mapsto \mathrm{tr}_{\mathbb{C}/\mathbb{R}}\, \varphi(\sqrt{\Delta})^{-1}(h_{G^{\mathbb{Q}},\varphi}(\sqrt{-1})x, y)_\varphi$ is symmetric and positive definite on W_φ for each $\varphi \in \Phi$, and similarly for $W_\varphi^\flat$.

We next define Shimura data $(\widetilde{H}, \{h_{\widetilde{H}}\})$, $(\widetilde{G}, \{h_{\widetilde{G}}\})$, and $(\widetilde{HG}, \{h_{\widetilde{HG}}\})$. For this, note that Φ induces an identification

$$Z^{\mathbb{Q}}(\mathbb{R}) \cong \big\{\, (z_\varphi) \in (\mathbb{C}^\times)^\Phi \bigm| |z_\varphi| = |z_{\varphi'}| \text{ for all } \varphi, \varphi' \in \Phi \,\big\}.$$

In this way, we define $h_{Z^{\mathbb{Q}}} \colon \mathbb{C}^\times \to Z^{\mathbb{Q}}(\mathbb{R})$ to be the diagonal embedding, *pre-composed with complex conjugation.* We then obtain the desired Shimura data by defining the Shimura homomorphisms

$$h_{\widetilde{H}} \colon \mathbb{C}^\times \xrightarrow{(h_{Z^{\mathbb{Q}}}, h_{H^{\mathbb{Q}}})} \widetilde{H}(\mathbb{R}), \quad h_{\widetilde{G}} \colon \mathbb{C}^\times \xrightarrow{(h_{Z^{\mathbb{Q}}}, h_{G^{\mathbb{Q}}})} \widetilde{G}(\mathbb{R}), \quad h_{\widetilde{HG}} \colon \mathbb{C}^\times \xrightarrow{(h_{Z^{\mathbb{Q}}}, h_{H^{\mathbb{Q}}}, h_{G^{\mathbb{Q}}})} \widetilde{HG}(\mathbb{R}).$$

It is easy to see that $(\widetilde{H}, \{h_{\widetilde{H}}\})$, $(\widetilde{G}, \{h_{\widetilde{G}}\})$, and $(\widetilde{HG}, \{h_{\widetilde{HG}}\})$ have common reflex field $E \subset \mathbb{C}$ characterized by

$$\mathrm{Aut}(\mathbb{C}/E) = \{\, \sigma \in \mathrm{Aut}(\mathbb{C}) \mid \sigma \circ \Phi = \Phi \text{ and } \sigma \circ \varphi_0 = \varphi_0 \,\}. \tag{3.1}$$

Note F is a subfield of E via φ_0 (possibly proper when $F/\mathbb{Q}$ is not Galois). We therefore obtain *canonical models* over E of the Shimura varieties

$$\mathrm{Sh}_{K_{\widetilde{H}}}\big(\widetilde{H}, \{h_{\widetilde{H}}\}\big), \quad \mathrm{Sh}_{K_{\widetilde{G}}}\big(\widetilde{G}, \{h_{\widetilde{G}}\}\big), \quad \mathrm{Sh}_{K_{\widetilde{HG}}}\big(\widetilde{HG}, \{h_{\widetilde{HG}}\}\big),$$

where $K_{\widetilde{H}}$, resp. $K_{\widetilde{G}}$, resp. $K_{\widetilde{HG}}$ varies through the open compact subgroups of $\widetilde{H}(\mathbb{A}_f)$, resp. $\widetilde{G}(\mathbb{A}_f)$, resp. $\widetilde{HG}(\mathbb{A}_f)$.

The morphisms (2.2) are obviously compatible with the Shimura data $\{h_{\widetilde{H}}\}$ and $\{h_{\widetilde{G}}\}$, resp. $\{h_{\widetilde{H}}\}$ and $\{h_{\widetilde{HG}}\}$. We therefore obtain injective morphisms of Shimura varieties, i.e. injective morphisms of pro-varieties, in the sense of [9, Prop. 1.15],

$$\mathrm{Sh}\big(\widetilde{H}, \{h_{\widetilde{H}}\}\big) \hookrightarrow \mathrm{Sh}\big(\widetilde{G}, \{h_{\widetilde{G}}\}\big) \quad \text{and} \quad \mathrm{Sh}\big(\widetilde{H}, \{h_{\widetilde{H}}\}\big) \hookrightarrow \mathrm{Sh}\big(\widetilde{HG}, \{h_{\widetilde{HG}}\}\big). \tag{3.2}$$

Remark 3.1. The above Shimura varieties are related to other Shimura varieties, as follows.

(i) The pair $(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\})$ is a Shimura datum, and there are morphisms of Shimura data

$$\big(\widetilde{H}, \{h_{\widetilde{H}}\}\big) \longrightarrow \big(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\}\big), \quad \big(\widetilde{G}, \{h_{\widetilde{G}}\}\big) \longrightarrow \big(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\}\big), \quad \big(\widetilde{HG}, \{h_{\widetilde{HG}}\}\big) \longrightarrow \big(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\}\big)$$

induced by the natural projections to $Z^{\mathbb{Q}}$. These induce morphisms of Shimura varieties

$$\begin{aligned} \mathrm{Sh}\big(\widetilde{H}, \{h_{\widetilde{H}}\}\big) &\longrightarrow \mathrm{Sh}\big(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\}\big), \\ \mathrm{Sh}\big(\widetilde{G}, \{h_{\widetilde{G}}\}\big) &\longrightarrow \mathrm{Sh}\big(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\}\big), \\ \mathrm{Sh}\big(\widetilde{HG}, \{h_{\widetilde{HG}}\}\big) &\longrightarrow \mathrm{Sh}\big(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\}\big), \end{aligned} \tag{3.3}$$

which identify

$$\mathrm{Sh}\big(\widetilde{HG}, \{h_{\widetilde{HG}}\}\big) \cong \mathrm{Sh}\big(\widetilde{H}, \{h_{\widetilde{H}}\}\big) \times_{\mathrm{Sh}(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\})} \mathrm{Sh}\big(\widetilde{G}, \{h_{\widetilde{G}}\}\big).$$

The reflex field of $(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\})$ is the reflex field of the CM type Φ, i.e. the fixed field of the group $\{\sigma \in \mathrm{Aut}(\mathbb{C}) \mid \sigma \circ \Phi = \Phi\}$, which is manifestly contained in E.

(ii) There are morphisms of Shimura data

$$\big(\widetilde{H}, \{h_{\widetilde{H}}\}\big) \longrightarrow \big(H^{\mathbb{Q}}, \{h_{H^{\mathbb{Q}}}\}\big) \quad \text{and} \quad \big(\widetilde{G}, \{h_{\widetilde{G}}\}\big) \longrightarrow \big(G^{\mathbb{Q}}, \{h_{G^{\mathbb{Q}}}\}\big),$$

both induced by the natural projections, which induce morphisms of Shimura varieties

$$\mathrm{Sh}\big(\widetilde{H}, \{h_{\widetilde{H}}\}\big) \longrightarrow \mathrm{Sh}\big(H^{\mathbb{Q}}, \{h_{H^{\mathbb{Q}}}\}\big) \quad \text{and} \quad \mathrm{Sh}\big(\widetilde{G}, \{h_{\widetilde{G}}\}\big) \longrightarrow \mathrm{Sh}\big(G^{\mathbb{Q}}, \{h_{G^{\mathbb{Q}}}\}\big).$$

(iii) One may also introduce Shimura data

$$\big(\mathrm{Res}_{F_0/\mathbb{Q}} H, \{h_H\}\big), \quad \big(\mathrm{Res}_{F_0/\mathbb{Q}} G, \{h_G\}\big), \quad \big(\mathrm{Res}_{F_0/\mathbb{Q}}(H \times G), \{h_{H\times G}\}\big),$$

where h_H, resp. h_G, resp. $h_{H\times G}$ is defined by composing $h_{\widetilde{H}}$, resp. $h_{\widetilde{G}}$, resp. $h_{\widetilde{HG}}$ with the projection to the second factor in (2.1). Note that the restrictions of h_H, h_G, and $h_{H\times G}$ to the subgroup $\mathbb{R}^\times$ of $\mathbb{C}^\times$ are trivial. In particular, the corresponding Shimura varieties are not of PEL type. These are the Shimura varieties that appear in Gan–Gross–Prasad [12, §27]. The product decompositions in (2.1) induce product decompositions of Shimura data,

$$\begin{aligned} \big(\widetilde{H}, \{h_{\widetilde{H}}\}\big) &= \big(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\}\big) \times \big(\mathrm{Res}_{F/F_0} H, \{h_H\}\big), \\ \big(\widetilde{G}, \{h_{\widetilde{G}}\}\big) &= \big(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\}\big) \times \big(\mathrm{Res}_{F/F_0} G, \{h_G\}\big), \\ \big(\widetilde{HG}, \{h_{\widetilde{HG}}\}\big) &= \big(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\}\big) \times \big(\mathrm{Res}_{F_0/\mathbb{Q}}(H \times G), \{h_{H\times G}\}\big). \end{aligned} \tag{3.4}$$

In [12, §27], the Shimura variety $\mathrm{Sh}(\mathrm{Res}_{F_0/\mathbb{Q}} G, \{h_G\})$ is considered over its reflex field, which is F, embedded into $\mathbb{C}$ via φ_0. By contrast, in the present paper, we consider the Shimura variety

$\mathrm{Sh}(\widetilde{G},\{h_{\widetilde{G}}\})$ over E, which is visibly the join of the reflex fields of the two factors in the product decomposition (3.4). The natural projections then induce morphisms of Shimura varieties,

$$\begin{aligned}\mathrm{Sh}\big(\widetilde{H},\{h_{\widetilde{H}}\}\big) &\longrightarrow \mathrm{Sh}\big(\mathrm{Res}_{F_0/\mathbb{Q}} H,\{h_H\}\big),\\ \mathrm{Sh}\big(\widetilde{G},\{h_{\widetilde{G}}\}\big) &\longrightarrow \mathrm{Sh}\big(\mathrm{Res}_{F_0/\mathbb{Q}} G,\{h_G\}\big),\\ \mathrm{Sh}\big(\widetilde{HG},\{h_{\widetilde{HG}}\}\big) &\longrightarrow \mathrm{Sh}\big(\mathrm{Res}_{F_0/\mathbb{Q}}(H\times G),\{h_{H\times G}\}\big).\end{aligned} \tag{3.5}$$

Remark 3.2. Let us finally make our Shimura varieties more concrete. We consider the case of $\mathrm{Sh}(\widetilde{G},\{h_{\widetilde{G}}\})$; the other Shimura varieties are analogous. In terms of the product decomposition $\widetilde{G}_{\mathbb{R}}\cong Z^{\mathbb{Q}}_{\mathbb{R}}\times\prod_{\varphi\in\Phi}\mathrm{U}(W_\varphi)$ induced by by (2.1), the conjugacy class $\{h_{\widetilde{G}}\}$ is the product of $\{h_{Z^{\mathbb{Q}}}\}$ with the $\mathrm{U}(W_\varphi)(\mathbb{R})$-conjugacy class $\{h_{G,\varphi}\}$ for each $\varphi\in\Phi$, where $h_{G,\varphi}$ denotes the φ-component of the cocharacter h_G defined in Remark 3.1(iii). The conjugacy class $\{h_{Z^{\mathbb{Q}}}\}$ consists of a single element; so does $\{h_{G,\varphi}\}$ for $\varphi\neq\varphi_0$, since in this case $h_{G,\varphi}$ is the trivial cocharacter. For $\varphi=\varphi_0$, in terms of the basis for W_{φ_0} chosen above, h_{G,φ_0} is the cocharacter

$$h_{G,\varphi_0}\colon z\longmapsto \mathrm{diag}(z/\overline{z},1,\dots,1).$$

The conjugacy class $\{h_{G,\varphi_0}\}$ then identifies with the open subset $\mathcal{D}_{\varphi_0}\subset\mathbb{P}(W_{\varphi_0})(\mathbb{C})$ of positive-definite lines for the hermitian form (send $h\in\{h_{G,\varphi_0}\}$ to the -1-eigenspace of $h(\sqrt{-1})$; we remark that $\mathcal{D}_{\varphi_0}$ is also isomorphic to the open unit ball in $\mathbb{C}^{n-1}$). Thus for $K_{\widetilde{G}}\subset\widetilde{G}(\mathbb{A}_f)$ an open compact subgroup, we obtain the presentation

$$\mathrm{Sh}_{K_{\widetilde{G}}}\big(\widetilde{G},\{h_{\widetilde{G}}\}\big)(\mathbb{C})=\widetilde{G}(\mathbb{Q})\big\backslash\big[\mathcal{D}_{\varphi_0}\times\widetilde{G}(\mathbb{A}_f)/K_{\widetilde{G}}\big],$$

where the action of $\widetilde{G}(\mathbb{Q})$ is diagonal by the translation action on $\widetilde{G}(\mathbb{A}_f)$ and by the action on $\mathcal{D}_{\varphi_0}$ given via

$$\widetilde{G}(\mathbb{Q})\longrightarrow\widetilde{G}_{\mathrm{ad}}(\mathbb{R})\longrightarrow\mathrm{PU}(W_{\varphi_0})(\mathbb{R}).$$

3.2. **The moduli problem over** E**.** In this subsection we define moduli problems on the category of schemes over $\operatorname{Spec}E$ for the three Shimura varieties above. Since this is almost identical in each case, let us do this for $\mathrm{Sh}(\widetilde{G},\{h_{\widetilde{G}}\})$, and only indicate briefly the modifications needed for the other two (mostly for $\mathrm{Sh}(\widetilde{HG},\{h_{\widetilde{HG}}\})$). We will only consider open compact subgroups $K_{\widetilde{G}}\subset\widetilde{G}(\mathbb{A}_f)$ which, with respect to the product decomposition (2.1), are of the form

$$K_{\widetilde{G}}=K_{Z^{\mathbb{Q}}}\times K_G, \tag{3.6}$$

where $K_G\subset G(\mathbb{A}_{F_0,f})$ is an open compact subgroup and where $K_{Z^{\mathbb{Q}}}\subset Z^{\mathbb{Q}}(\mathbb{A}_f)$ is the unique maximal compact subgroup

$$K_{Z^{\mathbb{Q}}}:=Z^{\mathbb{Q}}(\widehat{\mathbb{Z}})=\big\{\,z\in(O_F\otimes\widehat{\mathbb{Z}})^\times\bigm|\mathrm{Nm}_{F/F_0}(z)\in\widehat{\mathbb{Z}}^\times\,\big\}. \tag{3.7}$$

(Note that $Z^{\mathbb{Q}}$ is defined over $\operatorname{Spec}\mathbb{Z}$ in an obvious way.)

Before doing this, let us first introduce an auxiliary moduli problem M_0 over E. In fact, for use in the construction of integral models later, we will define a moduli problem $\mathcal{M}_0$ over O_E whose generic fiber will be M_0. For a locally noetherian O_E-scheme S, we define $\mathcal{M}_0(S)$ to be the groupoid of triples (A_0,ι_0,λ_0), where

- A_0 is an abelian variety over S with an O_F-action $\iota_0\colon O_F\to\mathrm{End}(A_0)$, which satisfies the Kottwitz condition of signature $((0,1)_{\varphi\in\Phi})$, i.e.,

$$\mathrm{char}\big(\iota(a)\mid\mathrm{Lie}\,A_0\big)=\prod_{\varphi\in\Phi}\big(T-\overline{\varphi}(a)\big)\quad\text{for all}\quad a\in O_F; \tag{3.8}$$

and

- λ_0 is a principal polarization of A_0 whose Rosati involution induces on O_F, via ι_0, the nontrivial Galois automorphism of F/F_0 .

A morphism between two objects (A_0,ι_0,λ_0) and $(A_0',\iota_0',\lambda_0')$ is an O_F-linear isomorphism $\mu_0\colon A_0\to A_0'$ under which λ_0' pulls back to λ_0. Then $\mathcal{M}_0$ is a Deligne–Mumford stack, finite and étale over $\operatorname{Spec}O_E$, cf. [18, Prop. 3.1.2].[4] (In fact $\mathcal{M}_0$ is defined over the ring of integers in

[4]Strictly speaking loc. cit. is stated only for CM algebras and CM types Φ which are of a rather special sort, but the proof relies only on the very general Th. 2.2.1 in [18] and applies equally well to our situation.

the reflex field of Φ, which is contained in O_E, cf. Remark 3.1(i).) We let M_0 denote the generic fiber of $\mathcal{M}_0$. Then $M_0 \otimes_E \mathbb{C}$ is isomorphic to $\mathrm{Sh}_{K_{Z^{\mathbb{Q}}}}(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\})$, provided $\mathcal{M}_0$ is non-empty.

In order to circumvent the problem of non-emptiness of $\mathcal{M}_0$, we also introduce the following variant of $\mathcal{M}_0$, cf. [18, Def. 3.1.1]. Fix a non-zero ideal $\mathfrak{a}$ of O_{F_0}. Then we introduce the Deligne–Mumford stack $\mathcal{M}_0^{\mathfrak{a}}$ of triples $(A_0, \iota_0, \lambda_0)$ as before, except that we replace the condition that λ_0 is principal by the condition that $\ker \lambda_0 = A_0[\mathfrak{a}]$. Then, again, $\mathcal{M}_0^{\mathfrak{a}}$ is finite and étale over $\mathrm{Spec}\, O_E$ and $\mathcal{M}_0^{\mathfrak{a}} \otimes_E \mathbb{C}$ is isomorphic to $\mathrm{Sh}_{K_{Z^{\mathbb{Q}}}}(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\})$, provided that $\mathcal{M}_0^{\mathfrak{a}}$ is non-empty, cf. [18, Prop. 3.1.2].

Remark 3.3. (i) Given finitely many prime numbers $p_1, \ldots, p_r$, there always exists $\mathfrak{a}$ relatively prime to $p_1, \ldots, p_r$ such that $\mathcal{M}_0^{\mathfrak{a}}$ is non-empty.

(ii) If F/F_0 is ramified at some finite place, then $\mathcal{M}_0^{\mathfrak{a}}$ is non-empty for any $\mathfrak{a}$, cf. [18, proof of Prop. 3.1.6]. A special case of this is when $F = F_0 K$, where K is an imaginary quadratic field and the discriminants of $K/\mathbb{Q}$ and $F_0/\mathbb{Q}$ are relatively prime. We further remark that in the context of the global integral models we define in Section 5 below, we will eventually impose conditions on the hermitian spaces W and $W^\flat$ that force F/F_0 to be ramified at some finite place, cf. Remark 5.2.

In the following, we fix an ideal $\mathfrak{a}$ such that $\mathcal{M}_0^{\mathfrak{a}}$ is non-empty. We denote its generic fiber by $M_0^{\mathfrak{a}}$. We now define a groupoid $M_{K_{\widetilde{G}}}(\widetilde{G})$ fibered over the category of locally noetherian schemes over E. Here, to lighten notation, we have suppressed the ideal $\mathfrak{a}$. For such a scheme S, the objects of $M_{K_G}(\widetilde{G})(S)$ are collections $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta})$, where

- $(A_0, \iota_0, \lambda_0)$ is an object of $M_0^{\mathfrak{a}}(S)$;
- A is an abelian scheme over S, up to isogeny, with an F-action $\iota\colon F \to \mathrm{End}^\circ(A)$ satisfying the Kottwitz condition of signature $((1, n-1)_{\varphi_0}, (0, n)_{\varphi \in \Phi \smallsetminus \{\varphi_0\}})$, i.e.,

$$\mathrm{char}\big(\iota(a) \mid \mathrm{Lie}\, A\big) = \big(T - \varphi_0(a)\big)\big(T - \varphi_0(\bar{a})\big)^{n-1} \prod_{\varphi \in \Phi \smallsetminus \{\varphi_0\}} \big(T - \varphi(\bar{a})\big)^n \quad \text{for all} \quad a \in O_F; \tag{3.9}$$

- λ is a polarization of A, in the sense of [30, fn. p. 111], whose Rosati involution induces on F, via ι, the nontrivial Galois automorphism of F/F_0; and
- $\overline{\eta}$ is a $K_{\widetilde{G}}$-level structure, by which we mean a K_G-orbit (equivalently, a $K_{\widetilde{G}}$-orbit, where $K_{\widetilde{G}}$ acts through its projection $K_{\widetilde{G}} \to K_G$) of $\mathbb{A}_{F,f}$-linear isometries

$$\eta\colon \widehat{V}(A_0, A) \simeq -W \otimes_F \mathbb{A}_{F,f};$$

comp. [30, Rem. 4.2]. Here

$$\widehat{V}(A_0, A) := \mathrm{Hom}_F\big(\widehat{V}(A_0), \widehat{V}(A)\big),$$

endowed with its natural hermitian form h_A taking values in $\mathbb{A}_{F,f}$. Recall [30, §2.3] that this form is defined by the formula

$$h_A(x, y) = \lambda_0^{-1} \circ y^\vee \circ \lambda \circ x \in \mathrm{End}_{\mathbb{A}_{F,f}}\big(\widehat{V}(A_0)\big) = \mathbb{A}_{F,f}. \tag{3.10}$$

A morphism between two objects $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta})$, and $(A_0', \iota_0', \lambda_0', A', \iota', \lambda', \overline{\eta}')$ is given by an isomorphism $\mu_0\colon (A_0, \iota_0, \lambda_0) \xrightarrow{\sim} (A_0', \iota_0', \lambda_0')$ in $M_0^{\mathfrak{a}}(S)$ and an F-linear isogeny $\mu\colon A \to A'$ pulling λ' back to λ and $\overline{\eta}'$ back to $\overline{\eta}$.

Remark 3.4. Let $r\colon \mathrm{Hom}(F, \mathbb{C}) \to \{0, 1, n-1, n\}$, $\varphi \mapsto r_\varphi$, be the function defined by

$$r_\varphi := \begin{cases} 1, & \varphi = \varphi_0; \\ 0, & \varphi \in \Phi \smallsetminus \{\varphi_0\}; \\ n - r_{\overline{\varphi}}, & \varphi \notin \Phi. \end{cases} \tag{3.11}$$

Then the Kottwitz condition (3.9) is

$$\mathrm{char}\big(\iota(a) \mid \mathrm{Lie}\, A\big) = \prod_{\varphi \in \mathrm{Hom}(F, \mathbb{C})} \big(T - \varphi(a)\big)^{r_\varphi} \quad \text{for all} \quad a \in O_F,$$

comp. [48, (8.4)].

The following proposition is a special case of Deligne's description of Shimura varieties of PEL type.

Proposition 3.5. *$M_{K_{\widetilde{G}}}(\widetilde{G})$ is a Deligne–Mumford stack smooth of relative dimension $n-1$ over $\operatorname{Spec} E$. The coarse moduli scheme of $M_{K_{\widetilde{G}}}(\widetilde{G})$ is a quasi-projective scheme over $\operatorname{Spec} E$, naturally isomorphic to the canonical model of $\mathrm{Sh}_{K_{\widetilde{G}}}(\widetilde{G},\{h_{\widetilde{G}}\})$. For $K_{\widetilde{G}}$ sufficiently small, the forgetful morphism $M_{K_{\widetilde{G}}}(\widetilde{G}) \to M_0^{\mathfrak{a}}$ is relatively representable.*

Proof. We will content ourselves with exhibiting a map, which turns out to be an isomorphism,

$$M_{K_{\widetilde{G}}}(\widetilde{G}) \otimes_E \mathbb{C} \longrightarrow \mathrm{Sh}_{K_{\widetilde{G}}}\big(\widetilde{G},\{h_{\widetilde{G}}\}\big).$$

By Remark 3.1(iii), the target is the product of Shimura varieties

$$\mathrm{Sh}_{K_{Z^{\mathbb{Q}}}}\big(Z^{\mathbb{Q}},\{h_{Z^{\mathbb{Q}}}\}\big) \times \mathrm{Sh}_{K_G}\big(\operatorname{Res}_{F/F_0} G,\{h_G\}\big).$$

For the map into the first factor, we simply compose the forgetful map $M_{K_{\widetilde{G}}}(\widetilde{G})\otimes_E\mathbb{C} \to M_0^{\mathfrak{a}}\otimes_E\mathbb{C}$ with the isomorphism $M_0^{\mathfrak{a}} \otimes_E \mathbb{C} \simeq \mathrm{Sh}_{K_{Z^{\mathbb{Q}}}}(Z^{\mathbb{Q}},\{h_{Z^{\mathbb{Q}}}\})$.

To explain the map into the second factor, let $(A_0,\iota_0,\lambda_0,A,\iota,\lambda,\overline{\eta})$ be a $\mathbb{C}$-valued point of $M_{K_{\widetilde{G}}}(\widetilde{G})$. Let $\mathcal{H} := H_1(A,\mathbb{Q})$ and $\mathcal{H}_0 := H_1(A_0,\mathbb{Q})$. The polarization λ endows $\mathcal{H}$ with a $\mathbb{Q}$-valued alternating form $\langle\ ,\ \rangle$ satisfying $\langle \iota(a)x,y\rangle = \langle x,\iota(\overline{a})y\rangle$ for all $a \in F$, and such that the induced form $x,y \mapsto \langle\sqrt{-1}\cdot x,y\rangle$ on $\mathcal{H}\otimes_{\mathbb{Q}}\mathbb{R}$ is symmetric and positive definite, where multiplication by $\sqrt{-1}$ is defined in terms of the right-hand side of the canonical isomorphism $\mathcal{H}_{\mathbb{R}} \cong \operatorname{Lie} A$. Similarly, λ_0 endows $\mathcal{H}_0$ with a Riemann form $\langle\ ,\ \rangle_0$.

Let $V(A_0,A) := \operatorname{Hom}_F(\mathcal{H}_0,\mathcal{H})$. Then $V(A_0,A)$ is an F-vector space of the same dimension as W, and we make it into an F/F_0-hermitian space by defining the pairing (α,β) to be the composite

$$\begin{array}{c}\mathcal{H}_0 \xrightarrow{\ \alpha\ } \mathcal{H} \longrightarrow \mathcal{H}^\vee \xrightarrow{\ \beta^\vee\ } \mathcal{H}_0^\vee \longrightarrow \mathcal{H}_0,\\ x \longmapsto \langle x,-\rangle\end{array}$$

where the checks denote $\mathbb{Q}$-linear duals and the last arrow is the inverse of $y \mapsto \langle y,-\rangle_0$; this composite is an F-linear endomorphism of the one-dimensional F-vector space $\mathcal{H}_0$, and hence identifies with an element in F. Clearly $V(A_0,A)\otimes_F\mathbb{A}_{F,f} \cong \widehat{V}(A_0,A)$ as hermitian spaces, and hence, by the existence of a level structure, $V(A_0,A)\otimes_F\mathbb{A}_{F,f} \simeq -W\otimes_F\mathbb{A}_{F,f}$. Furthermore, it is easy to see that the Kottwitz condition (3.9) implies that $V(A_0,A)_\varphi$ has signature $(n-1,1)$ if $\varphi = \varphi_0$ and $(n,0)$ if $\varphi\in\Phi\smallsetminus\{\varphi_0\}$. Hence, by the Hasse principle for hermitian spaces, $V(A_0,A)$ and $-W$ are isomorphic. Choose an isometry $j\colon V(A_0,A) \xrightarrow{\sim} -W$. Using the complex structures on $\mathcal{H}_{0,\mathbb{R}}$ and $\mathcal{H}_{\mathbb{R}}$, let $z\in\mathbb{C}^\times$ act on $V(A_0,A)$ by sending the F-linear map α to $z\alpha z^{-1}$. This defines a homomorphism $\mathbb{C}^\times \to \mathrm{U}(V(A_0,A))(\mathbb{R})$, and composing this with $j_*\colon \mathrm{U}(V(A_0,A))(\mathbb{R}) \xrightarrow{\sim} \mathrm{U}(-W)(\mathbb{R}) = \mathrm{U}(W)(\mathbb{R})$ gives an element in $\{h_G\}$. The level structure $\overline{\eta}$ corresponds to an element of $(\operatorname{Res}_{F_0/\mathbb{Q}} G)(\mathbb{A}_f)/K_G$, and eliminating the choice of j corresponds to dividing out by the action of $(\operatorname{Res}_{F_0/\mathbb{Q}} G)(\mathbb{Q})$. $\square$

An analogous description holds for the model $M_{K_{\widetilde{H}}}(\widetilde{H})$ of the Shimura variety $\mathrm{Sh}_{K_{\widetilde{H}}}(\widetilde{H},\{h_{\widetilde{H}}\})$ (replace n by $n-1$, and W by $W^\flat$).

There is also an analog for the Shimura variety $\mathrm{Sh}_{K_{\widetilde{HG}}}(\widetilde{HG},\{h_{\widetilde{HG}}\})$. In this case we take the level subgroup to be of the form

$$K_{\widetilde{HG}} = K_{Z^{\mathbb{Q}}} \times K_H \times K_G, \tag{3.12}$$

where as always $K_{Z^{\mathbb{Q}}}$ is the subgroup (3.7), and $K_H \subset H(\mathbb{A}_{F_0,f})$ and $K_G \subset G(\mathbb{A}_{F_0,f})$ are open compact subgroups. The value of the corresponding moduli functor $M_{K_{\widetilde{HG}}}(\widetilde{HG})$ on a locally noetherian scheme S over E is the set of isomorphism classes of tuples

$$\big(A_0,\iota_0,\lambda_0,A^\flat,\iota^\flat,\lambda^\flat,A,\iota,\lambda,\overline{(\eta^\flat,\eta)}\big),$$

where the last entry is a pair of $\mathbb{A}_{F,f}$-linear isometries

$$\eta^\flat\colon \widehat{V}(A_0,A^\flat) \simeq -W^\flat\otimes_F\mathbb{A}_{F,f} \quad\text{and}\quad \eta\colon \widehat{V}(A_0,A) \simeq -W\otimes_F\mathbb{A}_{F,f},$$

modulo the action of $K_H \times K_G$. In other words, the moduli functor $M_{K_{\widetilde{HG}}}(\widetilde{HG})$ is simply the fibered product $M_{K_{\widetilde{H}}}(\widetilde{H}) \times_{M_0^{\mathfrak{a}}} M_{K_{\widetilde{G}}}(\widetilde{G})$.

In terms of these moduli problems, the injective morphisms (3.2) can be described as follows. Assume that $K_H \subset H(\mathbb{A}_{F_0,f}) \cap K_G$. Then the first morphism of Shimura varieties in (3.2) arises by base change from E to $\mathbb{C}$ from the functor morphism

$$\begin{array}{ccc} M_{K_{\widetilde{H}}}(\widetilde{H}) & \longrightarrow & M_{K_{\widetilde{G}}}(\widetilde{G}) \\ \big(A_0, \iota_0, \lambda_0, A^\flat, \iota^\flat, \lambda^\flat, \overline{\eta}^\flat\big) & \longmapsto & \big(A_0, \iota_0, \lambda_0, A^\flat \times A_0, \iota^\flat \times \iota_0, \lambda^\flat \times \lambda_0(u), \overline{\eta}\big). \end{array} \tag{3.13}$$

Here $\lambda_0(u) = -(u,u)\lambda_0$, and the isomorphism $\eta \colon \widehat{V}(A_0, A^\flat \times A_0) \simeq -W \otimes_F \mathbb{A}_{F,f} \bmod K_G$ is given, with respect to the decomposition

$$\mathrm{Hom}_F\big(\widehat{V}(A_0), \widehat{V}(A^\flat \times A_0)\big) \cong \mathrm{Hom}_F\big(\widehat{V}(A_0), \widehat{V}(A^\flat) \oplus \widehat{V}(A_0)\big) \cong \widehat{V}(A_0, A^\flat) \oplus \mathbb{A}_{F,f},$$

by the trivialization

$$\eta^\flat \colon \widehat{V}(A_0, A^\flat) \simeq -W^\flat \otimes_F \mathbb{A}_{F,f} \bmod K_H$$

in the first summand, and by identifying the basis element 1 in the second summand with $u \otimes 1 \in -W \otimes_F \mathbb{A}_{F,f}$. The morphism (3.13) is finite and unramified.

The second morphism in (3.2) then arises from the graph morphism of (3.13),

$$M_{K_{\widetilde{H}}}(\widetilde{H}) \longrightarrow M_{K_{\widetilde{HG}}}(\widetilde{HG}) = M_{K_{\widetilde{H}}}(\widetilde{H}) \times_{M_0^{\mathfrak{a}}} M_{K_{\widetilde{G}}}(\widetilde{G}). \tag{3.14}$$

The morphism (3.14) is a closed embedding.

Remark 3.6. For $F_0 \neq \mathbb{Q}$, the Shimura varieties above are compact. For $F_0 = \mathbb{Q}$, it may happen that the Shimura variety $M_{K_{\widetilde{HG}}}(\widetilde{HG})$ is non-compact. In fact, this will be automatic when $n \geq 3$. In this case, we will need to use its canonical toroidal compactification, cf. [19, §2].

4. Semi-global integral models

Fix a prime number p and an embedding $\widetilde{\nu} \colon \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$. This determines a p-adic place ν of E and, via φ_0, places v_0 of F_0 and w_0 of F. In this section we are going to define "semi-global" integral models over $O_{E,(\nu)}$ of the moduli spaces introduced in Section 3, in the case of various level structures at p. We denote by S_p the set of places v of F_0 over p. Throughout this section, we assume that the ideal $\mathfrak{a}$ occurring in the definition of $\mathcal{M}_0^{\mathfrak{a}}$ is prime to p, cf. Remark 3.3(i).

4.1. Hyperspecial level at v_0.

In this case we assume that the place v_0 is unramified over p, and that v_0 either splits in F or is inert in F and the hermitian space W_{v_0} is split. We also assume $p \neq 2$ if there is any $v \in S_p$ which is non-split in F. We are going to define smooth models over $O_{E,(\nu)}$.

For each $v \in S_p$, choose a vertex lattice Λ_v in the $F_v/F_{0,v}$-hermitian space W_v. By our case assumptions, we may and will take Λ_{v_0} to be self-dual. Recalling the subgroup $K_{\widetilde{G}} = K_{Z^{\mathbb{Q}}} \times K_G$ from (3.6), we take K_G to be of the form

$$K_G = K_G^p \times K_{G,p},$$

where $K_G^p \subset G(\mathbb{A}_{F_0,f}^p)$ is arbitrary, and where the level subgroup at p is the product

$$K_{G,p} := \prod_{v \in S_p} K_{G,v} \subset G(F_0 \otimes \mathbb{Q}_p) = \prod_{v \in S_p} G(F_{0,v}), \tag{4.1}$$

with $K_{G,v}$ the stabilizer of Λ_v in $G(F_{0,v})$.

We formulate a moduli problem over $\operatorname{Spec} O_{E,(\nu)}$ as follows. To each locally noetherian $O_{E,(\nu)}$-scheme S, we associate the set of isomorphism classes of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}^p)$, where $(A_0, \iota_0, \lambda_0)$ is an object of $\mathcal{M}_0^{\mathfrak{a}}(S)$. Furthermore:

- (A, ι) is an abelian scheme over S up to isogeny prime to p, with an $O_F \otimes \mathbb{Z}_{(p)}$-action ι, satisfying the Kottwitz condition (3.9) of signature $((1, n-1)_{\varphi_0}, (0, n)_{\varphi \in \Phi \smallsetminus \{\varphi_0\}})$.

• λ is a polarization on A whose Rosati involution induces on $O_F \otimes \mathbb{Z}_{(p)}$ the non-trivial Galois automorphism of F/F_0, subject to the following condition. First note that the action of the ring $O_{F_0} \otimes \mathbb{Z}_p \cong \prod_{v \in S_p} O_{F_0,v}$ on the p-divisible group $A[p^\infty]$ induces a decomposition of p-divisible groups,

$$A[p^\infty] = \prod_{v \in S_p} A[v^\infty]. \tag{4.2}$$

Since Ros_λ is trivial on O_{F_0}, λ induces a polarization $\lambda_v \colon A[v^\infty] \to A^\vee[v^\infty] \cong A[v^\infty]^\vee$ for each v. The condition we impose is that $\ker \lambda_v$ is contained in $A[\iota(\pi_v)]$ of rank $\#(\Lambda_v^\vee/\Lambda_v)$ for each $v \in S_p$.

• $\overline{\eta}^p$ is a K_G^p-orbit of $\mathbb{A}_{F,f}^p$-linear isometries

$$\eta^p \colon \widehat{V}^p(A_0, A) \simeq -W \otimes_F \mathbb{A}_{F,f}^p, \tag{4.3}$$

where

$$\widehat{V}^p(A_0, A) := \mathrm{Hom}_F\big(\widehat{V}^p(A_0), \widehat{V}^p(A)\big),$$

and the hermitian form on $\widehat{V}^p(A_0, A)$ is the obvious prime-to-p analog of (3.10).

We also impose for each $v \neq v_0$ over p the *sign condition* and the *Eisenstein condition*. Let us explain these conditions.

The sign condition at v is only non-empty when v does not split in F, in which case it demands that at every point s of S,

$$\mathrm{inv}_v^r(A_{0,s}, \iota_{0,s}, \lambda_{0,s}, A_s, \iota_s, \lambda_s) = \mathrm{inv}_v(-W_v). \tag{4.4}$$

Here the left-hand side is the sign factor defined in (A.5) and (A.8) in Appendix A (in the definition of (A.8), one may use the embedding $\widetilde{\nu}$ fixed at the beginning of this section). The right-hand side is the Hasse invariant of the hermitian space $-W_v$ defined above in (1.4). Note that by Proposition A.1, the left-hand side of (4.4) is a locally constant function in s.

The Eisenstein condition is only non-empty when the base scheme S has non-empty special fiber. In this case, we may even base change via $\widetilde{\nu} \colon O_{E,(\nu)} \to \overline{\mathbb{Z}}_p$ (the ring of integers in $\overline{\mathbb{Q}}_p$) and pass to completions and assume that S is a scheme over $\mathrm{Spf}\, \overline{\mathbb{Z}}_p$. Similarly to (4.2), there is a decomposition of the p-divisible group $A[p^\infty]$,

$$A[p^\infty] = \prod_{w|p} A[w^\infty], \tag{4.5}$$

where the indices range over the places w of F lying over p. Since we assume that p is locally nilpotent on S, there is a natural isomorphism

$$\mathrm{Lie}\, A \cong \mathrm{Lie}\, A[p^\infty] = \bigoplus_{w|p} \mathrm{Lie}\, A[w^\infty].$$

For each place w, by the Kottwitz condition (3.9), the p-divisible group $A[w^\infty]$ is of height $n \cdot [F_w : \mathbb{Q}_p]$ and dimension

$$\dim A[w^\infty] = \sum_{\varphi \in \mathrm{Hom}(F_w, \overline{\mathbb{Q}}_p)} r_\varphi. \tag{4.6}$$

Here r_φ is as in (3.11), and we have used the embedding $\widetilde{\nu} \colon \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ to identify

$$\mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}) \simeq \mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}_p),$$

which in turn identifies

$$\big\{\, \varphi \in \mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}) \bigm| w_\varphi = w \,\big\} \simeq \mathrm{Hom}_{\mathbb{Q}_p}(F_w, \overline{\mathbb{Q}}_p), \tag{4.7}$$

where w_φ denotes the p-adic place in F induced by $\widetilde{\nu} \circ \varphi$.

Now suppose that w lies over a place v different from v_0. Then the action of F on $A[w^\infty]$ is of a *banal* signature type, in the sense that each integer r_φ occurring in (4.6) is equal to 0 or n, cf. Appendix B. Let $\pi = \pi_w$ be a uniformizer in F_w, and let $F_w^t \subset F_w$ be the maximal unramified subextension of $\mathbb{Q}_p$. For each $\psi \in \mathrm{Hom}_{\mathbb{Q}_p}(F_w^t, \overline{\mathbb{Q}}_p)$, let

$$A_\psi := \big\{\, \varphi \in \mathrm{Hom}_{\mathbb{Q}_p}(F_w, \overline{\mathbb{Q}}_p) \bigm| \varphi|_{F_w^t} = \psi \text{ and } r_\varphi = n \,\big\}.$$

Set

$$Q_{A_\psi}(T) := \prod_{\varphi\in A_\psi} \bigl(T-\varphi(\pi)\bigr) \in \overline{\mathbb{Z}}_p[T].$$

The Eisenstein condition at v demands the identity of endomorphisms of $\operatorname{Lie} A[w^\infty]$, for each of the one or two places w over v,

$$Q_{A_\psi}\bigl(\iota(\pi) \mid \operatorname{Lie} A[w^\infty]\bigr) = 0 \quad \text{for all} \quad \psi \in \operatorname{Hom}_{\mathbb{Q}_p}(F_w^t, \overline{\mathbb{Q}}_p). \tag{4.8}$$

This condition is the analog in our context of the condition with the same name in [48]. The Kottwitz condition implies that the Eisenstein condition at v is automatically satisfied when the place(s) w over v are unramified over p, cf. Lemma B.3.

A morphism between two objects $(A_0,\iota_0,\lambda_0,A,\iota,\lambda,\overline{\eta}^p)$ and $(A_0',\iota_0',\lambda_0',A',\iota',\lambda',\overline{\eta}'^p)$ is given by an isomorphism $(A_0,\iota_0,\lambda_0) \xrightarrow{\sim} (A_0',\iota_0',\lambda_0')$ in $\mathcal{M}_0^{\mathfrak{a}}(S)$ and a quasi-isogeny $A\to A'$ which induces an isomorphism

$$A[p^\infty] \xrightarrow{\sim} A'[p^\infty],$$

compatible with ι and ι', with λ and λ', and with $\overline{\eta}^p$ and $\overline{\eta}^{p\prime}$.

Theorem 4.1. *The moduli problem just formulated is representable by a Deligne–Mumford stack $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ smooth over $\operatorname{Spec} O_{E,(\nu)}$. For $K_{\widetilde{G}}^p$ small enough, $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ is relatively representable over $\mathcal{M}_0^{\mathfrak{a}}$. Furthermore, the generic fiber $\mathcal{M}_{K_{\widetilde{G}}} \times_{\operatorname{Spec} O_{E,(\nu)}} \operatorname{Spec} E$ is canonically isomorphic to $M_{K_{\widetilde{G}}}(\widetilde{G})$.*

Proof. Representability and relative representability are standard, cf. [27, p. 391]. Smoothness follows as usual from the theory of local models, cf. [40]. More precisely, the local model for $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ decomposes into a product of local models, one for each of the abelian schemes A_0 and A in the moduli problem. The local model corresponding to A_0 is étale because $\mathcal{M}_0^{\mathfrak{a}}$ is. Now, under (4.7) we have

$$\operatorname{Hom}_{\mathbb{Q}}(F,\overline{\mathbb{Q}}) \simeq \coprod_{v\in S_p} \operatorname{Hom}_{\mathbb{Q}_p}(F_v,\mathbb{Q}_p). \tag{4.9}$$

In this way the completion E_ν identifies with the join of the local reflex fields E_{Φ_v} and $E_{r|_v}$ in $\overline{\mathbb{Q}}_p$ as v varies through S_p, where $\Phi_v := \Phi \cap \operatorname{Hom}_{\mathbb{Q}_p}(F_v,\overline{\mathbb{Q}}_p)$ in terms of the identification (4.9), and where $r|_v\colon \operatorname{Hom}_{\mathbb{Q}_p}(F_v,\overline{\mathbb{Q}}_p)\to\mathbb{Z}$ denotes the restriction of the function r to the v-summand on the right-hand side of (4.9). The local model M corresponding to A then decomposes as

$$M = \prod_{v\in S_p} M_v \times_{\operatorname{Spec} O_{E_{r|_v}}} \operatorname{Spec} O_{E_\nu}. \tag{4.10}$$

Here for $v\neq v_0$, by the Kottwitz condition (3.9), $M_v = M(F_v/F_{0,v}, r|_v, \Lambda_v)$ is a *banal* local model, i.e. of the form defined in Appendix B. Hence $M_v = \operatorname{Spec} O_{E_{r|_v}}$ by Lemmas B.1 and B.4. (The same is true for the local model corresponding to A_0 at every $v\in S_p$, by these lemmas and Remark B.2.) The local model M_{v_0} is smooth by [15].

It remains to prove the last assertion. Let S be a scheme over E, and let $(A_0,\iota_0,\lambda_0,A,\iota,\lambda,\overline{\eta}^p)$ be a point of $\mathcal{M}_{K_{\widetilde{G}}}(S)$. We want to associate to this a point of $M_{K_{\widetilde{G}}}(S)$, i.e., we want to add the p-component of $\overline{\eta}$. The product of hermitian spaces $W\otimes_{\mathbb{Q}}\mathbb{Q}_p = \prod_{v\in S_p} W_v$ contains the lattice $\prod_{v\in S_p}\Lambda_v$, where Λ_v is a vertex lattice in W_v. By assumption on the polarization λ, the product of hermitian spaces $\widehat{V}_p(A_0,A) := \operatorname{Hom}_F(V_p(A_0),V_p(A)) = \prod_{v\in S_p}\widehat{V}_v(A_0,A)$ contains $\operatorname{Hom}_{O_F}(T_p(A_0),T_p(A))$ as a product of vertex lattices, where the factor at each v is of the same type as Λ_v. Since $p\neq 2$ when there are non-split places in S_p, it follows that, if there exists an isometry $\eta_p\colon \widehat{V}_p(A_0,A)\simeq -W\otimes_{\mathbb{Q}}\mathbb{Q}_p$ at all, then there also exists one that maps these two vertex lattices of identical type into one another, and the class modulo $K_{G,p}$ of such an isometry is then uniquely determined.

Hence we are reduced to showing that there exists an isometry η_p, i.e., the equality of Hasse invariants $\operatorname{inv}_v(\widehat{V}_v(A_0,A)) = \operatorname{inv}_v(-W_v)$ for all $v\in S_p$. By the Hasse principle and the product formula for hermitian spaces, it suffices to prove that for any $\mathbb{C}$-valued point $(A_0,\iota_0,\lambda_0,A,\iota,\lambda,\overline{\eta}^p)$ of $\mathcal{M}_{K_{\widetilde{G}}}$,

$$\operatorname{inv}_v\bigl(V(A_0,A)_v\bigr) = \operatorname{inv}_v(-W_v) \quad \text{for all} \quad v\neq v_0,$$

where v runs through all places of F_0, including the archimedean ones. Here, as in the proof of Proposition 3.5, $V(A_0, A) = \mathrm{Hom}_F(\mathcal{H}_0, \mathcal{H})$, where $\mathcal{H}_0 = H_1(A_0, \mathbb{Q})$ and $\mathcal{H} = H_1(A, \mathbb{Q})$. For the non-archimedean places not lying over p, this follows from the existence of the level structure; for the places $v \in S_p \smallsetminus \{v_0\}$, this follows from the sign condition (4.4) at v; and finally, for the archimedean places, this follows from the fact that the signatures of $V(A_0, A)$ and $-W$ at all archimedean places are identical, cf. the proof of Proposition 3.5. $\square$

We analogously define the DM stacks $\mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H})$ and $\mathcal{M}_{K_{\widetilde{HG}}}(\widetilde{HG})$ over $\operatorname{Spec} O_{E,(\nu)}$, comp. the end of Section 3.2. Both are again smooth over $\operatorname{Spec} O_{E,(\nu)}$.

Let us now assume that the special vector $u \in W$ has norm $(u, u) \in O_{F_0,(p)}^\times$. Then we obtain a finite unramified morphism, resp. a closed embedding, in analogy with (3.13), resp. (3.14),

$$\mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H}) \longrightarrow \mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G}) \quad \text{and} \quad \mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H}) \hookrightarrow \mathcal{M}_{K_{\widetilde{HG}}}(\widetilde{HG}). \tag{4.11}$$

For this we assume that $K_H^p \subset H(\mathbb{A}_{F_0,f}^p) \cap K_G^p$. Furthermore, we assume for each $v \in S_p$, the lattices in W_v and $W_v^\flat$ satisfy the relation

$$\Lambda_v = \Lambda_v^\flat \oplus O_{F,v} u. \tag{4.12}$$

For the lattice in $W_v^\flat \oplus W_v$, we take the direct sum $\Lambda_v^\flat \oplus \Lambda_v$.

We end this subsection by defining Hecke correspondences attached to adelic elements prime to p. We first consider the case of $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$. Fix $g \in \widetilde{G}(\mathbb{A}_f^p)$. Let $K_{\widetilde{G}}^p := Z^{\mathbb{Q}}(\widehat{\mathbb{Z}}^p) \times K_G^p$ and $K_{\widetilde{G},p} := Z^{\mathbb{Q}}(\mathbb{Z}_p) \times K_{G,p}$, and set

$$K_{\widetilde{G}}'^p := K_{\widetilde{G}}^p \cap g K_{\widetilde{G}}^p g^{-1} \quad \text{and} \quad K_{\widetilde{G}}' := K_{\widetilde{G}}'^p \times K_{\widetilde{G},p}.$$

Then we obtain in the standard way a diagram of finite étale morphisms,

$$\begin{tikzcd} & \mathcal{M}_{K'_{\widetilde{G}}}(\widetilde{G}) \arrow[dl, "\mathrm{nat}_1"'] \arrow[dr, "\mathrm{nat}_g"] & \\ \mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G}) & & \mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G}), \end{tikzcd} \tag{4.13}$$

which we view as a correspondence from $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ to itself. Note that for a central element $g = z \in Z(G)(\mathbb{A}_{F_0,f}^p) = \{z \in (\mathbb{A}_{F,f}^p)^\times \mid \mathrm{Nm}_{F/F_0}(z) = 1\}$, the diagram (4.13) collapses to a map

$$\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G}) \xrightarrow{z} \mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G}), \tag{4.14}$$

and this induces an action of $Z(G)(\mathbb{A}_{F_0,f}^p)$ on $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$.

The cases of $\mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H})$ and $\mathcal{M}_{K_{\widetilde{HG}}}(\widetilde{HG})$ are completely analogous, simply replacing $\widetilde{G}$ everywhere by $\widetilde{H}$ and $\widetilde{HG}$, respectively. For later use, we record the diagram of finite étale morphisms we obtain for $\widetilde{HG}$:

$$\begin{tikzcd} & \mathcal{M}_{K'_{\widetilde{HG}}}(\widetilde{HG}) \arrow[dl, "\mathrm{nat}_1"'] \arrow[dr, "\mathrm{nat}_g"] & \\ \mathcal{M}_{K_{\widetilde{HG}}}(\widetilde{HG}) & & \mathcal{M}_{K_{\widetilde{HG}}}(\widetilde{HG}). \end{tikzcd} \tag{4.15}$$

Remark 4.2. It is possible to extend the above definitions to open compact subgroups $K_G^\star = K_G^p \times K_{G,p}^\star$ of $K_G = K_G^p \times K_{G,p}$ which are smaller away from v_0. More precisely, under the product decomposition (4.1), let $K_{G,p}^\star$ be of the form

$$K_{G,p}^\star = K_{G,p}^{\star v_0} \times K_{G,v_0}, \tag{4.16}$$

where $K_{G,p}^{\star v_0}$ is a subgroup of finite index of $\prod_{v \neq v_0} K_{G,v}$. In this case, we add a *level structure* $\overline{\eta}_p^{v_0}$ to the data $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}^p)$ above. Let us explain what is meant by this, concentrating on the most interesting case when the base scheme S is a scheme over $\operatorname{Spf} \overline{\mathbb{Z}}_p$, comp. [32, §8].

Let $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}^p) \in \mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$. For a place $v \in S_p \smallsetminus \{v_0\}$, consider the lisse local system $\mathcal{V}(A_0[v^\infty], A[v^\infty])$ in F_v-modules on S given at a geometric point $\overline{s} \to S$ by the Frobenius invariants,

$$\mathcal{V}(A_0[v^\infty], A[v^\infty])_{\overline{s}} := \mathrm{Hom}_{F_v}\big(M(A_0[v^\infty]_{\overline{s}}), M(A[v^\infty]_{\overline{s}})\big)^{\mathrm{Frob}=1},$$

comp. Remark 8.18. Here $M(A_0[v^\infty]_{\overline{s}})$ and $M(A[v^\infty]_{\overline{s}})$ denote the *rational* Dieudonné modules of $A_0[v^\infty]_{\overline{s}}$ and $A[v^\infty]_{\overline{s}}$. By the Kottwitz condition (3.9), the p-divisible groups $A_0[v^\infty]$ and $A[v^\infty]$, with their actions by O_{F_0} and their polarizations $\lambda_{0,v}$ and λ_v, are banal in the sense of Appendix B. Therefore, when n is even, the sign condition at v implies that the typical fiber of this local system is isomorphic to $-W_v$, cf. Remark 8.18. The same holds trivially when v is split in F. If n is odd, and the typical fiber is not isomorphic to $-W_v$, then we choose once and for all a non-norm unit in O_{F_v} and use it to modify the hermitian form of the local system. In all cases, we have now obtained a local system on S with typical fiber $-W_v$, which we denote by $V_p(A_0[v^\infty], A[v^\infty])$. Then a level structure $\overline{\eta}_p^{v_0}$ is a $K_{G,p}^{\star v_0}$-orbit of $\prod_{v \neq v_0} F_v$-linear isometries

$$\eta_p^{v_0} \colon \prod_{v \neq v_0} V_p(A_0[v^\infty], A[v^\infty]) \simeq \prod_{v \neq v_0} (-W_v). \tag{4.17}$$

This defines the moduli problem $\mathcal{M}_{K^\star_{\widetilde{G}}}$, where $K^\star_{\widetilde{G}} := K_{Z^{\mathbb{Q}}} \times K^\star_G$, which is obviously relatively representable over $\mathcal{M}_{K_{\widetilde{G}}}$. Furthermore, the level structure $\overline{\eta}_p^{v_0}$ is redundant when $K^\star_{G,p} = K_{G,p}$, so that our new definition of $\mathcal{M}_{K^\star_{\widetilde{G}}}$ coincides in this case with our old definition. Indeed, this follows since any two vertex lattices of identical type in $-W_v$ are conjugate, comp. the argument in the proof of Theorem 4.1 for proving the isomorphism $\mathcal{M}_{K_{\widetilde{G}}} \times_{\operatorname{Spec} O_{E,(\nu)}} \operatorname{Spec} E \simeq M_{K_{\widetilde{G}}}(\widetilde{G})$. We still have to see that the generic fiber $\mathcal{M}_{K^\star_{\widetilde{G}}} \times_{\operatorname{Spec} O_{E,(\nu)}} \operatorname{Spec} E$ is canonically isomorphic to $M_{K^\star_{\widetilde{G}}}$. By Lemmas B.1 and B.4, for every geometric point $\overline{s}$ of S, $A_0[v^\infty]_{\overline{s}}$ and $A[v^\infty]_{\overline{s}}$ have canonical liftings $\widetilde{A}_0[v^\infty]_{\overline{s}}$ and $\widetilde{A}[v^\infty]_{\overline{s}}$ to the Witt vectors $W(\kappa(\overline{s}))$. Taking into account our modification of $\mathcal{V}(A_0[v^\infty], A[v^\infty])$ when n is odd and v non-split, the Fontaine functor induces a bijection

$$\mathrm{Isom}_{F_v}\Big(V_p\big(\widetilde{A}_0[v^\infty]_{\overline{s}}, \widetilde{A}[v^\infty]_{\overline{s}}\big), -W_v\Big) \simeq \mathrm{Isom}_{F_v}\big(V_p(A_0[v^\infty], A[v^\infty])_{\overline{s}}, -W_v\big),$$

where $V_p(\widetilde{A}_0[v^\infty]_{\overline{s}}, \widetilde{A}[v^\infty]_{\overline{s}})$ is formed from the rational Tate modules of the generic fibers of $\widetilde{A}_0[v^\infty]_{\overline{s}}$ and $\widetilde{A}[v^\infty]_{\overline{s}}$, cf. Proposition A.1 (this uses banality to recover the filtrations on the rational Dieudonné modules). Therefore a level structure in the usual sense on the rational Tate modules in the generic fiber corresponds uniquely to a level structure in the special fiber as defined above and this constructs the desired canonical isomorphism. We conclude that Theorem 4.1 holds with $K^\star_{\widetilde{G}}$ in place of $K_{\widetilde{G}}$.

We analogously define subgroups $K^\star_{\widetilde{H}}$ and $K^\star_{\widetilde{HG}}$, and moduli stacks $\mathcal{M}_{K^\star_{\widetilde{H}}}(\widetilde{H})$ and $\mathcal{M}_{K^\star_{\widetilde{HG}}}(\widetilde{HG})$. In order to construct a finite unramified morphism, resp. a closed embedding, covering the corresponding maps in (4.11), we have to assume in addition to the conditions spelled out there that $K^{\star v_0}_{H,p} \subset H(F^{v_0}_{0,p}) \cap K^{\star v_0}_{G,p}$. Under these conditions, we obtain

$$\mathcal{M}_{K^\star_{\widetilde{H}}}(\widetilde{H}) \longrightarrow \mathcal{M}_{K^\star_{\widetilde{G}}}(\widetilde{G}) \quad \text{and} \quad \mathcal{M}_{K^\star_{\widetilde{H}}}(\widetilde{H}) \hookrightarrow \mathcal{M}_{K^\star_{\widetilde{HG}}}(\widetilde{HG}). \tag{4.18}$$

Also, we obtain additional Hecke correspondences for elements $g \in \widetilde{G}(F^{v_0}_{0,p})$, where

$$\widetilde{G}(F^{v_0}_{0,p}) := \Bigg\{ (z_v, g_v) \in \prod_{v \in S_p \smallsetminus \{v_0\}} \big(F_v^\times \times \mathrm{GU}(F_v)\big) \ \Bigg| \ \mathrm{Nm}_{F_v/F_{0,v}}(z_v) = c(g_v) \Bigg\}.$$

4.2. **Split level at v_0.** We continue with the setup and assumptions of the previous subsection, except we now allow v_0 to be ramified over p. In addition, we assume that v_0 splits in F, say into w_0 and another place $\overline{w}_0$. We are again going to define smooth integral models over $O_{E,(\nu)}$.

We define the moduli functor $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ as follows. To each locally noetherian $O_{E,(\nu)}$-scheme S, we associate the set of isomorphism classes of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}^p)$ as in the previous subsection, except that we impose the further condition corresponding to w_0 that when p is locally nilpotent on S, the p-divisible group $A[w_0^\infty]$ is a Lubin–Tate group of type $r|_{w_0}$, in the sense of [48, §8] (note that this involves the *Eisenstein condition* of loc. cit.). Here $r|_{w_0}$ is the restriction of the function r on $\mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}})$ to $\mathrm{Hom}_{\mathbb{Q}_p}(F_{w_0}, \overline{\mathbb{Q}}_p)$, in the sense of the identification (4.7). We note that if v_0 is unramified over p, then this further Eisenstein condition is redundant, and the moduli functor $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ is the same as the one defined in the previous subsection, cf. [48].

Theorem 4.3. *The moduli problem just formulated is representable by a Deligne–Mumford stack $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ smooth over $\mathrm{Spec}\, O_{E,(\nu)}$. For K_G^p small enough, $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ is relatively representable over $\mathcal{M}_0^{\mathfrak{a}}$. Furthermore, the generic fiber $\mathcal{M}_{K_{\widetilde{G}}} \times_{\mathrm{Spec}\, O_{E,(\nu)}} \mathrm{Spec}\, E$ is canonically isomorphic to $M_{K_{\widetilde{G}}}(\widetilde{G})$.*

Proof. Same as the proof of Theorem 4.1, using in addition that the factor at w_0 for the local model for A for the newly introduced Eisenstein condition is smooth, cf. [48, §8]. □

We analogously define the DM stacks $\mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H})$ and $\mathcal{M}_{K_{\widetilde{HG}}}(\widetilde{HG})$ over $\mathrm{Spec}\, O_{E,(\nu)}$, comp. the end of Section 3.2. Both are again smooth over $\mathrm{Spec}\, O_{E,(\nu)}$. We then obtain a finite unramified morphism, resp. a closed embedding,

$$\mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H}) \longrightarrow \mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G}) \quad \text{and} \quad \mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H}) \hookrightarrow \mathcal{M}_{K_{\widetilde{HG}}}(\widetilde{HG}), \tag{4.19}$$

and Hecke correspondences exactly as in the previous subsection.

Remark 4.4. The definitions of the above moduli spaces can be extended to compact open subgroups $K_{\widetilde{G}}^\star$, $K_{\widetilde{H}}^\star$, and $K_{\widetilde{HG}}^\star$ which are smaller at p-adic places away from v_0, just as in Remark 4.2.

4.3. **Drinfeld level at v_0.** We continue with the setup and assumptions of Section 4.2, with v_0 split in F and possibly ramified over p. In this subsection we are going to define integral models over $O_{E,(\nu)}$ where we impose a *Drinfeld level structure* at v_0. To do this, we require that the *matching condition* between the CM type Φ and the chosen place ν of E is satisfied, which demands that

$$\{\, \varphi \in \mathrm{Hom}(F, \overline{\mathbb{Q}}) \mid w_\varphi = w_0 \,\} \subset \Phi, \tag{4.20}$$

where w_φ is the place of F induced by $\widetilde{\nu} \circ \varphi$, as in (4.7). We note that this condition only depends on the place ν of E induced by $\widetilde{\nu}$. When condition (4.20) is satisfied, we also say that the CM type Φ and the place ν of E are *matched*. Here are some examples in which the matching condition is guaranteed to hold.

Lemma 4.5. *The matching condition for Φ and ν is satisfied in each of the following two situations.*

(i) F is of the form KF_0 for an imaginary quadratic field $K/\mathbb{Q}$, Φ is the unique CM type induced from K containing φ_0, and p splits in K.

(ii) The place v_0 is of degree 1 over $\mathbb{Q}$.

Proof. The matching condition is obvious in (i), and in (ii) it holds because the left-hand side of (4.20) is the singleton set $\{\varphi_0\}$. □

Remark 4.6. We call the case (i) the *Harris–Taylor case*, cf. [17].

Now let m be a nonnegative integer. We define the level subgroup $K_G^m \subset G(\mathbb{A}_{F_0,f})$ in exactly the same way as K_G in Section 4.1 (subject to the choice of certain vertex lattices Λ_v for $v \in S_p$), except in the v_0-factor, we take K_{G,v_0}^m to be the principal congruence subgroup modulo $\mathfrak{p}_{v_0}^m$ inside K_{G,v_0}. In particular, K_G coincides with K_G^m for $m = 0$. We define $K_{\widetilde{G}}^m := K_{Z^{\mathbb{Q}}} \times K_G^m$ as in (3.6).

We now define the moduli functor $\mathcal{M}_{K_{\widetilde{G}}^m}(\widetilde{G})$ over $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$. Let $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}^p) \in \mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})(S)$. Consider the factors occurring in the decomposition (4.5) of the p-divisible group $A[p^\infty]$,

$$A[v_0^\infty] = A[w_0^\infty] \times A[\overline{w}_0^\infty]. \tag{4.21}$$

When p is locally nilpotent on S, the p-divisible group $A[w_0^\infty]$ satisfies the Kottwitz condition of type $r|_{w_0}$ for the action of O_{F,w_0} on its Lie algebra, in the sense of the previous subsection. In fact, in this case the matching condition (4.20) and the Kottwitz condition (3.9) imply that

$A[w_0^\infty]$ is a one-dimensional formal O_{F,w_0}-module of (absolute) height $n\cdot[F_{w_0}:\mathbb{Q}_p]$, whereas the second factor in (4.21) is identified via the polarization λ with the (absolute) dual of the first factor (and has dimension $n\cdot[F_{w_0}:\mathbb{Q}_p]-1$). Analogously,

$$A_0[v_0^\infty]=A_0[w_0^\infty]\times A_0[\overline{w}_0^\infty],$$

where, by (4.20) and the Kottwitz condition on A_0, the p-divisible group $A_0[w_0^\infty]$ with O_{F,w_0}-action is étale of height $[F_{w_0}:\mathbb{Q}_p]$, whereas $A_0[\overline{w}_0^\infty]$ is identified with the dual of $A_0[w_0^\infty]$.

In analogy with the prime-to-p theory, we introduce the finite flat group scheme over S,

$$T_{w_0}(A_0,A)[w_0^m]:=\underline{\mathrm{Hom}}_{O_{F,w_0}}(A_0[w_0^m],A[w_0^m]).$$

Note that as m varies, the right-hand side is naturally an inverse system under restriction of homomorphisms, and to make it into a directed system depends on the choice of uniformizer π_{w_0} of F_{0,w_0}. The colimit $T_{w_0}(A_0,A):=\varinjlim T_{w_0}(A_0,A)[w_0^m]$ is a 1-dimensional formal O_{F,w_0}-module since $A[w_0^\infty]$ is.

For the moduli problem $\mathcal{M}_{K^m_{\widetilde{G}}}(\widetilde{G})$, we equip the object $(A_0,\iota_0,\lambda_0,A,\iota,\lambda,\overline{\eta}^p)\in\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ with the following additional datum. Let $\Lambda_{v_0}=\Lambda_{w_0}\oplus\Lambda_{\overline{w}_0}$ denote the natural decomposition of the lattice Λ_{v_0} attached to the split place v_0. The additional datum is

- an O_{F,w_0}-linear homomorphism of finite flat group schemes,

$$\varphi\colon \pi_{w_0}^{-m}\Lambda_{w_0}/\Lambda_{w_0}\longrightarrow \underline{\mathrm{Hom}}_{O_{F,w_0}}(A_0[w_0^m],A[w_0^m]), \tag{4.22}$$

which is a Drinfeld w_0^m-structure on the target, cf. [17, §II.2].

Theorem 4.7. *The moduli problem $\mathcal{M}_{K^m_{\widetilde{G}}}(\widetilde{G})$ is relatively representable by a finite flat morphism to $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$. It is regular and flat over $\operatorname{Spec}O_{E,(\nu)}$. Furthermore, the generic fiber $\mathcal{M}_{K^m_{\widetilde{G}}}\times_{\operatorname{Spec}O_{E,(\nu)}}\operatorname{Spec}E$ is canonically isomorphic to $M_{K^m_{\widetilde{G}}}(\widetilde{G})$.*

Proof. After an étale base change, the subgroup $\mathcal{A}_0[w_0^m]$ of the universal abelian scheme $\mathcal{A}_0$ over $\mathcal{M}_0^{\mathfrak{a}}$ becomes constant, and then the proof of [17, Lem. III.4.1(4)(5)] applies. ☐

We analogously define the DM stack $\mathcal{M}_{K^m_{\widetilde{H}}}(\widetilde{H})$ over $\operatorname{Spec}O_{E,(\nu)}$, and obtain for it the analog of Theorem 4.7. We also define $\mathcal{M}_{K^m_{\widetilde{HG}}}(\widetilde{HG}):=\mathcal{M}_{K^m_{\widetilde{H}}}(\widetilde{H})\times_{\mathcal{M}_0^{\mathfrak{a}}}\mathcal{M}_{K^m_{\widetilde{G}}}(\widetilde{G})$, but we note that this stack is *not* regular for $m>0$. We then obtain a finite unramified morphism, resp. a closed embedding,

$$\mathcal{M}_{K^m_{\widetilde{H}}}(\widetilde{H})\longrightarrow\mathcal{M}_{K^m_{\widetilde{G}}}(\widetilde{G})\quad\text{and}\quad\mathcal{M}_{K^m_{\widetilde{H}}}(\widetilde{H})\hookrightarrow\mathcal{M}_{K^m_{\widetilde{HG}}}(\widetilde{HG}),$$

provided that $K_H^p\subset H(\mathbb{A}^p_{F_0,f})\cap K_G^p$ and that the lattices Λ_v and $\Lambda_v^\flat$ are as in (4.12). Indeed, the Drinfeld level structure $\varphi^\flat\colon\pi_{w_0}^{-m}\Lambda^\flat_{w_0}/\Lambda^\flat_{w_0}\to\underline{\mathrm{Hom}}_{O_{F,w_0}}(A_0[w_0^m],A^\flat[w_0^m])$ induces a Drinfeld level structure

$$\varphi\colon\pi_{w_0}^{-m}\Lambda_{w_0}/\Lambda_{w_0}\longrightarrow\underline{\mathrm{Hom}}_{O_{F,w_0}}\big(A_0[w_0^m],(A^\flat\times A_0)[w_0^m]\big)$$

by taking the direct sum of $\varphi^\flat$ and the O_{F,w_0}-linear homomorphism

$$\begin{aligned}\varphi_0\colon\ \pi_{w_0}^{-m}O_{F,w_0}u/O_{F,w_0}u&\longrightarrow\underline{\mathrm{Hom}}_{O_{F,w_0}}(A_0[w_0^m],A_0[w_0^m])\\ \pi_{w_0}^{-m}u&\longmapsto\mathrm{id},\end{aligned}\tag{4.23}$$

with respect to the natural decompositions in the source and target of φ.

In the present situation there are more Hecke correspondences than those defined for adelic elements prime to p in the previous two subsections, cf. [17, §III.4]. We again first treat the case for $\widetilde{G}$. With respect to the decomposition obtained from (2.1),

$$\widetilde{G}(\mathbb{Q}_p)=Z^{\mathbb{Q}}(\mathbb{Q}_p)\times\prod_{v\in S_p}G(F_{0,v}),$$

let $g \in G(F_{0,v_0})$, considered as an element in the left-hand side. In the special case that $g \in K_{G,v_0}$, since K^m_{G,v_0} is a normal subgroup of K_G, we get a diagram of finite flat morphisms

$$\begin{tikzcd} & \mathcal{M}_{K^m_{\widetilde{G}}}(\widetilde{G}) \arrow[dl, "\mathrm{nat}_1"'] \arrow[dr, "\mathrm{nat}_g"] & \\ \mathcal{M}_{K^m_{\widetilde{G}}}(\widetilde{G}) & & \mathcal{M}_{K^m_{\widetilde{G}}}(\widetilde{G}), \end{tikzcd}$$

analogously to (4.13). For general $g \in G(F_{0,v_0})$, choose m' and μ large enough that

$$\Lambda_{v_0} \supset \varpi_{v_0}^{m'} g\Lambda_{v_0} \supset \varpi_{v_0}^{m+m'} g\Lambda_{v_0} \supset \varpi_{v_0}^{\mu}\Lambda_{v_0}.$$

Then $K^{\mu}_{G,v_0} \subset K^m_{G,v_0} \cap gK^m_{G,v_0}g^{-1}$. Hence we obtain a diagram of finite flat morphisms

$$\begin{tikzcd} & \mathcal{M}_{K^{\mu}_{\widetilde{G}}}(\widetilde{G}) \arrow[dl, "\mathrm{nat}_1"'] \arrow[dr, "\mathrm{nat}_g"] & \\ \mathcal{M}_{K^m_{\widetilde{G}}}(\widetilde{G}) & & \mathcal{M}_{K^m_{\widetilde{G}}}(\widetilde{G}) \end{tikzcd} \tag{4.24}$$

as before. In terms of the moduli descriptions above, these morphisms are given as follows. Consider a point $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}^p, \varphi)$ of $\mathcal{M}_{K^{\mu}_{\widetilde{HG}}}(\widetilde{G})$. Then nat_1 sends this point to the point represented by $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}^p, \varphi|_{(\pi_{w_0}^{-m}\Lambda_{w_0}/\Lambda_{w_0})})$. To describe nat_g, let $C \subset A[w_0^{\mu}]$ be the unique closed subgroup scheme for which the set of $\varphi(x)$, $x \in \pi_{w_0}^{m+m'-\mu} g\Lambda_{w_0}/\Lambda_{w_0}$, is a complete set of sections, cf. [24, Cor. 1.10.3] (note that $\underline{\mathrm{Hom}}_{O_{F,w_0}}(A_0[w_0^{\mu}], A[w_0^{\mu}])$ is étale-locally isomorphic to $A[w_0^{\mu}]$). Let $C^{\perp} \subset A[\overline{w}_0^{\infty}]$ be the annihilator of C under the Weil pairing, and let A' be the quotient of A by the finite subgroup scheme $C \times C^{\perp}$. Let $\alpha\colon A \to A'$ be the corresponding O_F-linear isogeny. Then $\lambda = \varpi_{v_0}^{-\mu}\alpha^*(\lambda')$, where λ' is a principal polarization of A'. Furthermore A' comes with a level structure $\overline{\eta}'^p$, and the quadruple $(A', \iota', \lambda', \overline{\eta}'^p)$ is an object of $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$. We obtain a Drinfeld level w_0^m-structure φ' on $\underline{\mathrm{Hom}}_{O_{F,w_0}}(A_0[w_0^m], A'[w_0^m])$ via the following commutative diagram:

$$\begin{tikzcd} \pi_{w_0}^{-m}\Lambda_{w_0}/\Lambda_{w_0} \arrow[d, "\pi_{w_0}^{m+m'-\mu}g"', "\sim"] \arrow[dr, "\varphi'"] & \\ \pi_{w_0}^{m'-\mu}g\Lambda_{w_0}/\pi_{w_0}^{m+m'-\mu}g\Lambda_{w_0} \arrow[r, dashed] \arrow[d, phantom, "\cap"] & \underline{\mathrm{Hom}}(A_0[w_0^m], A'[w_0^m]) \arrow[d, phantom, "\cap"] \\ \pi_{w_0}^{-\mu}\Lambda_{w_0}/\pi_{w_0}^{m+m'-\mu}g\Lambda_{w_0} \arrow[r, dashed] & \underline{\mathrm{Hom}}(A_0[w_0^{\mu}], A[w_0^{\mu}]/C) \\ \pi_{w_0}^{-\mu}\Lambda_{w_0}/\Lambda_{w_0} \arrow[u, two heads] \arrow[r, "\varphi"] & \underline{\mathrm{Hom}}(A_0[w_0^{\mu}], A[w_0^{\mu}]). \arrow[u, two heads] \end{tikzcd}$$

Here the dashed arrows are induced by φ. Then the image of $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}^p, \varphi)$ under nat_g is $(A_0, \iota_0, \lambda_0, A', \iota', \lambda', \overline{\eta}'^p, \varphi')$. We remark that the above construction is independent of the auxiliary choices of m' and μ.

Again, as in (4.14), this defines an action of the center of $G(F_{0,v_0})$ on $\mathcal{M}_{K^m_{\widetilde{G}}}(\widetilde{G})$.

The above construction carries over in the obvious way with $\widetilde{H}$ in place of $\widetilde{G}$. Taking the product over $\mathcal{M}_0^{\mathfrak{a}}$ of the diagram (4.24) with the one attached to $\widetilde{H}$ and an element $h \in H(F_{0,v_0})$, we obtain, for μ sufficiently large, an analogous diagram of finite flat morphisms for $\widetilde{HG}$. We record this as the following diagram, where $g \in (H \times G)(F_{0,v_0})$:

$$\begin{tikzcd}[column sep=small] & \mathcal{M}_{K^{\mu}_{\widetilde{HG}}}(\widetilde{HG}) \arrow[dl, "\mathrm{nat}_1"'] \arrow[dr, "\mathrm{nat}_g"] & \\ \mathcal{M}_{K^{m}_{\widetilde{HG}}}(\widetilde{HG}) & & \mathcal{M}_{K^{m}_{\widetilde{HG}}}(\widetilde{HG}). \end{tikzcd} \tag{4.25}$$

Remark 4.8. As before, the above definitions and results extend readily to the case of subgroups $K^{\star}_{\widetilde{G}}$, $K^{\star}_{\widetilde{H}}$, and $K^{\star}_{\widetilde{HG}}$ which are smaller at p-adic places away from v_0, cf. Remark 4.2.

4.4. **AT parahoric level at v_0.** In this subsection, $p \neq 2$ and we assume that the place v_0 is of relative degree one over $\mathbb{Q}$. We again choose a vertex lattice $\Lambda_v \subset W_v$ for each $v \in S_p$, as in Section 4.1. We require that the pair (v_0, Λ_{v_0}) is of one of the following four types, which we call *AT types*.

(1) v_0 is inert in F and Λ_{v_0} is almost self-dual as an O_{F,v_0}-lattice.

(2) v_0 ramifies in F, n is even, and Λ_{v_0} is π_{v_0}-modular.

(3) v_0 ramifies in F, n is odd, and Λ_{v_0} is almost π_{v_0}-modular.

(4) v_0 ramifies in F, $n = 2$, and Λ_{v_0} is self-dual.

We refer to the end of the Introduction for the terminology on lattice types.

Again recalling the decomposition $K_{\widetilde{G}} = K_{Z^{\mathbb{Q}}} \times K_G$ from (3.6), we take the subgroup K_G to be of the form

$$K_G = K_G^p \times K_{G,p},$$

where $K_G^p \subset G(\mathbb{A}^p_{F_0,f})$ is arbitrary, and where $K_{G,p} \subset G(F_0 \otimes \mathbb{Q}_p)$ is given by

$$K_{G,p} = \prod_{v \in S_p} K_{G,v} \subset \prod_{v \in S_p} G(F_{0,v}),$$

with $K_{G,v}$ the stabilizer of Λ_v in $G(F_{0,v})$.

We define the moduli functor $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ over $\operatorname{Spec} O_{E,(\nu)}$ in exactly the same way as in Section 4.1. More precisely, to each $O_{E,(\nu)}$-scheme S, we associate the set of isomorphism classes of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}^p)$, where $(A_0, \iota_0, \lambda_0)$ is an object of $\mathcal{M}_0^{\mathfrak{a}}(S)$, where A is an abelian scheme over S up to isogeny prime to p, where ι is an $O_F \otimes \mathbb{Z}_{(p)}$-action on A satisfying the Kottwitz condition (3.9) of signature $((1, n-1)_{\varphi_0}, (0, n)_{\varphi \in \Phi \smallsetminus \{\varphi_0\}})$, and where λ is a polarization on A whose Rosati involution induces on $O_F \otimes \mathbb{Z}_{(p)}$ the non-trivial Galois automorphism of F/F_0, subject to the condition that under the decomposition (4.2) of the p-divisible group $A[p^\infty]$, $\ker \lambda_v$ is contained in $A[\iota(\pi_v)]$ of rank $\#(\Lambda_v^\vee / \Lambda_v)$ for all $v \in S_p$. Finally, $\overline{\eta}^p$ is a K_G^p-equivalence class of $\mathbb{A}^p_{F,f}$-linear isometries (4.3).

We also impose for each $v \neq v_0$ over p the sign condition (4.4) at v and the Eisenstein condition (4.8) at v. In addition, when the pair (v_0, Λ_{v_0}) is of AT type (2) or (3), we impose the following additional conditions. As with the Eisenstein condition, it suffices to impose the condition when the base scheme S lies over $\operatorname{Spf} \breve{\mathbb{Z}}_p$, where it takes the following form.

- If (v_0, Λ_{v_0}) is of type (2), then we impose the *wedge condition*

$$\bigwedge^2 \big(\iota(\pi_{v_0}) + \pi_{v_0} \mid \operatorname{Lie} A[v_0^\infty]\big) = 0 \tag{4.26}$$

and the *spin condition*

$$\text{the endomorphism } \iota(\pi_{v_0}) \mid \operatorname{Lie} A[v_0^\infty] \text{ is nonvanishing at each point of } S, \tag{4.27}$$

cf. [44, §6].

- If (v_0, Λ_{v_0}) is of type (3), then we impose the *refined spin condition* (7.9) of [44] on $\operatorname{Lie} A[v_0^\infty]$.

Remark 4.9. We note that the triple $(A[v_0^\infty], \iota[v_0^\infty], \lambda[v_0^\infty])$ arising from our moduli problem is of the type occurring in the moduli problem for one of the RZ spaces in [44, §5–8] (since we require that v_0 has relative degree one, the relative dual of $A[v_0^\infty]$ that is used in [44] is the Serre dual). One may ask whether one can replace the condition that v_0 has relative degree one by the condition that v_0 be unramified. This may be possible in view of [32] in the case of

AT type (1). However, loc. cit. shows for type (4) that even when v_0 is unramified, one needs to impose further conditions (the *Eisenstein conditions* in this context) to force flatness of the corresponding moduli schemes.

Theorem 4.10. *The moduli problem just formulated is representable by a Deligne–Mumford stack $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ flat over $\operatorname{Spec} O_{E,(\nu)}$. For K_G^p small enough, $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ is relatively representable over $\mathcal{M}_0^{\mathfrak{a}}$. The generic fiber $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G}) \times_{\operatorname{Spec} O_{E,(\nu)}} \operatorname{Spec} E$ is canonically isomorphic to $M_{K_{\widetilde{G}}}(\widetilde{G})$. Furthermore:*

(i) $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ is smooth over $\operatorname{Spec} O_{E,(\nu)}$ provided that (v_0, Λ_{v_0}) is of AT type (2) or (3).

(ii) $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ has semi-stable reduction over $\operatorname{Spec} O_{E,(\nu)}$ provided that (v_0, Λ_{v_0}) is of AT type (1) and E_ν is unramified over $\mathbb{Q}_p$.

Proof. The representability assertion and the assertion for the generic fiber are proved in the same way as in Theorem 4.1. The assertions concerning the local structure of $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ all reduce to statements about the local model. As in the proof of Theorem 4.1, the factor of the local model corresponding to A_0 is trivial, and the factor corresponding to A decomposes as in (4.10) into a further product of local models, one for each $v \in S_p$, with the factor at each $v \neq v_0$ trivial. The local model M_{v_0} is smooth over $\operatorname{Spec} O_{E_{r|v_0}}$ in types (2) [43, Prop. 3.10] and (3) [49, Th. 1.4], and has semi-stable reduction $\operatorname{Spec} O_{E_{r|v_0}}$ in types (1) [44, pf. of Th. 5.1] and (4) [44, §8]. The last assertion in (ii) follows because semi-stable reduction is preserved under an unramified base extension. □

Remark 4.11. The unramifiedness condition on E_ν in part (ii) of Theorem 4.10 is always satisfied in the Harris–Taylor case (cf. Remark 4.6), and in fact $E_\nu \cong F_{v_0} \cong \mathbb{Q}_{p^2}$ in this case.

Remark 4.12. In the case of AT type (4), we have $E_{r|v_0} = \mathbb{Q}_p$ and the local model M_{v_0} has semi-stable reduction, comp. [44]. However, the extension $E_\nu/\mathbb{Q}_p$ is always ramified, and therefore $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ is not regular over the place ν in this case. For this reason, we will exclude AT type (4) when considering arithmetic intersections.

We analogously define the DM stack $\mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H})$ over $\operatorname{Spec} O_{E,(\nu)}$ when the pair $(v_0, \Lambda_{v_0}^\flat)$ is of AT type. To define $\mathcal{M}_{K_{\widetilde{HG}}}(\widetilde{HG})$, we take as before the lattice $\Lambda_v^\flat \oplus \Lambda_v$ in $W_v^\flat \oplus W$ for each v, but the relation between $\Lambda_v^\flat$ and Λ_v that we allow can be more complicated. Furthermore, the definition of the analog of the morphisms (4.11) requires more care.

Let us first suppose that (v_0, Λ_{v_0}) is not of AT type (2). In this case, we assume that the lattices Λ_v and $\Lambda_v^\flat$ satisfy $\Lambda_v = \Lambda_v^\flat \oplus O_{F,v} u$ for all v. We also assume that $(u, u) \in O_{F_0,v}^\times$ for all v, with the single exception of $v = v_0$ when (v_0, Λ_{v_0}) is of AT type (1), in which case we impose that $\operatorname{ord}_{v_0}(u, u) = 1$. Provided that $K_H^p \subset H(\mathbb{A}_{F_0,f}^p) \cap K_G^p$, we then obtain a finite unramified morphism, resp. a closed embedding,

$$\mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H}) \longrightarrow \mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G}) \quad \text{and} \quad \mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H}) \hookrightarrow \mathcal{M}_{K_{\widetilde{HG}}}(\widetilde{HG}), \tag{4.28}$$

as in (4.11).

Remark 4.13. The unit (u, u) was chosen very carefully in [44] because in loc. cit. we made a definite choice between the two isomorphic RZ spaces $\mathcal{N}_n^{(0)}$ and $\mathcal{N}_n^{(1)}$ in the odd ramified case (more precisely, a definite choice of the framing object). Here we make no such choice, and therefore (u, u) can be an arbitrary unit at ramified places.

Now suppose that (v_0, Λ_{v_0}) is of AT type (2). Then we cannot define such simple embeddings, and it is necessary to consider more complicated diagrams involving additional spaces, cf. [44, §12]. In fact we will consider two variants. For both variants, we assume that $(u, u) \in O_{F_0,(p)}^\times$, and that $\Lambda_{v_0}^\flat$ and Λ_{v_0} are related by a chain of inclusions

$$\pi_{v_0}(\Lambda_{v_0}^\flat \oplus O_{F,v_0} u)^\vee \subset^1 \Lambda_{v_0} \subset^1 \Lambda_{v_0}^\flat \oplus O_{F,v_0} u. \tag{4.29}$$

Note that (4.29) is equivalent to the condition that $\Lambda_{v_0}^\flat$ is almost π_{v_0}-modular in $W_{v_0}^\flat$, and that Λ_{v_0} is one of the two π_{v_0}-modular lattices contained in $\Lambda_{v_0}^\flat \oplus O_{F,v_0} u$.

Variant 1: In the first variant, we continue to assume that for all $v \neq v_0$, we have $\Lambda_v = \Lambda_v^\flat \oplus O_{F,v}u$. We define $\mathcal{P}_{\widetilde{G}}$ to be the moduli stack defined in the same way as $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$, except that at the place v_0,

- $\ker \lambda_{v_0}$ has rank p^{n-2}; and
- when p is locally nilpotent on the base scheme, $\operatorname{Lie} A[v_0^\infty]$ satisfies the condition (9.2) of [44].

Now let $\mathcal{L}$ denote the self-dual multichain of $O_F \otimes \mathbb{Z}_p$-lattices in $W \otimes \mathbb{Q}_p = \bigoplus_{v \in S_p} W_v$ generated by Λ_{v_0} and $\Lambda_{v_0}^\flat \oplus O_{F,v_0}u$ (and its dual), and by Λ_v for all $v \neq v_0$. We define $\mathcal{P}'_{\widetilde{G}}$ to be the moduli stack of tuples $(A_0, \iota_0, \lambda_0, \mathbf{A}, \boldsymbol{\lambda}, \overline{\eta}^p)$, where $(A_0, \iota_0, \lambda_0)$ is an object of $\mathcal{M}_0^{\mathfrak{a}}$, $\mathbf{A} = \{A_\Lambda\}$ is an $\mathcal{L}$-set of abelian varieties, $\boldsymbol{\lambda}$ is a $\mathbb{Q}$-homogeneous principal polarization of $\mathbf{A}$, and $\overline{\eta}^p$ is a K_G^p-equivalence class of $\mathbb{A}_{F,f}^p$-linear isometries $\eta^p \colon \widehat{V}^p(A_0, \mathbf{A}) \simeq -W \otimes_F \mathbb{A}_{F,f}^p$, cf. [47, Def. 6.9]. We require that A_Λ satisfies the Kottwitz condition (3.9) for all Λ. We further require that over a base on which p is locally nilpotent, when the v_0-summand of Λ is Λ_{v_0}, $\operatorname{Lie} A_\Lambda[v_0^\infty]$ satisfies the wedge condition (4.26) and the spin condition (4.27) above; and when the v_0-component of Λ is $\Lambda_{v_0}^\flat \oplus O_{F,v_0}u$, $\operatorname{Lie} A_\Lambda[v_0^\infty]$ satisfies the condition (9.2) of [44]. We obtain a diagram

$$\begin{tikzcd}[column sep=small]
 & & \mathcal{P}'_{\widetilde{G}} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H}) \arrow[r] & \mathcal{P}_{\widetilde{G}} & & \mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G}).
\end{tikzcd} \tag{4.30}$$

Here the lower left morphism is defined in the usual way, i.e. analogously to (3.13), provided that $K_H^p \subset H(\mathbb{A}_{F_0,f}^p) \cap K_G^p$. It is again finite and unramified. The arrows π_1 and π_2 are induced by $\mathbf{A} \mapsto A_{(\Lambda_{v_0}^\flat \oplus O_{F,v_0}u) \oplus \bigoplus_{v \neq v_0} \Lambda_v}$ and $\mathbf{A} \mapsto A_{\Lambda_{v_0} \oplus \bigoplus_{v \neq v_0} \Lambda_v}$, respectively.

Lemma 4.14. *The morphism π_1 is finite étale of degree 2.*

Proof. The morphism is obviously proper. It is also finite because each geometric fiber has precisely two points. This last assertion follows over the complex numbers by looking at the homology of the abelian varieties in play, and in positive characteristic by looking at their Dieudonné modules. The question of étaleness reduces to the local models, which are isomorphic by [44, Prop. 9.12(ii)]. □

The morphism π_2 is proper, and it is finite étale over the generic fiber of degree $(p^n-1)/(p-1)$. However, π_2 is not finite when $n \geq 4$, cf. [44, Rem. 9.5].

Now define

$$\mathcal{P}_{\widetilde{HG}} := \mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H}) \times_{\mathcal{M}_0^{\mathfrak{a}}} \mathcal{P}_{\widetilde{G}} \quad \text{and} \quad \mathcal{P}'_{\widetilde{HG}} := \mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H}) \times_{\mathcal{M}_0^{\mathfrak{a}}} \mathcal{P}'_{\widetilde{G}}.$$

Applying the functor $\mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H}) \times_{\mathcal{M}_0^{\mathfrak{a}}} -$ to the rightmost three spaces in (4.30), we obtain a diagram

$$\begin{tikzcd}[column sep=small]
 & & \mathcal{P}'_{\widetilde{HG}} \arrow[dl] \arrow[dr] & \\
\mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H}) \arrow[r, hook] & \mathcal{P}_{\widetilde{HG}} & & \mathcal{M}_{K_{\widetilde{HG}}}(\widetilde{HG}).
\end{tikzcd} \tag{4.31}$$

Here the lower left embedding is the graph of the one in (4.30), again provided that $K_H^p \subset H(\mathbb{A}_{F_0,f}^p) \cap K_G^p$. Of course, the oblique arrows inherit the properties of the corresponding ones above under base change. Set

$$\mathcal{M}_{K'_{\widetilde{H}}}(\widetilde{H}) := \mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H}) \times_{\mathcal{P}_{\widetilde{HG}}} \mathcal{P}'_{\widetilde{HG}}.$$

Note that the generic fiber of $\mathcal{M}_{K'_{\widetilde{H}}}(\widetilde{H})$ is equal to $M_{K'_{\widetilde{H}}}(\widetilde{H})$, where $K'_{\widetilde{H}} = K_{Z^{\mathbb{Q}}} \times K_H^p \times K'_{H,p}$, with $K'_{H,v} = K_{H,v}$ at all places $v \neq v_0$, and K'_{H,v_0} the simultaneous stabilizer of $\Lambda_{v_0}^\flat$ and Λ_{v_0}.

Lemma 4.15. *The morphism*

$$\mathcal{M}_{K'_{\widetilde{H}}}(\widetilde{H}) \longrightarrow \mathcal{M}_{K_{\widetilde{HG}}}(\widetilde{HG})$$

induced by (4.31) *is a closed embedding.*

Proof. The proof of [44, Prop. 12.1] applies. $\square$

Variant 2: For the second variant, in addition to the place v_0, we allow there to be places $v_1, \dots, v_{m-1} \in S_p$ for which the lattice Λ_{v_i} is π_{v_i}-modular. For each $i = 0, \dots, m-1$, we then assume that the relation (4.29) holds with v_i in place of v_0. For all $v \neq v_0, \dots, v_{m-1}$, we again assume that $\Lambda_v = \Lambda_v^\flat \oplus O_{F,v} u$. We then define the stack $\mathcal{P}_{\widetilde{G}}$ exactly as above. We also define $\mathcal{P}'_{\widetilde{G}}$ exactly as above, except we now take $\mathcal{L}$ to be the self-dual multichain of $O_F \otimes \mathbb{Z}_p$-lattices in $W \otimes \mathbb{Q}_p = \bigoplus_{v \in S_p} W_v$ generated by the lattices Λ_{v_i} and $\Lambda_{v_i}^\flat \oplus O_{F_{v_i}} u$ for each $i = 0, \dots, m-1$, and by Λ_v for all $v \neq v_0, \dots, v_{m-1}$. (To be clear, the conditions above on the Lie algebra of the p-divisible group when p is locally nilpotent on the base still only involve the place v_0.)

In complete analogy with (4.30), there is a diagram

$$\begin{tikzcd} & \mathcal{P}'_{\widetilde{G}} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H}) \arrow[r] & \mathcal{P}_{\widetilde{G}} & \mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G}), \end{tikzcd} \tag{4.32}$$

where the lower left morphism is defined provided that $K_H^p \subset H(\mathbb{A}_{F_0,f}^p) \cap K_G^p$, and where the arrows π_1 and π_2 are induced by

$$\mathbf{A} \longmapsto A_{\Lambda'_{v_1} \oplus \cdots \oplus \Lambda'_{v_m} \oplus \bigoplus_{v \neq v_1, \dots, v_m} \Lambda_v} \quad \text{and} \quad \mathbf{A} \longmapsto A_{\Lambda_{v_1} \oplus \cdots \oplus \Lambda_{v_m} \oplus \bigoplus_{v \neq v_1, \dots, v_m} \Lambda_v},$$

respectively. The proof of Lemma 4.14 transposes to yield the following.

Lemma 4.16. *The morphism π_1 is finite étale of degree 2^m.* $\square$

In analogy with Variant 1, the morphism π_2 is proper, and finite étale over the generic fiber of degree $[(p^n - 1)/(p-1)]^m$. However, it is again not finite when $n \geq 4$.

Finally, we again define

$$\mathcal{P}_{\widetilde{HG}} := \mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H}) \times_{\mathcal{M}_0^{\mathfrak{a}}} \mathcal{P}_{\widetilde{G}} \quad \text{and} \quad \mathcal{P}'_{\widetilde{HG}} := \mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H}) \times_{\mathcal{M}_0^{\mathfrak{a}}} \mathcal{P}'_{\widetilde{G}},$$

and we apply the functor $\mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H}) \times_{\mathcal{M}_0^{\mathfrak{a}}} -$ to obtain

$$\begin{tikzcd} & \mathcal{P}'_{\widetilde{HG}} \arrow[dl] \arrow[dr] & \\ \mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H}) \arrow[r, hook] & \mathcal{P}_{\widetilde{HG}} & \mathcal{M}_{K_{\widetilde{HG}}}(\widetilde{HG}). \end{tikzcd} \tag{4.33}$$

Here for the lower left embedding we assume, as always, that $K_H^p \subset H(\mathbb{A}_{F_0,f}^p) \cap K_G^p$. Set

$$\mathcal{M}_{K'_{\widetilde{H}}}(\widetilde{H}) := \mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H}) \times_{\mathcal{P}_{\widetilde{HG}}} \mathcal{P}'_{\widetilde{HG}}.$$

Note that the generic fiber of $\mathcal{M}_{K'_{\widetilde{H}}}(\widetilde{H})$ is equal to $M_{K'_{\widetilde{H}}}(\widetilde{H})$, where $K'_{\widetilde{H}} = K_{Z^{\mathbb{Q}}} \times K_H^p \times K'_{H,p}$, with $K'_{H,v} = K_{H,v}$ at all places $v \neq v_0, \dots, v_{m-1}$, and K'_{H,v_i} the simultaneous stabilizer of $\Lambda_{v_i}^\flat$ and Λ_{v_i} at all places v_i for $i = 0, \dots, m-1$. As in the case of Lemma 4.15, we obtain the following.

Lemma 4.17. *The morphism*

$$\mathcal{M}_{K'_{\widetilde{H}}}(\widetilde{H}) \longrightarrow \mathcal{M}_{K_{\widetilde{HG}}}(\widetilde{HG})$$

induced by (4.33) *is a closed embedding.* $\square$

We note that if (v_0, Λ_{v_0}) is of type (2) or (3), then the spaces $\mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H})$ and $\mathcal{M}_{K_{\widetilde{HG}}}(\widetilde{HG})$ are smooth. If (v_0, Λ_{v_0}) is of type (1), then $\mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H})$ is smooth, and $\mathcal{M}_{K_{\widetilde{HG}}}(\widetilde{HG})$ has semi-stable reduction provided that E_ν is unramified over $\mathbb{Q}_p$.

In these cases, one can define Hecke correspondences prime to p, as at the end of Subsection 4.1.

Remark 4.18. In analogy with the above cases in which (v_0, Λ_{v_0}) is of AT type (2) and $(v_0, \Lambda^\flat_{v_0})$ is of AT type (3), one may also consider a situation in which (v_0, Λ_{v_0}) is hyperspecial and $(v_0, \Lambda^\flat_{v_0})$ is of AT type (1). One obtains a closed embedding analogous to the one in Lemma 4.15, where the source is a finite covering of $\mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H})$.

Remark 4.19. As before, the above definitions and results extend readily to the case of subgroups $K^\star_{\widetilde{G}}$, $K^\star_{\widetilde{H}}$, and $K^\star_{\widetilde{HG}}$ which are smaller at p-adic places away from v_0, cf. Remark 4.2.

5. Global integral models

In this section, we define integral models of the above moduli spaces over $\operatorname{Spec} O_E$. We will take $\mathfrak{a} = O_{F_0}$, i.e., we will assume that $\mathcal{M}_0 = \mathcal{M}_0^{O_{F_0}}$ is non-empty. Recall from Remark 3.3(ii) that this hypothesis is satisfied whenever F/F_0 is ramified at some finite place, a condition which we will eventually impose below in the context of arithmetic intersections, cf. Remark 5.2.

5.1. **Trivial level structure.** In this subsection, we are going to define integral models over $\operatorname{Spec} O_E$ of the previously defined moduli spaces in the case that the open compact subgroup is the stabilizer of a lattice of a certain form. Let us start with the case of $\widetilde{G}$. We consider the following finite set of finite places of F_0,

$$S^W_{\mathrm{AT}} := \{\, v \mid v \text{ is either inert in } F \text{ and } W_v \text{ is non-split, or } v \text{ ramifies in } F\,\}.$$

Let

$$\mathfrak{d}^W_{\mathrm{AT}} := \prod_{v \in S^W_{\mathrm{AT}}} \mathfrak{q}_v \subset O_F,$$

where $\mathfrak{q}_v$ denotes the (unique) prime in O_F determined by $v \in S^W_{\mathrm{AT}}$. We fix an O_F-lattice Λ in W with

$$\Lambda \subset \Lambda^\vee \subset (\mathfrak{d}^W_{\mathrm{AT}})^{-1}\Lambda. \tag{5.1}$$

We assume that the triple $(F/F_0, W, \Lambda)$ satisfies the following conditions.

(1) All finite places v of F_0 of residue characteristic p such that $F_{0,v}/\mathbb{Q}_p$ is ramified, or such that $p = 2$, are split in F.

(2) All places $v \in S^W_{\mathrm{AT}}$ are of degree one over $\mathbb{Q}$, and the pair (v, Λ_v) is isomorphic to one of the AT types (1)–(4) in Section 4.4.

As a consequence, for any finite place ν of E, denoting by v_0 the place of F_0 induced by ν via φ_0, the pair (v_0, Λ_{v_0}) is of the type considered in one of the four subsections Section 4.1–4.4. Associated to these data is the open compact subgroup

$$K^\circ_G := \bigl\{\, g \in G(\mathbb{A}_{F_0,f}) \bigm| g(\Lambda \otimes_{O_F} \widehat{O}_F) = \Lambda \otimes_{O_F} \widehat{O}_F \,\bigr\},$$

and as usual we define $K^\circ_{\widetilde{G}} := K_{Z^\mathbb{Q}} \times K^\circ_G$.

We formulate a moduli problem over $\operatorname{Spec} O_E$ as follows. To each O_E-scheme S, we associate the category of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda)$, where $(A_0, \iota_0, \lambda_0)$ is an object of $\mathcal{M}_0(S)$. Furthermore,

- (A, ι) is an abelian scheme over S, with O_F-action ι satisfying the Kottwitz condition (3.9) of signature $((1, n-1)_{\varphi_0}, (0, n)_{\varphi \in \Phi \smallsetminus \{\varphi_0\}})$; and
- λ is a polarization whose Rosati involution induces on O_F the non-trivial Galois automorphism of F/F_0.

We impose the sign condition that at every point s of S,

$$\operatorname{inv}^r_v(A_{0,s}, \iota_{0,s}, \lambda_{0,s}, A_s, \iota_s, \lambda_s) = \operatorname{inv}_v(-W_v), \tag{5.2}$$

for every finite prime v of F_0 which is non-split in F. Furthermore, we impose that for any finite place ν of E, denoting by p its residue characteristic, the triple up to isogeny prime to p over $S \times_{\operatorname{Spec} O_E} \operatorname{Spec} O_{E,(\nu)}$,

$$(A \otimes \mathbb{Z}_{(p)}, \iota \otimes \mathbb{Z}_{(p)}, \lambda \otimes \mathbb{Z}_{(p)}),$$

satisfies the conditions in the semi-global moduli problem for ν defined in Section 4.

The morphisms in this category are the isomorphisms.

Theorem 5.1. *The moduli problem just formulated is representable by a Deligne–Mumford stack $\mathcal{M}_{K^\circ_{\widetilde{G}}}(\widetilde{G})$ flat over $\operatorname{Spec} O_E$. For every place ν of E, the base change $\mathcal{M}_{K^\circ_{\widetilde{G}}} \times_{\operatorname{Spec} O_E} \operatorname{Spec} O_{E,(\nu)}$ is canonically isomorphic to the semi-global moduli space defined in one of Sections 4.1, 4.2, or 4.4 above. Hence:*

(i) $\mathcal{M}_{K^\circ_{\widetilde{G}}}(\widetilde{G})$ is smooth of relative dimension $n-1$ over the open subscheme of $\operatorname{Spec} O_E$ obtained by removing all places ν for which the induced pair (v_0, Λ_{v_0}) is of AT type (1) or (4) in Section 4.4.

(ii) $\mathcal{M}_{K^\circ_{\widetilde{G}}}(\widetilde{G})$ has semi-stable reduction over the open subscheme of $\operatorname{Spec} O_E$ obtained by removing all places ν for which either (v_0, Λ_{v_0}) is of AT type (4), or is of AT type (1) and for which E_ν is ramified over $\mathbb{Q}_p$. □

Replacing W by $W^\flat$ and choosing a lattice $\Lambda^\flat \subset W^\flat$ analogously to above, we define the DM stack $\mathcal{M}_{K^\circ_{\widetilde{H}}}(\widetilde{H})$, where $K^\circ_{\widetilde{H}} := K_{Z^\mathbb{Q}} \times K^\circ_H$ with $K^\circ_H \subset H(\mathbb{A}_{F_0,f})$ the stabilizer of $\Lambda^\flat \otimes_{O_F} \widehat{O}_F$. Similarly, we define $\mathcal{M}_{K^\circ_{\widetilde{HG}}}(\widetilde{HG})$, where $K^\circ_{\widetilde{HG}} := K_{Z^\mathbb{Q}} \times K^\circ_H \times K^\circ_G$, and where we impose the following additional conditions on $\Lambda^\flat$ and Λ. We first require that $\Lambda^\flat_v$ is self-dual for all v which are split or inert in F (i.e. we require that $S^{W^\flat}_{\mathrm{AT}}$ consists of exactly the finite places of F_0 which ramify in F), and that $\Lambda^\flat_v$ and Λ_v are π_v-modular or almost π_v-modular for all v which ramify in F (i.e. $S^{W^\flat}_{\mathrm{AT}}$ and S^W_{AT} contain no v for which $(v, \Lambda^\flat_v)$ or (v, Λ_v) is of AT type (4)).

Remark 5.2. The conditions that we have just imposed on $\Lambda^\flat$ place non-trivial constraints on the extension F/F_0 and on the hermitian space $W^\flat$. Let $d := [F_0 : \mathbb{Q}]$.

First consider the case when $n = 2m+1$ is odd. Then our assumptions on $\Lambda^\flat$ imply that $W^\flat$ is split at all finite places of F_0. On the other hand, at each archimedean place φ, the Hasse invariant $\operatorname{inv}_\varphi(W^\flat_\varphi)$ is equal to $(-1)^{m-1}$ if $\varphi = \varphi_0$, and to $(-1)^m$ if $\varphi \neq \varphi_0$. Hence the product formula (1.5) imposes the congruence

$$dm \equiv 1 \bmod 2. \tag{5.3}$$

In particular, since this requires d to be odd, the extension F/F_0 is forced to be ramified at at least one finite place (otherwise the product formula for the norm residue symbol $(-1, F/F_0)$ would fail). On the other hand, if the congruence (5.3) is satisfied, then the hermitian space $W^\flat$ will exist and admit a lattice $\Lambda^\flat$ as above.

Now consider the case when $n = 2m$ is even. If F/F_0 is ramified at some finite place, then a hermitian space $W^\flat$ admitting a lattice $\Lambda^\flat$ as above will exist for any d and m. On the other hand, we claim that that F/F_0 being everywhere unramified is again disallowed. Indeed, the same argument applies: our assumptions on $\Lambda^\flat$ would again imply that $W^\flat$ is split at all finite places of F_0, and the Hasse invariants at the infinite places are again given by $(-1)^{m-1}$ at φ_0 and by $(-1)^m$ at each $\varphi \neq \varphi_0$. Hence we again obtain the congruence $dm \equiv 1 \bmod 2$, forcing d to be odd.

In work in preparation, we plan to handle more cases of an AT conjecture which will allow us to weaken the constraints on the extension F/F_0 and on the lattices Λ and $\Lambda^\flat$; this will then also remove these constraints on the hermitian spaces W and $W^\flat$. More precisely, we will allow places v which are inert in F and of relative degree one over $\mathbb{Q}$ such that Λ_v is self-dual and $\Lambda^\flat_v$ is almost self-dual, cf. Remark 4.18. In light of [32], it may be possible to allow here v to be unramified over $\mathbb{Q}$.

Let us continue with the conditions we impose on $\Lambda^\flat$ and Λ. When n is odd, we require that

$$\Lambda = \Lambda^\flat \oplus O_F u,$$

and that (u,u) is a unit at each finite place v unless v is inert in F and W_v is non-split, in which case $\operatorname{ord}_v(u,u) = 1$ (and hence (v, Λ_v) is of AT type (1)). In this case, we have closed embeddings

$$\mathcal{M}_{K^\circ_{\widetilde{H}}}(\widetilde{H}) \hookrightarrow \mathcal{M}_{K^\circ_{\widetilde{G}}}(\widetilde{G}) \quad \text{and} \quad \mathcal{M}_{K^\circ_{\widetilde{H}}}(\widetilde{H}) \hookrightarrow \mathcal{M}_{K^\circ_{\widetilde{HG}}}(\widetilde{HG}) \tag{5.4}$$

completely analogous to those we have considered before, e.g. (4.11), (4.19), and (4.28).

Now suppose that n is even, and let $v_1, \dots, v_m$ be the places of F_0 which ramify in F, cf. Remark 5.2. By our assumptions, each $(v_i, \Lambda^\flat_{v_i})$ is of AT type (3). For each i, we further require that (u,u) is a unit at v_i, and that Λ_{v_i} is one of the two lattices for which the relation (4.29) holds with v_i in place of v_0. Then (v_i, Λ_{v_i}) is indeed of AT type (2). At the split and inert places v, we again require that $\Lambda_v = \Lambda_v^\flat \oplus O_{F,v}u$, where (u,u) is a unit at v unless v is inert in F and W_v is non-split, in which case $\mathrm{ord}_v(u,u) = 1$. Then we obtain natural global analogs of the diagrams (4.32) and (4.33),

$$\begin{tikzcd} & \mathcal{R}'_{\widetilde{G}} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{M}_{K^\circ_{\widetilde{H}}}(\widetilde{H}) \hookrightarrow \mathcal{R}_{\widetilde{G}} & & \mathcal{M}_{K^\circ_{\widetilde{G}}}(\widetilde{G}) \end{tikzcd}$$

and

$$\begin{tikzcd} & \mathcal{R}'_{\widetilde{HG}} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{M}_{K^\circ_{\widetilde{H}}}(\widetilde{H}) \hookrightarrow \mathcal{R}_{\widetilde{HG}} & & \mathcal{M}_{K^\circ_{\widetilde{HG}}}(\widetilde{HG}). \end{tikzcd}$$

Here the global analogs of the auxiliary spaces appearing in (4.32) and (4.33) are defined in the obvious way. Similarly, we obtain a closed embedding

$$\mathcal{M}_{K^{\circ\prime}_{\widetilde{H}}}(\widetilde{H}) := \mathcal{M}_{K^\circ_{\widetilde{H}}}(\widetilde{H}) \times_{\mathcal{R}_{\widetilde{HG}}} \mathcal{R}'_{\widetilde{HG}} \hookrightarrow \mathcal{M}_{K^\circ_{\widetilde{HG}}}(\widetilde{HG}). \tag{5.5}$$

We note that $\mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H})$ and $\mathcal{M}_{K_{\widetilde{HG}}}(\widetilde{HG})$ are smooth over the open subscheme of $\mathrm{Spec}\, O_E$ obtained by removing all places ν for which the induced pair (v_0, Λ_{v_0}) is of AT type (1). Furthermore, $\mathcal{M}_{K^\circ_{\widetilde{HG}}}(\widetilde{HG})$ has semi-stable reduction over the open subscheme of $\mathrm{Spec}\, O_E$ obtained by removing all places ν for which E_ν is ramified over $\mathbb{Q}_p$.

5.2. **Drinfeld level structure.** We continue with the setup of the previous subsection. In particular, we have the lattice $\Lambda \subset W$ satisfying the relation (5.1) and the assumptions that follow it. Let

$$\begin{aligned} \Sigma^{\mathrm{spl}} &:= \{\, \text{places } v \text{ of } F_0 \mid v \text{ splits in } F \,\}, \\ \Sigma^{\mathrm{spl},\Phi} &:= \{\, v \in \Sigma^{\mathrm{spl}} \mid \text{every place } \nu \text{ of } E \text{ above } v \text{ matches } \Phi \,\}, \end{aligned} \tag{5.6}$$

cf. (4.20). In addition, we fix a function

$$\mathbf{m} \colon \Sigma^{\mathrm{spl}} \longrightarrow \mathbb{Z}_{\geq 0} \tag{5.7}$$

with finite support contained in $\Sigma^{\mathrm{spl},\Phi}$.

Associated to these data is the open compact subgroup

$$K_G^{\mathbf{m}} := \big\{\, g \in G(\mathbb{A}_{F_0,f}) \bigm| g(\Lambda \otimes_{O_F} \widehat{O}_F) = \Lambda \otimes_{O_F} \widehat{O}_F \text{ and } g \equiv \mathrm{id} \bmod N(\mathbf{m}) \,\big\},$$

where

$$N(\mathbf{m}) := \prod_{v \in \Sigma^{\mathrm{spl}}} \mathfrak{p}_v^{\mathbf{m}(v)}.$$

As usual, we define $K_{\widetilde{G}}^{\mathbf{m}} := K_{Z^{\mathbb{Q}}} \times K_G^{\mathbf{m}}$ as in (3.6). Note that if $\mathbf{m} = 0$, then $K_G^{\mathbf{m}} = K_G^\circ$ and $K_{\widetilde{G}}^{\mathbf{m}} = K_{\widetilde{G}}^\circ$.

The subgroup $K_{\widetilde{G}}^{\mathbf{m}}$ defines a moduli stack $M_{K_{\widetilde{G}}^{\mathbf{m}}}(\widetilde{G})$ as in Section 3.2, which maps via a finite flat morphism to $M_{K^\circ_{\widetilde{G}}}(\widetilde{G})$. We then define $\mathcal{M}_{K_{\widetilde{G}}^{\mathbf{m}}}(\widetilde{G})$ to be the normalization of $\mathcal{M}_{K^\circ_{\widetilde{G}}}(\widetilde{G})$ in $M_{K_{\widetilde{G}}^{\mathbf{m}}}(\widetilde{G})$.

Theorem 5.3. *$\mathcal{M}_{K_{\widetilde{G}}^{\mathbf{m}}}(\widetilde{G})$ is a regular Deligne–Mumford stack finite and flat over $\mathcal{M}_{K^\circ_{\widetilde{G}}}(\widetilde{G})$. For $N(\mathbf{m})$ big enough, $\mathcal{M}_{K_{\widetilde{G}}^{\mathbf{m}}}(\widetilde{G})$ is relatively representable over $\mathcal{M}_0$. For every finite place ν of E, the base change $\mathcal{M}_{K_{\widetilde{G}}^{\mathbf{m}}} \times_{\mathrm{Spec}\, O_E} \mathrm{Spec}\, O_{E,(\nu)}$ is canonically isomorphic to the semi-global moduli space defined in one of Sections 4.1, 4.3, or 4.4 above.*

Note that in the last assertion of this theorem, in the case that $\mathbf{m}$ is not identically zero, we are implicitly appealing to the definitions in Remarks 4.2, 4.8, and 4.19. We analogously define the DM stacks $\mathcal{M}_{K^{\mathbf{m}}_{\widetilde{H}}}(\widetilde{H})$ and $\mathcal{M}_{K^{\mathbf{m}}_{\widetilde{HG}}}(\widetilde{HG})$ over $\operatorname{Spec} O_E$.

To define embeddings between the stacks we have introduced, we make the same assumptions on the lattices and on (u, u) as in Section 5.1. When n is odd, we analogously obtain closed embeddings

$$\mathcal{M}_{K^{\mathbf{m}}_{\widetilde{H}}}(\widetilde{H}) \hookrightarrow \widetilde{\mathcal{M}}_{K^{\mathbf{m}}_{\widetilde{G}}}(\widetilde{G}) \quad \text{and} \quad \mathcal{M}_{K^{\mathbf{m}}_{\widetilde{H}}}(\widetilde{H}) \hookrightarrow \widetilde{\mathcal{M}}_{K^{\mathbf{m}}_{\widetilde{HG}}}(\widetilde{HG}). \tag{5.8}$$

When n is even, we analogously obtain a closed embedding

$$\mathcal{M}_{K'^{\mathbf{m}}_{\widetilde{H}}}\big(\widetilde{H}\big) \hookrightarrow \mathcal{M}_{K^{\mathbf{m}}_{\widetilde{HG}}}\big(\widetilde{HG}\big). \tag{5.9}$$

Also, in all cases we obtain Hecke correspondences for elements $g \in \prod'_{v\in\Sigma^{\mathrm{spl},\Phi}}(H \times G)(F_{0,v})$ (restricted direct product),

$$\begin{tikzcd}[column sep=small] & \mathcal{M}_{K^{\boldsymbol{\mu}}_{\widetilde{HG}}}(\widetilde{HG}) \arrow[dl, "\mathrm{nat}_1"'] \arrow[dr, "\mathrm{nat}_g"] & \\ \mathcal{M}_{K^{\mathbf{m}}_{\widetilde{HG}}}(\widetilde{HG}) & & \mathcal{M}_{K^{\mathbf{m}}_{\widetilde{HG}}}(\widetilde{HG}). \end{tikzcd} \tag{5.10}$$

6. The Arithmetic Gan–Gross–Prasad conjecture

In this section, we state a version of the Arithmetic Gan–Gross–Prasad conjecture [12]. It is based on some widely open standard conjectures about algebraic cycles.

6.1. **Standard conjectures on height pairing.** Consider the category $\mathscr{V}$ of smooth proper varieties over a number field E, and let $H^*\colon \mathscr{V} \to \mathrm{grVec}_K$ be a Weil cohomology theory with coefficients in a field K of characteristic zero. Let X be an object in $\mathscr{V}$, and let $\mathrm{Ch}^i(X)$ be the group of codimension-i algebraic cycles in X modulo rational equivalence. We have a cycle class map

$$\mathrm{cl}_i\colon \mathrm{Ch}^i(X)_{\mathbb{Q}} \longrightarrow H^{2i}(X).$$

Its kernel is the group of cohomologically trivial cycles, denoted by $\mathrm{Ch}^i(X)_{\mathbb{Q},0}$. We take the Weil cohomology theory H^* as either the Betti cohomology $H^*\big(X(\mathbb{C}), \mathbb{Q}\big)$, or, for a prime ℓ, the ℓ-adic cohomology $H^*(X \otimes_E \overline{E}, \mathbb{Q}_\ell)$, endowed with its continuous $\mathrm{Gal}(\overline{E}/E)$-action. Comparison theorems between Betti cohomology and étale cohomology show that the subspace $\mathrm{Ch}^i(X)_{\mathbb{Q},0}$ is independent of the choice of these two.

We are going to base ourselves on the following conjectures of Beilinson and Bloch, cf. [21, §2].

Conjecture 6.1. *There exists a regular proper flat model $\mathcal{X}$ of X over* $\operatorname{Spec} O_E$.

Let $\mathcal{X}$ be such a model, and consider its ith Chow group $\mathrm{Ch}^i(\mathcal{X})_{\mathbb{Q}}$. Restriction to the generic fiber defines a map

$$\mathrm{Ch}^i(\mathcal{X})_{\mathbb{Q}} \longrightarrow \mathrm{Ch}^i(X)_{\mathbb{Q}}.$$

Let $\mathrm{Ch}^i_{\mathrm{fin}}(\mathcal{X})_{\mathbb{Q}}$ be the kernel of this map (cycles supported on "finite fibers"), and let $\mathrm{Ch}^i(\mathcal{X})_{\mathbb{Q},0}$ be the pre-image of $\mathrm{Ch}^i(X)_{\mathbb{Q},0}$. We are going to use the Arakelov pairing defined in Gillet–Soulé [14, §4.2.10],

$$(\ ,\ )_{\mathrm{GS}}\colon \mathrm{Ch}^i(\mathcal{X})_{\mathbb{Q},0} \times \mathrm{Ch}^{d+1-i}(\mathcal{X})_{\mathbb{Q},0} \longrightarrow \mathbb{R}, \quad d := \dim X. \tag{6.1}$$

Let $\mathrm{Ch}^{d+1-i}_{\mathrm{fin}}(\mathcal{X})^{\perp}_{\mathbb{Q}} \subset \mathrm{Ch}^i(\mathcal{X})_{\mathbb{Q},0}$ be the orthogonal complement of $\mathrm{Ch}^{d+1-i}_{\mathrm{fin}}(\mathcal{X})_{\mathbb{Q}}$ under the pairing (6.1).

Conjecture 6.2. *The natural map $\mathrm{Ch}^{d+1-i}_{\mathrm{fin}}(\mathcal{X})^{\perp}_{\mathbb{Q}} \to \mathrm{Ch}^i(X)_{\mathbb{Q},0}$ is surjective.*

Assuming Conjecture 6.1 and 6.2 above, the height pairing of Beilinson and Bloch

$$(\ ,\ )_{\mathrm{BB}}\colon \mathrm{Ch}^i(X)_{\mathbb{Q},0} \times \mathrm{Ch}^{d+1-i}(X)_{\mathbb{Q},0} \longrightarrow \mathbb{R} \tag{6.2}$$

can be defined as follows. Lift the elements $c_1 \in \mathrm{Ch}^i(X)_{\mathbb{Q},0}$ and $c_2 \in \mathrm{Ch}^{d+1-i}(X)_{\mathbb{Q},0}$ to $\widetilde{c}_1 \in \mathrm{Ch}^{d+1-i}_{\mathrm{fin}}(\mathcal{X})^{\perp}_{\mathbb{Q}}$ and $\widetilde{c}_2 \in \mathrm{Ch}^{i}_{\mathrm{fin}}(\mathcal{X})^{\perp}_{\mathbb{Q}}$, respectively. Define

$$(c_1, c_2)_{\mathrm{BB}} := (\widetilde{c}_1, \widetilde{c}_2)_{\mathrm{GS}}. \tag{6.3}$$

It is easy to see that this is independent of the choices of the liftings.

Remark 6.3. Assuming Conjectures 6.1 and 6.2, the pairing (6.2) is independent of the choice of the (regular proper flat) integral model $\mathcal{X}$, cf. [34, Lem. 1.5].

Remark 6.4. Assume that there exists a smooth proper model $\mathcal{X}$ of X over O_E. Then by [33, Th. 6.11], Conjecture 6.2 holds for $\mathcal{X}$ and therefore the intersection product (6.2) is defined; again, by [34, Lem. 1.5], the intersection product is independent of the choice of $\mathcal{X}$ (assumed to be smooth and proper).

6.2. **Cohomology and Hecke–Kunneth projectors.** We apply these considerations to the Shimura varieties defined in Section 3.[5] In order to simplify notation, we write K for $K_{\widetilde{HG}}$ in $\mathrm{Sh}_{K_{\widetilde{HG}}}(\widetilde{HG})$ throughout the rest of this section.

Denote by $\mathscr{H}_K$ the Hecke algebra of bi-K-invariant $\mathbb{Q}$-valued functions with compact support, with multiplication given by the convolution product,

$$\mathscr{H}_K := C_c^\infty\big(\widetilde{HG}(\mathbb{A}_f)/\!\!/K, \mathbb{Q}\big). \tag{6.4}$$

The variety $\mathrm{Sh}_K(\widetilde{HG})$ is equipped with a collection of algebraic correspondences ($\mathbb{Q}$-linear combinations of algebraic cycles on the self-product of $\mathrm{Sh}(\widetilde{HG})_K$ with itself), the Hecke correspondences associated to $f \in \mathscr{H}_K$.

We also introduce some subalgebras. Let S be a finite set of non-archimedean places of $\mathbb{Q}$ containing all bad places and such that $K = \prod_p K_p$; thus for all $p \notin S$ the group $\widetilde{HG} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is quasi-split and K_p is hyperspecial. We define

$$\mathscr{H}_K^S \subset \mathscr{H}_K \tag{6.5}$$

as the subspace spanned by pure tensors $f = \otimes_p f_p \in \mathscr{H}_K$ such that $f_p = \mathbf{1}_{K_p}$ for all $p \in S$. This clearly defines a subalgebra, and we have

$$\mathscr{H}_K^S \simeq \sideset{}{'}\bigotimes_{p \notin S} \mathscr{H}_{K_p}$$

(restricted tensor product).

Recall from (2.1) the product decomposition

$$\widetilde{HG}(\mathbb{Q}_p) = Z^{\mathbb{Q}}(\mathbb{Q}_p) \times \prod_{v|p} G_W(F_{0,v}). \tag{6.6}$$

For $p \notin S$, we have a hyperspecial subgroup $K_p = Z^{\mathbb{Q}}(\mathbb{Z}_p) \times K_{HG,p}$ of $\widetilde{HG}(\mathbb{Q}_p)$ and

$$K_{HG,p} = \prod_{v|p} K_{HG,v},$$

where $K_{HG,v}$ is a hyperspecial compact open subgroup of $(H \times G)(F_{0,v})$. We define

$$\mathscr{H}_K^{S,\mathrm{spl}} \subset \mathscr{H}_K^S$$

as the subspace spanned by all $f = \otimes_p f_p \in \mathscr{H}_K^S$ such that $f_p = \phi_p \otimes \bigotimes_{v|p} f_v$, where $\phi_p = \mathbf{1}_{K_{Z^{\mathbb{Q}},p}}$ for all p, and $f_v = \mathbf{1}_{K_{HG,v}}$ unless v is split in F. This defines a subalgebra of $\mathscr{H}_K^S$, and we have

$$\mathscr{H}_K^{S,\mathrm{spl}} \simeq \bigotimes_{v \in \Sigma^{S,\mathrm{spl}}} \mathscr{H}_{HG,K_{HG,v}}, \tag{6.7}$$

where $\mathscr{H}_{HG,K_{HG,v}} := C_c^\infty((H \times G)(F_{0,v})/\!\!/K_{HG,v}, \mathbb{Q})$, and where $\Sigma^{S,\mathrm{spl}}$ is the set of places in Σ^{spl} that do not lie over S. Here Σ^{spl} is as in (5.6).

[5] Note that these spaces can in fact be (and in particular cases of interest to us, are) DM stacks, but we will suppress this point in our discussion throughout the rest of the paper. The extension of the usual intersection theory to DM stacks is supplied by Gillet's paper [13].

We introduce a subalgebra of $\mathscr{H}_K^{S,\mathrm{spl}}$ by further demanding in the definition of $\mathscr{H}_K^{S,\mathrm{spl}}$ that $f_v = \mathbf{1}_{K_{HG,v}}$ unless v is of degree one over $\mathbb{Q}$. We denote this subalgebra by $\mathscr{H}_K^{S,\deg=1}$ and we have

$$\mathscr{H}_K^{S,\deg=1} \simeq \bigotimes_{v\in\Sigma^{S,\deg=1}} \mathscr{H}_{HG,K_{HG,v}}. \tag{6.8}$$

Here

$$\Sigma^{S,\deg=1} := \big\{\, v \in \Sigma^{S,\mathrm{spl}} \bigm| \deg_{\mathbb{Q}} v = 1 \,\big\}. \tag{6.9}$$

The last two subalgebras $\mathscr{H}_K^{S,\mathrm{spl}}$ and $\mathscr{H}_K^{S,\deg=1}$ are introduced in order to apply the automorphic Chebotarev density theorem of Ramakrishnan [41, Cor. B] and [42, Th. A]. It is clear that the two subalgebras are in the center of the full Hecke algebra $\mathscr{H}_K$.

The Hodge conjecture implies that the Kunneth projector from $\bigoplus_{i\in\mathbb{Z}} H^i(\mathrm{Sh}_K(\widetilde{HG}),\mathbb{Q})$ to each summand $H^i(\mathrm{Sh}_K(\widetilde{HG}),\mathbb{Q})$, or to the primitive cohomology, is induced by algebraic cycles (with $\mathbb{Q}$-linear combinations). Morel and Suh [38] prove a partial result on the algebraicity of Kunneth projectors for Shimura varieties (the so-called "standard sign conjecture"), conditional on Arthur's conjecture. Here we improve on their result in our setting of unitary groups with the help of the density theorem of Ramakrishnan.

Remark 6.5. The hypotheses on which [38] is based are in fact known in our setting: 1) The stabilization of the twisted trace formula is known by the work of Waldspurger. 2) The Arthur conjecture on the expression of the discrete spectrum in terms of discrete Arthur parameters is known (cf. Mok [37] for quasi-split unitary groups, and Kaletha–Minguez–Shin–White [23] (and its sequels) for inner forms of unitary groups). Here we note that our group $\widetilde{HG}$ is a product of a unitary group and a torus by (2.1). 3) The comparison between the Adams–Johnson classification and the Arthur classification of cohomological Arthur parameters of real groups is known by Arancibia–Moeglin–Renard [1].

Theorem 6.6. *Let S be a finite set of non-archimedean places of $\mathbb{Q}$ containing all bad places. Let $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$. Then there exists f^ε in $\mathscr{H}_K^{S,\mathrm{spl}}$ such that the associated Hecke correspondence induces the projector to the even, resp. odd, degree cohomology,*

$$\bigoplus_{i\in\mathbb{Z}} H^i\big(\mathrm{Sh}_K(\widetilde{HG}),\mathbb{Q}\big) \longrightarrow \bigoplus_{i\equiv\varepsilon \bmod 2} H^i\big(\mathrm{Sh}_K(\widetilde{HG}),\mathbb{Q}\big).$$

Definition 6.7. We set $f^+ = f^0$ and $f^- = f^1$, and call them the even, resp. odd, *Hecke–Kunneth projectors.*

Remark 6.8. We are informed by Ramakrishnan that a slightly stronger result than [42, Th. A] holds (namely, one may replace the set $\Sigma^1(K/F)$ of primes of K of degree one over F in loc. cit. by the subset of primes of K that are of degree one over $\mathbb{Q}$). This will allow us to replace $\mathscr{H}_K^{S,\mathrm{spl}}$ by its subalgebra $\mathscr{H}_K^{S,\deg=1}$ in Theorem 6.6. In fact, if $F/\mathbb{Q}$ is Galois, then this follows directly from [42, Th. A].

Proof of Theorem 6.6. The argument follows that of [38], with the density theorem of Ramakrishnan [41] and [42, Th. A] as new ingredient. To indicate the modifications, we outline the proof in loc. cit. It suffices to prove the assertions after tensoring with $\mathbb{C}$. We henceforth consider $H^i(\mathrm{Sh}_K(\widetilde{HG}),\mathbb{C})$ with the action by $\mathscr{H}_{K,\mathbb{C}} = \mathscr{H}_K \otimes \mathbb{C}$.

By Matsushima's formula [6, VII], we have a decomposition of the Betti cohomology[6] into a finite direct sum, equivariant for the action of $\mathscr{H}_K$

$$H^*\big(\mathrm{Sh}_K(\widetilde{HG}),\mathbb{C}\big) \simeq \bigoplus_{\pi=\pi_f\otimes\pi_\infty} m_{\mathrm{disc}}(\pi)\big(\pi_f^K \otimes H^*(\widetilde{\mathfrak{hg}}, K_\infty;\pi_\infty)\big), \tag{6.10}$$

where the Hecke algebra acts on the right hand side through the space π_f^K. Here

[6]When the Shimura variety $\mathrm{Sh}_K(\widetilde{HG})$ is non-compact, one has to replace $H^*(\mathrm{Sh}_K(\widetilde{HG}),\mathbb{C})$ by the image of the Betti cohomology of the toroidal compactification in the Betti cohomology of $\mathrm{Sh}_K(\widetilde{HG})$. This also coincides with the intersection cohomology of the Baily–Borel compactification of $\mathrm{Sh}_K(\widetilde{HG})$.

- π runs through the set $\Pi_{\mathrm{disc}}(\widetilde{HG})$ of irreducible representations of $\widetilde{HG}(\mathbb{A})$ in the discrete automorphic spectrum $L^2_{\mathrm{disc}}(Z^{\mathbb{Q}}(\mathbb{R})\widetilde{HG}(\mathbb{Q})\backslash\widetilde{HG}(\mathbb{A}))$, and $m_{\mathrm{disc}}(\pi)$ is the multiplicity of π;[7]
- $\widetilde{\mathfrak{hg}}$ is the complex Lie algebra of $\widetilde{HG}(\mathbb{R})$;
- K_∞ is the centralizer of $h_{\widetilde{HG}}$ in $\widetilde{HG}(\mathbb{R})$;[8] and
- π_f^K denotes the invariants of π_f under K.

We have a decomposition of the discrete spectrum as $\widetilde{HG}(\mathbb{A})$-modules (cf. [2], [26, §8]),

$$L^2_{\mathrm{disc}}\big(Z^{\mathbb{Q}}(\mathbb{R})\widetilde{HG}(\mathbb{Q})\backslash\widetilde{HG}(\mathbb{A})\big) \simeq \bigoplus_{\psi}\bigoplus_{\pi\in\Pi_\psi} m(\psi,\pi)\cdot\pi.$$

Here the sum is taken over all equivalence classes of global Arthur parameters ψ, with trivial associated quasi-character of $Z^{\mathbb{Q}}(\mathbb{R})$. Furthermore, Π_ψ denotes the Arthur packet attached to ψ, and $m(\psi,\pi)$ denotes the Arthur multiplicity. Hence we may rewrite the decomposition (6.10) according to global Arthur parameters as

$$H^*\big(\mathrm{Sh}_K(\widetilde{HG}),\mathbb{C}\big) \simeq \bigoplus_{\psi} V_{\psi,K}, \tag{6.11}$$

where we set

$$V_{\psi,K} := \bigoplus_{\pi=\pi_f\otimes\pi_\infty\in\Pi_\psi} m(\psi,\pi)\big(\pi_f^K\otimes H^*(\widetilde{\mathfrak{hg}},K_\infty;\pi_\infty)\big).$$

The isomorphism (6.11) is $\mathscr{H}_K$-equivariant. Moreover, there is a canonical Lefschetz class (coming from the cup product with the Killing form) which induces an $\mathrm{SL}_2(\mathbb{C})$-action on the graded vector space $H^*(\mathrm{Sh}_K(\widetilde{HG}),\mathbb{C})$. Correspondingly there is an $\mathrm{SL}_2(\mathbb{C})$-action on the graded vector space $H^*(\widetilde{\mathfrak{hg}},K_\infty;\pi_\infty)$. The decomposition respects the $\mathrm{SL}_2(\mathbb{C})$-action and the grading on both sides of (6.11). We refer to [2, Prop. 9.1] for details.

In general the definition of Arthur parameter involves the hypothetical automorphic Langlands group. In the case of classical groups, Arthur avoids the Langlands group by using cuspidal (or isobaric) automorphic representations of general linear groups as substitute parameters [3]. In our setting, our group is a product $\widetilde{HG} = Z^{\mathbb{Q}}\times HG$, where

$$HG := \mathrm{Res}_{F_0/\mathbb{Q}}(H\times G), \tag{6.12}$$

cf. (2.1). Accordingly, we will write an Arthur parameter ψ as a pair (ψ_0,ψ_1) where ψ_0 and ψ_1 are Arthur parameters for the two factors $Z^{\mathbb{Q}}$ and HG and those are defined in terms of cuspidal (or isobaric) automorphic representations of general linear groups.

By the product decomposition (3.4) we have

$$\mathrm{Sh}_K(\widetilde{HG})_{\mathbb{C}} \simeq \mathrm{Sh}_{K_{Z^{\mathbb{Q}}}}(Z^{\mathbb{Q}})_{\mathbb{C}} \times_{\mathrm{Spec}\,\mathbb{C}} \mathrm{Sh}_{K_{HG}}(HG)_{\mathbb{C}}$$

and an induced isomorphism (note that the first factor above is zero-dimensional)

$$H^*\big(\mathrm{Sh}_K(\widetilde{HG}),\mathbb{C}\big) \simeq H^0\big(\mathrm{Sh}_{K_{Z^{\mathbb{Q}}}}(Z^{\mathbb{Q}}),\mathbb{C}\big)\otimes H^*\big(\mathrm{Sh}_{K_{HG}}(HG),\mathbb{C}\big). \tag{6.13}$$

We may therefore replace $\mathrm{Sh}_K(\widetilde{HG})$ by $\mathrm{Sh}_{K_{HG}}(HG)$ in the conclusion of Theorem 6.6. We record the decomposition similar to (6.11),

$$H^*\big(\mathrm{Sh}_{K_{HG}}(HG),\mathbb{C}\big) \simeq \bigoplus_{\psi_1} V_{\psi_1,K_{HG}}. \tag{6.14}$$

We will prove the following proposition after the end of the proof of Theorem 6.6.

Proposition 6.9. *Let $\pi_{\Sigma^{S,\mathrm{spl}}}$ be an irreducible admissible representation of $(H\times G)(\mathbb{A}_{\Sigma^{S,\mathrm{spl}}})$. Then there is at most one (global) Arthur parameter ψ_1 such that $\pi_{\Sigma^{S,\mathrm{spl}}}$ can be completed to an irreducible constituent π of Π_{ψ_1}.*

By [38, §4], this proposition implies the following statement.

[7] When the group $\widetilde{HG}$ is anisotropic modulo its center, the quotient is compact and then $L^2_{\mathrm{disc}} = L^2$.

[8] In particular, $Z^{\mathbb{Q}}(\mathbb{R})$ is contained in K_∞.

Corollary 6.10. *With the same notation as in Proposition 6.9, let*

$$\Pi_\infty(\pi_{\Sigma^{S,\mathrm{spl}}}) := \big\{\, \pi_\infty \in \Pi\big(HG(\mathbb{R})\big) \bigm| \textit{there exists } \pi_f \textit{ such that } \pi_f \otimes \pi_\infty \in \Pi_{\mathrm{disc}}(HG) \,\big\}.$$

Here $\Pi(HG(\mathbb{R}))$ *denotes the set of equivalence classes of irreducible admissible representations of* $HG(\mathbb{R})$, *and* π_f *runs through all irreducible admissible representations of* $HG(\mathbb{A}_f)$ *which complete* $\pi_{\Sigma^{S,\mathrm{spl}}}$.

Then the degree i *modulo* 2 *such that* $H^i(\mathfrak{hg}, K_\infty; \pi_\infty) \neq 0$ *is constant (in* $\mathbb{Z}/2\mathbb{Z}$*) as* π_∞ *varies through* $\Pi_\infty(\pi_{\Sigma^{S,\mathrm{spl}}})$ *and* i *varies through* $\mathbb{Z}$. □

We denote the constant i mod 2 above by $\varepsilon(\psi_1) \in \mathbb{Z}/2\mathbb{Z}$, where ψ_1 is the unique Arthur parameter from Proposition 6.9.

Now we return to the decomposition (6.14). The (finitely many) nonzero direct summands $V_{\psi_1, K_{HG}}$ are distinct $\mathscr{H}^{S,\mathrm{spl}}_{K,\mathbb{C}}$-modules. Therefore, for each ψ_1, there exists $f_{\psi_1} \in \mathscr{H}^{S,\mathrm{spl}}_{K,\mathbb{C}}$ that induces the projector to $V_{\psi_1, K_{HG}}$. Now set

$$f^+ := \sum_{\substack{\psi_1 \\ \varepsilon(\psi_1)=0}} f_{\psi_1} \quad \text{and} \quad f^- := \sum_{\substack{\psi_1 \\ \varepsilon(\psi_1)=1}} f_{\psi_1}.$$

This completes the proof of Theorem 6.6. □

Proof of Proposition 6.9. In our case of the product of unitary groups HG, the global Arthur parameter $\psi_1 = \otimes_v \psi_{1,v}$ is defined in terms of isobaric automorphic representations on general linear groups $(\mathrm{GL}_{n-1} \times \mathrm{GL}_n)(\mathbb{A}_F)$ [3]. By the automorphic Chebotarev density theorem of Ramakrishnan [41, 42], an isobaric automorphic representation $\Pi = \otimes_w \Pi_w$ of $\mathrm{GL}_n(\mathbb{A}_F)$ is determined by the collection Π_w for all places w of F of relative degree one over F_0, away from any finite set of places. □

6.3. **Arithmetic diagonal cycles.** We now apply the considerations of Sections 6.1 and 6.2 to the canonical model $M_K(\widetilde{HG})$ of $\mathrm{Sh}_K(\widetilde{HG})$ over E. When $K = K_{\widetilde{HG}}$ is as in (3.12), we have given in Section 3.2 a moduli interpretation of $M_K(\widetilde{HG})$.

We consider the cycle class map in degree $n-1$ (for the Betti cohomology),

$$\mathrm{cl}_{n-1} \colon \mathrm{Ch}^{n-1}\big(M_K(\widetilde{HG})\big)_{\mathbb{Q}} \longrightarrow H^{2(n-1)}\big(\mathrm{Sh}_K(\widetilde{HG}), \mathbb{Q}\big).$$

Note that $\dim M_K(\widetilde{HG})) = 2n-3$ is odd. In particular the above cohomology group is *not* in the middle degree, but just above.

Let $K_{\widetilde{H}}$ be a compact open subgroup of $\widetilde{H}(\mathbb{A}_f)$ contained in $K \cap \widetilde{H}(\mathbb{A}_f)$. We have a finite and unramified morphism

$$M_{K_{\widetilde{H}}}(\widetilde{H}) \longrightarrow M_K(\widetilde{HG}).$$

The proper push-forward defines a cycle class $[M_{K_{\widetilde{H}}}(\widetilde{H})] \in \mathrm{Ch}^{n-1}(M_K(\widetilde{HG}))_{\mathbb{Q}}$. We now fix a Haar measure on $\widetilde{H}(\mathbb{A})$ such that the volume $\mathrm{vol}(K_{\widetilde{H}}) \in \mathbb{Q}$. We then define the normalized class

$$z_K = \mathrm{vol}(K_{\widetilde{H}})[M_{K_{\widetilde{H}}}(\widetilde{H})] \in \mathrm{Ch}^{n-1}\big(M_K(\widetilde{HG})\big)_{\mathbb{Q}}, \tag{6.15}$$

which is independent of the choice of the group $K_{\widetilde{H}}$. We call z_K the *arithmetic diagonal cycle* since it lies in the arithmetic middle dimension, i.e., $2\dim z_K = \dim M_K(\widetilde{HG}) + 1$. Let $\mathcal{Z}_K$ be the cyclic Hecke submodule of $\mathrm{Ch}^{n-1}(M_K(\widetilde{HG}))_{\mathbb{Q}}$ generated by z_K.

Let $f^- \in \mathscr{H}^{S,\mathrm{spl}}_K$ be an odd Hecke–Kunneth projector, cf. Definition 6.7. We obtain a map to the Chow group of cohomologically trivial cycles,

$$R(f^-) \colon \mathrm{Ch}^{n-1}\big(M_K(\widetilde{HG})\big)_{\mathbb{Q}} \longrightarrow \mathrm{Ch}^{n-1}\big(M_K(\widetilde{HG})\big)_{\mathbb{Q},0}.$$

The map respects the action of the Hecke algebra $\mathscr{H}_K$ since f^- lies in the center of $\mathscr{H}_K$. We obtain a cohomologically trivial cycle in the Chow group,

$$z_{K,0} = R(f^-)(z_K) \in \mathrm{Ch}^{n-1}\big(M_K(\widetilde{HG})\big)_{\mathbb{Q},0}. \tag{6.16}$$

Accordingly we call $z_{K,0}$ the *cohomologically trivial arithmetic diagonal cycle*.

Remark 6.11. (i) Conjecturally, for any smooth projective variety X over a number field, it should be true that, given i, if a correspondence induces the zero endomorphism on $H^{2i-1}(X)$, then the induced endomorphism on $\mathrm{Ch}^i(X)_{\mathbb{Q},0}$ is zero, cf. [22, Conj. 2.1(d)] and [4, Conj. 5.2]. If this were true for $X = M_K(\widetilde{HG})$ and $i = n-1$, then the element $z_{K,0}$ would be independent of the choice of $f^- \in \mathscr{H}_K$. In particular, the endomorphism $R(f^-)$ would define a projector, and induce a decomposition of $\mathscr{H}_K$-modules

$$\mathrm{Ch}^{n-1}\big(M_K(\widetilde{HG})\big)_{\mathbb{Q}} = \mathrm{Ch}^{n-1}\big(M_K(\widetilde{HG})\big)_{\mathbb{Q},0} \oplus \mathrm{Im}(\mathrm{cl}_{n-1}). \tag{6.17}$$

One expects that the summands in the decomposition (6.17) have finite dimension. The action of the Hecke algebra $\mathscr{H}_K$ on both summands on the right-hand side of (6.17) should be semisimple and have the form

$$\begin{aligned} \mathrm{Ch}^{n-1}(M_K(\widetilde{HG}))_{\mathbb{C},0} &= \bigoplus_{\pi} \mathrm{Ch}^{n-1}(M_K(\widetilde{HG}))_{\mathbb{C},0}[\pi_f^K], \\ \mathrm{Im}(\mathrm{cl}_{n-1})_{\mathbb{C}} &= \bigoplus_{\sigma} \mathrm{Im}(\mathrm{cl}_{n-1})_{\mathbb{C}}[\sigma_f^K]. \end{aligned} \tag{6.18}$$

Here π runs through all *automorphic* representations contributing to $H^{2n-3}(\mathrm{Sh}_K(\widetilde{HG}), \mathbb{C})$, and σ runs through all *automorphic* representations contributing to $H^{2n-2}(\mathrm{Sh}_K(\widetilde{HG}), \mathbb{C})$. Also, $\mathrm{Ch}^{n-1}(M_K(\widetilde{HG}))_{\mathbb{C},0}[\pi_f^K]$ denotes the π_f^K-isotypic component of the $\mathscr{H}_K$-module, i.e., the image under the evaluation map, which is injective,

$$\pi_f^K \otimes \mathrm{Hom}_{\mathscr{H}_K}\big(\pi_f^K, \mathrm{Ch}^{n-1}(M_K(\widetilde{HG}))_{\mathbb{C},0}\big) \longrightarrow \mathrm{Ch}^{n-1}\big(M_K(\widetilde{HG})\big)_{\mathbb{C},0}.$$

Then

$$\mathrm{Ch}^{n-1}\big(M_K(\widetilde{HG})\big)_{\mathbb{C},0}[\pi_f^K] \simeq \pi_f^K \otimes \mathrm{Hom}_{\mathscr{H}_K}\big(\pi_f^K, \mathrm{Ch}^{n-1}(M_K(\widetilde{HG}))_{\mathbb{C},0}\big). \tag{6.19}$$

Similar definitions apply to $\mathrm{Im}(\mathrm{cl}_{n-1})_{\mathbb{C}}[\sigma_f^K]$.

(ii) In [54] the *Chow–Kunneth decomposition* was used to modify the arithmetic diagonal cycle to make its cohomology class trivial. However, it is difficult to show the existence of a Chow–Kunneth decomposition except in some special cases. Therefore, the procedure above is preferable.

6.4. **The Arithmetic Gan–Gross–Prasad conjecture, for a fixed level $K \subset \widetilde{HG}(\mathbb{A}_f)$.** Let $\mathcal{Z}_{K,0}$ denote the cyclic Hecke submodule of $\mathrm{Ch}^{n-1}(M_K(\widetilde{HG}))_{\mathbb{Q},0}$ generated by $z_{K,0}$ or, equivalently, the image of $\mathcal{Z}_K$ under the map $R(f^-)$. We would like to decompose as $\mathscr{H}_K$-modules,

$$\mathcal{Z}_{K,0} \subset \mathrm{Ch}^{n-1}\big(M_K(\widetilde{HG})\big)_{\mathbb{C},0}.$$

In the conjectural decomposition (6.18), we only consider the *tempered* part of the spectrum. The non-tempered part is also interesting but will be postponed to the future.

Let π be an automorphic representation of $\widetilde{HG}(\mathbb{A})$ with trivial restriction to the central subgroup $Z^{\mathbb{Q}}(\mathbb{A})$. By (2.1) we may consider π as an automorphic representation of $(H \times G)(\mathbb{A}_{F_0})$. Let R be the tensor product representation of the L-group of $H \times G$ defined in [12, §22]. The L-function $L(s, \pi, R)$ depends only on the Arthur parameter ψ of π.

To explain this L-function, we write a tempered Arthur parameter $\psi = \psi^{(n-1)} \boxtimes \psi^{(n)}$ formally as

$$\psi^{(n-1)} = \bigoplus_i \psi_i^{(n-1)}, \quad \psi^{(n)} = \bigoplus_j \psi_j^{(n)}, \tag{6.20}$$

where $\psi_i^{(n-1)}$ correspond to *distinct* cuspidal automorphic representations of $\mathrm{GL}_{N_i^{(n-1)}}$ such that $\sum N_i^{(n-1)} = n-1$, and similarly for $\psi^{(n)}$. Then the L-function in question equals

$$L(s, \pi, R) := \prod_{i,j} L\big(s, \psi_i^{(n-1)} \boxtimes \psi_j^{(n)}\big), \tag{6.21}$$

where each factor is a Rankin–Selberg convolution.

We assume that π lies in the packet Π_ψ of a cohomological tempered packet ψ. Here "cohomological" is a condition on the archimedean component ψ_∞ and refers to the trivial coefficient system.

The first version of the Arithmetic Gan–Gross–Prasad conjecture can now be stated as follows.

Conjecture 6.12. *Let $K \subset \widetilde{HG}(\mathbb{A}_f)$ be an open compact subgroup. Let π be as above, i.e., with trivial restriction to the central subgroup $Z^{\mathbb{Q}}(\mathbb{A})$ and lying in a cohomological tempered Arthur packet. Consider the following conditions on π.*

(a) $\dim \mathrm{Hom}_{\mathscr{H}_K}(\pi_f^K, \mathcal{Z}_{K,0}) = 1$.[9]

(b) The order of vanishing $\mathrm{ord}_{s=1/2} L(s,\pi,R)$ *equals one, the space* $\mathrm{Hom}_{\widetilde{H}(\mathbb{A}_f)}(\pi_f, \mathbb{C})$ *is one-dimensional, and its generator does not vanish on the subspace* $\pi_f^K \subset \pi_f$.

(c) $\mathrm{Hom}_{\mathscr{H}_K}(\pi_f^K, \mathrm{Ch}^{n-1}(M_K(\widetilde{HG}))_{\mathbb{C},0}) \neq 0$.

Then (a) and (b) are equivalent and imply (c). If $E = F$, then (a), (b) and (c′) are equivalent, where

(c′) $\dim \mathrm{Hom}_{\mathscr{H}_K}(\pi_f^K, \mathrm{Ch}^{n-1}(M_K(\widetilde{HG}))_{\mathbb{C},0}) = 1$, *the space* $\mathrm{Hom}_{\widetilde{H}(\mathbb{A}_f)}(\pi_f, \mathbb{C})$ *is one-dimensional, and its generator does not vanish on the subspace* $\pi_f^K \subset \pi_f$.

Remark 6.13. If $E = F$, the equivalence between (b) and (c′) is part of the Beilinson–Bloch conjecture [4, 5] that generalizes the Birch–Swinnerton-Dyer conjecture.

We would like to test the conjecture quantitatively through height pairings. Now we assume that Conjectures 6.1 and 6.2 hold for $M_K(\widetilde{HG})$. For instance, they can be verified when $M_K(\widetilde{HG})$ has everywhere good reduction,[10] cf. Remark 6.4. In particular, we have the Beilinson–Bloch height pairing (6.2) between cohomologically trivial cycles for $i = n-1$. We extend it to a hermitian form on $\mathrm{Ch}^{n-1}(M_K(\widetilde{HG}))_{\mathbb{C},0}$. Pairing against the distinguished element $z_{K,0} \in \mathrm{Ch}^{n-1}(M_K(\widetilde{HG}))_{\mathbb{Q},0}$ then defines a linear functional

$$\begin{aligned} \ell_K\colon\ \mathrm{Ch}^{n-1}\big(M_K(\widetilde{HG})\big)_{\mathbb{C},0} &\longrightarrow \mathbb{C} \\ z &\longmapsto (z, z_{K,0})_{\mathrm{BB}}. \end{aligned}$$

By Remark 6.11, the functional ℓ_K should be independent of the choice of f^-.

Let $\mathcal{Z}_{K,0}[\pi_f^K]$ be the π_f^K-isotypic component of $\mathcal{Z}_{K,0}$ as an $\mathscr{H}_K$-module, so that

$$\mathcal{Z}_{K,0}[\pi_f^K] \simeq \pi_f^K \otimes \mathrm{Hom}_{\mathscr{H}_K}(\pi_f^K, \mathcal{Z}_{K,0}).$$

The second version of the Arithmetic Gan–Gross–Prasad conjecture in terms of the height pairing can be stated as follows.

Conjecture 6.14. *Let $K \subset \widetilde{HG}(\mathbb{A}_f)$ be an open compact subgroup. Let π be as above. Then the following conditions on π are equivalent.*

(a) $\ell_K|_{\mathcal{Z}_{K,0}[\pi_f^K]} \neq 0$.

(b) The order of vanishing $\mathrm{ord}_{s=1/2} L(s,\pi,R)$ *equals one, the space* $\mathrm{Hom}_{\widetilde{H}(\mathbb{A}_f)}(\pi_f, \mathbb{C})$ *is one-dimensional, and its generator does not vanish on the subspace* $\pi_f^K \subset \pi_f$.

(c) $\ell_K|_{\mathrm{Ch}^{n-1}(M_K(\widetilde{HG}))_{\mathbb{C},0}[\pi_f^K]} \neq 0$.

Remark 6.15. Our formulation differs in several aspects from [12, Conj. 27.1]. First, in [12], the Shimura varieties are associated to unitary groups, whereas here we consider Shimura varieties associated to groups which differ from those in loc. cit. by a central subgroup, cf. Remark 3.1(iii). Correspondingly, the varieties in loc. cit. are defined over F, whereas our varieties are defined over the extension E of F. As a consequence, we cannot predict the dimension of $\mathrm{Hom}_{\mathscr{H}_K}(\pi_f^K, \mathrm{Ch}^{n-1}(M_K(\widetilde{HG}))_{\mathbb{C},0})$ in Conjecture 6.12(b) when $F \neq E$ (in the version of loc. cit., this space is one-dimensional if it is non-zero). Note, however, that if $F = F_0K$ for an imaginary quadratic field K and the CM type is induced from K (as in [17]), then $F = E$. Second, we

[9] Note that we always have $\dim \mathrm{Hom}_{\mathscr{H}_K}(\pi_f^K, \mathcal{Z}_{K,0}) \leq 1$, because $\mathcal{Z}_{K,0}$ is a cyclic $\mathscr{H}_K$-module.

[10] Instances of everywhere good reduction can in fact be constructed, cf. Remark 6.19.

exploit that the standard sign conjecture is known in our case, and we use it to construct the cohomologically trivial diagonal cycle $z_{K,0}$ and the corresponding linear functional ℓ_K that occur in our version of the conjecture. Third, we work with a fixed level K and specify the compact open subgroup K over which the linear functional ℓ_K should be nonzero. Finally, we note that, in the terminology of [12], we are only considering the case of a trivial local system $\mathcal{F}$.

Remark 6.16. The space $\mathrm{Hom}_{\widetilde{H}(\mathbb{A}_f)}(\pi_f, \mathbb{C})$ is at most one-dimensional. It is one-dimensional if and only if the member π in the packet Π_ψ is as prescribed by the local Gan–Gross–Prasad conjecture [12, Conj. 17.3]. The local Gan–Gross–Prasad conjecture [12, Conj. 17.1] predicts that there is a unique π in the packet Π_ψ such that $\mathrm{Hom}_{\widetilde{H}(\mathbb{A}_f)}(\pi_f, \mathbb{C}) \neq 0$ (in which case $\dim \mathrm{Hom}_{\widetilde{H}(\mathbb{A}_f)}(\pi_f, \mathbb{C}) = 1$).

Remark 6.17. Let us restrict our attention to the *tempered part* in the decomposition in the first line of (6.18),

$$\mathrm{Ch}^{n-1}\big(M_K(\widetilde{HG})\big)_{\mathbb{C},0,\mathrm{temp}} = \bigoplus_{\pi \text{ tempered}} \mathrm{Ch}^{n-1}\big(M_K(\widetilde{HG})\big)_{\mathbb{C},0}[\pi_f^K].$$

Let $\mathcal{Z}_{K,0,\mathrm{temp}}$ be the Hecke submodule of $\mathrm{Ch}^{n-1}(M_K(\widetilde{HG}))_{\mathbb{C},0,\text{ temp}}$ generated by $z_{K,0}$. Then Conjecture 6.12 (together with the expectations in Remark 6.11(i)) implies that when $E = F$,

$$\mathcal{Z}_{K,0,\mathrm{temp}} = \bigoplus_{\pi} \mathrm{Ch}^{n-1}\big(M_K(\widetilde{HG})\big)_{\mathbb{C},0}[\pi_f^K],$$

where the sum runs over all tempered automorphic representations π such that

$$\mathrm{ord}_{s=1/2}\, L(\pi, s, R) = 1$$

and such that the space $\mathrm{Hom}_{\widetilde{H}(\mathbb{A}_f)}(\pi_f, \mathbb{C})$ is one-dimensional, with generator not vanishing on the subspace $\pi_f^K \subset \pi_f$.

Remark 6.18. A parallel question is to investigate the structure of the $\mathscr{H}_K$-submodule in $H^{2(n-2)}(M_K(\widetilde{HG}), \mathbb{C})$ generated by the cohomology class $\mathrm{cl}_{n-1}(z_K)$. However, since every automorphic representation contributing to the cohomology in off-middle degree must be non-tempered, the answer to such a question must involve the non-tempered version of the Gan–Gross–Prasad conjecture. We hope to return to this in the future.

Remark 6.19. As remarked above, the height pairing is defined unconditionally if $M_K(\widetilde{HG})$ has everywhere good reduction. To construct such cases, let us assume now that $K = K_{\widetilde{HG}} = K_{Z^{\mathbb{Q}}} \times K_H \times K_G$, where $K_{Z^{\mathbb{Q}}}$ is the usual maximal compact subgroup (3.7), K_G is the stabilizer of a lattice Λ in W, and K_H is the stabilizer of a lattice $\Lambda^\flat$ in $W^\flat$. We make the following assumptions on the field extensions $F/F_0/\mathbb{Q}$:

- Each finite place v of F_0 which is ramified over $\mathbb{Q}$ or of residue characteristic 2 is split in F.
- Each finite place v of F_0 which ramifies in F is of degree one over $\mathbb{Q}$.

We also make the following assumptions on the hermitian spaces W and $W^\flat$. We distinguish the case when n is odd from the case when n is even, cf. Remark 5.2.

When $n = 2m + 1$ is odd, we impose that

- W is split at all finite places of F_0 which are inert in F; and
- $W^\flat$ is split at all finite places, which forces m and $[F_0 : \mathbb{Q}]$ to be odd, cf. Remark 5.2.

Then we choose Λ_v to be self-dual when v is split or inert in F, and almost π_v-modular when v is ramified in F. Furthermore, we choose $\Lambda_v^\flat$ to be self-dual when v is inert in F and π_v-modular when v is ramified in F. Such lattices exist, even when we impose that $\Lambda = \Lambda^\flat \oplus O_F u$ with $(u, u) \in O_{F_0}^\times$. With these definitions, $M_K(\widetilde{HG})$ has everywhere good reduction.

When $n = 2m$ is even, we impose that

- W is split at all finite places of F_0, which again forces m and $[F_0 : \mathbb{Q}]$ to be odd, cf. Remark 5.2; and
- $W^\flat$ is split at all finite places of F_0 which are inert in F.

Now we choose Λ_v to be self-dual when v is split or inert in F, and π_v-modular when v is ramified in F. Furthermore, we choose $\Lambda_v^\flat$ to be self-dual when v is inert in F and almost π_v-modular when v is ramified in F. Such lattices exist, even when we impose that there exists $u \in W$ with $(u, u) \in O_{F_0}^\times$ such that, for all inert finite places $\Lambda_v = \Lambda_v^\flat \oplus O_{F,v}u$, and such that, for all ramified places, $\Lambda_v^\flat$ and Λ_v are related by a chain of inclusions

$$\pi_v(\Lambda_v^\flat \oplus O_{F,v}u)^\vee \subset^1 \Lambda_v \subset^1 \Lambda_v^\flat \oplus O_{F,v}u,$$

cf. (4.29). With these definitions, $M_K(\widetilde{HG})$ has everywhere good reduction.

One may ask in this connection whether Conjecture 6.12 is non-empty for $M_K(\widetilde{HG})$ (with everywhere good reduction). By our expectations in Remark 6.11(i), this comes down to asking whether there are representations $\pi \in \Pi_{\mathrm{disc}}(\widetilde{HG})$ with $\pi_f^K \neq 0$ which contribute to the cohomology $H^{2n-3}(\mathrm{Sh}_K(\widetilde{HG}), \mathbb{C})$. Chenevier [8] has indicated to us a method of producing such π for low values of n, when F/F_0 is everywhere unramified. The method should also apply when F/F_0 is ramified once one has a better understanding of the local Langlands correspondence for unitary groups in ramified cases.

Remark 6.20. Even though we can define the height pairing unconditionally in the everywhere good reduction case, we are only able to calculate the height pairing in terms of the arithmetic intersection pairing. This leads to Conjecture 8.2 below, where we can only allow a certain smaller Hecke algebra $\mathscr{H}_K^{\mathrm{spl},\Phi}$ which, however, is not too small, in the sense that it contains $\mathscr{H}_K^{S,\deg=1}$. This is the reason we insist on having a Hecke–Kunneth projector in $\mathscr{H}_K^{S,\deg=1}$, and the stronger Chebotarev density result of Ramakrishnan, cf. Remark 6.8.

Remark 6.21. Assume $n = 2$. In this case, the Beilinson–Bloch pairing is defined unconditionally and coincides with the Néron–Tate height. Conjecture 6.14 is closely related to the Gross–Zagier formula in [51]. It would be interesting to clarify this relation.

7. L-functions and the relative trace formula

In this section, we recall certain distributions on G' that appear in the context of the relative trace formula. For test functions with some local hypotheses, we follow [55, §3.1] and [57, §2.1–§2.4]. In general, our definition relies on the truncation of relative trace formulas of Zydor [59].

On one hand, these distributions are related to L-functions via the Rankin–Selberg theory (for $\mathrm{GL}_{n-1} \times \mathrm{GL}_n$) of Jacquet, Piatetski-Shapiro, and Shalika [20]. On the other hand, they will serve as the analytic side in our conjectures on arithmetic intersection numbers formulated in the next section.

7.1. **The L-function.** Let $\Pi = \Pi_1 \boxtimes \Pi_2$ be a cuspidal automorphic representation of $G'(\mathbb{A}_{F_0})$, where Π_1, Π_2 are automorphic representations of $\mathrm{GL}_{n-1}(\mathbb{A}_F)$ and $\mathrm{GL}_n(\mathbb{A}_F)$ respectively. Let $L(s, \Pi_1 \boxtimes \Pi_2)$ be the Rankin–Selberg convolution L-function. This is an entire function in $s \in \mathbb{C}$ and it satisfies a functional equation of the form

$$L(s, \Pi_1 \boxtimes \Pi_2) = \epsilon(s, \Pi_1 \boxtimes \Pi_2) L(1 - s, \Pi_1^\vee \boxtimes \Pi_2^\vee),$$

cf. [20]. Here $\Pi_i^\vee$ denotes the contragredient representation of Π_i.

The L-function $L(s, \Pi_1 \boxtimes \Pi_2)$ is represented by an integral. Let $\varphi = \otimes_v \varphi_v \in \Pi = \otimes_v \Pi_v$ be a decomposable vector. Consider the integral

$$\lambda(\varphi, s) := \int_{H_1'(F_0)\backslash H_1'(\mathbb{A}_{F_0})} \varphi(h) |\det(h)|_F^s \, dh, \quad s \in \mathbb{C}. \tag{7.1}$$

Then by [20] we have a decomposition

$$\lambda(\varphi, s) = C \cdot L\Big(s + \frac{1}{2}, \Pi_1 \boxtimes \Pi_2\Big) \prod_v \lambda_v(\varphi_v, s).$$

Here the left-hand side is an entire function in $s \in \mathbb{C}$, and the local factors $\lambda(\varphi_v, s)$ have the following properties.

(1) For every $\varphi_v \in \Pi_v$, the function $s \mapsto \lambda_v(\varphi_v, s)$ is entire, and there exists φ_v° such that $\lambda_v(\varphi_v^\circ, s) \equiv 1$.

(2) For any decomposable $\varphi = \otimes_v \varphi_v \in \Pi = \otimes_v \Pi_v$, we have $\lambda_v(\varphi_v, s) \equiv 1$ for all but finitely many v.

(3) C is a non-zero constant depending only on the choices of Haar measures.

It follows that if $L\left(\frac{1}{2}, \Pi_1 \boxtimes \Pi_2\right) = 0$ (e.g., Π_1 and Π_2 are self-dual, and $\epsilon(1/2, \Pi_1 \boxtimes \Pi_2) = -1$), then we have

$$\frac{d}{ds}\Big|_{s=0} \int_{H_1'(F_0)\backslash H_1'(\mathbb{A}_{F_0})} \varphi(h)|\det(h)|_F^s\, dh = C \cdot L'\left(\frac{1}{2}, \Pi_1 \boxtimes \Pi_2\right) \prod_v \lambda(\varphi_v, 0).$$

We note that, if Π_1 and Π_2 are the substitute Arthur parameters in (6.20), then we may write the L-function in (6.21) as

$$L(s, \pi, R) = L\left(s, \Pi_1 \boxtimes \Pi_2\right).$$

7.2. **The global distribution on G'.** We briefly recall the global distribution on G' from [55, §3.1], [57, §2] (the notation in loc. cit. is slightly different). We denote by $A_{G'}$ and $A_{H_2'}$ the maximal F_0-split tori in the centers of G' and H_2', respectively. We have (via the embedding $H_2' \hookrightarrow G'$) an equality $A_{G'} = A_{H_2'}$ and both are isomorphic to $\mathbb{G}_{m,F_0} \times \mathbb{G}_{m,F_0}$.

Let $f' = \otimes_v f_v' \in \mathscr{H}(G'(\mathbb{A}_{F_0})) = C_c^\infty(G'(\mathbb{A}_{F_0}))$ be a pure tensor. We consider the associated automorphic kernel function

$$K_{f'}(x,y) := \int_{A_{G'}(F_0)\backslash A_{G'}(\mathbb{A}_{F_0})} \sum_{\gamma \in G'(F_0)} f'(x^{-1} z \gamma y)\, dz, \quad x, y \in G'(\mathbb{A}_{F_0}). \tag{7.2}$$

We define

$$J(f', s) := \int_{H_1'(F_0)\backslash H_1'(\mathbb{A}_{F_0})} \int_{A_{H_2'}(\mathbb{A}_{F_0}) H_2'(F_0)\backslash H_2'(\mathbb{A}_{F_0})} K_{f'}(h_1, h_2)\eta(h_2)|\det(h_1)|_F^s\, dh_1\, dh_2, \tag{7.3}$$

where the quadratic character $\eta\colon H_2'(\mathbb{A}_{F_0}) \to \{\pm 1\}$ is defined as follows: for $h_2 = (x_{n-1}, x_n) \in H_2'(\mathbb{A}_{F_0})$, with $x_i \in \mathrm{GL}_i(\mathbb{A}_{F_0})$,

$$\eta(h_2) = \eta_{F/F_0}\big(\det(x_{n-1})^n \det(x_n)^{n-1}\big).$$

Here $\eta_{F/F_0}\colon \mathbb{A}_{F_0}^\times \to \{\pm 1\}$ is the idele class character associated to the extension of global fields F/F_0.

The kernel function (7.2) has a spectral decomposition. The contribution of a cuspidal automorphic representation Π to the kernel function is given by

$$K_{\Pi, f'}(x,y) = \sum_{\varphi \in \mathrm{OB}(\Pi)} \big(\Pi(f')\varphi\big)(x)\, \overline{\varphi(y)},$$

where the sum runs over an orthonormal basis $\mathrm{OB}(\Pi)$ of Π. Correspondingly, a cuspidal automorphic representation Π contributes to the global distribution $J(f', s)$

$$J_\Pi(f', s) := \sum_{\varphi \in \mathrm{OB}(\Pi)} \lambda\big(\Pi(f')\varphi, s\big)\, \beta(\overline{\varphi}),$$

where β is the Flicker–Rallis period integral with respect to the subgroup H_2', cf. [58, §3.2]. It is expected from the endoscopic classification for unitary groups (cf. [37, 23]) that the period β does not vanish identically if and only if Π is in the image of base change from unitary groups associated to the quadratic extension F/F_0.

Proposition 7.1. *Let $\Pi = \Pi_1 \boxtimes \Pi_2$ be cuspidal, and assume that it is the base change of an automorphic representation π on the unitary group $H \times G$. If $L(1/2, \Pi_1 \boxtimes \Pi_2) = 0$, then*

$$\frac{d}{ds}\Big|_{s=0} J_\Pi(f', s) = L(1, \eta)^2 \frac{L'(1/2, \Pi_1 \boxtimes \Pi_2)}{L(1, \pi, \mathrm{Ad})} \prod_v J_{\Pi_v}(f_v'),$$

where $J_{\Pi_v}(f_v')$ is the local distribution defined in [58, §3, (3.31)], *and $L(1, \pi, \mathrm{Ad})$ is the adjoint L-function (cf.* [58, §3.4], [37], [23]*).*

Proof. It follows from [58, Prop. 3.6] that

$$J_\Pi(f', s) = L(1,\eta)^2 \frac{L(s+1/2, \Pi_1 \boxtimes \Pi_2)}{L(1,\pi,\mathrm{Ad})} \prod_v J_{\Pi_v}(f'_v, s). \tag{7.4}$$

Here the local distribution $J_{\Pi_v}(f'_v, s)$ is defined in an analogous way to that of its value at $s = 0$ given by [58, §3, (3.31)],

$$J_{\Pi_v}(f'_v, s) = \sum_{\{\varphi_v\}} \frac{\lambda_v^\natural(\Pi_v(f'_v)\varphi_v, s)\overline{\beta^\natural}(\varphi_v)}{\theta^\natural(\varphi_v, \varphi_v)}, \tag{7.5}$$

where the linear functional $\varphi_v \mapsto \lambda_v^\natural(\varphi_v, s)$ on Π_v is defined by [58, §3, (3.24)], using the Whittaker model. We refer to [58, §3] for the precise normalization of measures and the linear functionals $\beta^\natural$ and $\theta^\natural$. In particular, if $L(1/2, \Pi_1 \boxtimes \Pi_2) = 0$, then we have

$$\frac{d}{ds}\Big|_{s=0} J_\Pi(f', s) = L(1,\eta)^2 \frac{L'(1/2, \Pi_1 \boxtimes \Pi_2)}{L(1,\pi,\mathrm{Ad})} \prod_v J_{\Pi_v}(f'_v, 0).$$

Since $J_{\Pi_v}(f'_v, 0) = J_{\Pi_v}(f'_v)$ by definition, the proof is complete. □

Proposition 7.2. *Let $f' = \otimes_v f'_v \in \mathscr{H}(G'(\mathbb{A}_{F_0})) = C_c^\infty(G'(\mathbb{A}_{F_0}))$ be a pure tensor. Suppose that for a split place v the function f'_v is essentially the matrix coefficient of a supercuspidal representation (i.e., the function $g \mapsto \int_{F_{0,v}^\times \times F_{0,v}^\times} f'_v(zg)\, dz$ is a matrix coefficient of a supercuspidal representation). Then the integral* (7.3) *converges absolutely and it decomposes as*

$$J(f', s) = \sum_\Pi J_\Pi(f', s) = \sum_\Pi L(1,\eta)^2 \frac{L(s+1/2, \Pi_1 \boxtimes \Pi_2)}{L(1,\pi,\mathrm{Ad})} \prod_v J_{\Pi_v}(f'_v, s),$$

where the sum runs through the set of cuspidal automorphic representations $\Pi = \Pi_1 \boxtimes \Pi_2$ of $G'(\mathbb{A}_{F_0})$ coming by base change from automorphic representations π of unitary groups. Here the distribution $J_{\Pi_v}(f'_v, s)$ is defined by (7.5).

Proof. The spectral decomposition, i.e., the first equality, follows from the simple version of the relative trace formula in [57, Th. 2.3]. Note that in loc. cit., the test function f' is required to be "nice." However, the spectral decomposition only requires the existence of a place v where the function f'_v is essentially the matrix coefficient of a supercuspidal representation. Moreover, though only the case $s = 0$ is stated in loc. cit., the same proof works for all $s \in \mathbb{C}$. The second equality follows from Proposition 7.1. □

It follows that for f' as in Proposition 7.2, we have an expansion for the first derivative

$$\begin{aligned} \frac{d}{ds}\Big|_{s=0} J(f', s) = &\sum_{\substack{\Pi \\ \epsilon(\Pi)=-1}} L(1,\eta)^2 \frac{L'(1/2, \Pi_1 \boxtimes \Pi_2)}{L(1,\pi,\mathrm{Ad})} \prod_v J_{\Pi_v}(f'_v, 0) \\ &+ \sum_{\substack{\Pi \\ \epsilon(\Pi)=1}} L(1,\eta)^2 \frac{L(1/2, \Pi_1 \boxtimes \Pi_2)}{L(1,\pi,\mathrm{Ad})} \cdot \frac{d}{ds}\Big(\prod_v J_{\Pi_v}(f'_v, s)\Big)\Big|_{s=0}. \end{aligned} \tag{7.6}$$

Here $\epsilon(\Pi) = \epsilon(1/2, \Pi_1 \boxtimes \Pi_2)$ is the global root number for the Rankin–Selberg convolution.

Remark 7.3. We note that the contribution to the spectral decomposition from non-cuspidal automorphic representations is more complicated and we will not touch on this topic in this paper. A full spectral decomposition to remove the local restriction on f'_v in the last proposition is the work in progress by Zydor, and a coarse spectral decomposition has been obtained in [59].

Definition 7.4. Let λ be a place of F_0. A function $f'_\lambda \in C_c^\infty(G'(F_{0,\lambda}))$ *has regular support* if $\mathrm{supp}(f'_\lambda) \subset G'(F_{0,\lambda})_{\mathrm{rs}}$. A pure tensor $f' = \otimes_v f'_v \in \mathscr{H}(G'(\mathbb{A}_{F_0}))$ *has regular support at* λ if f'_λ has regular support.

Let us assume that f' has regular support at λ. Later we will assume that λ is non-archimedean. Then, by [55, Lem. 3.2] the integral (7.3) is absolutely convergent for all $s \in \mathbb{C}$, and admits a decomposition

$$J(f', s) = \sum_{\gamma \in G'(F_0)_{\mathrm{rs}}/H'_{1,2}(F_0)} \mathrm{Orb}(\gamma, f', s), \tag{7.7}$$

where each term is a product of local orbital integrals,

$$\mathrm{Orb}(\gamma, f', s) = \prod_v \mathrm{Orb}(\gamma, f'_v, s), \tag{7.8}$$

where in turn

$$\mathrm{Orb}(\gamma, f'_v, s) := \int_{H'_{1,2}(F_{0,v})} f'_v(h_1^{-1}\gamma h_2) |\det h_1|^s \eta(h_2)\, dh_1\, dh_2.$$

We set

$$J(f') := J(f', 0). \tag{7.9}$$

Then the decomposition (7.7) specializes to

$$J(f') = \sum_{\gamma \in G'(F_0)_{\mathrm{rs}}/H'_{1,2}(F_0)} \mathrm{Orb}(\gamma, f'),$$

where

$$\mathrm{Orb}(\gamma, f') := \mathrm{Orb}(\gamma, f', 0).$$

We introduce

$$J_v(f', s) := \sum_{\gamma \in G'(F_0)_{\mathrm{rs}}/H'_{1,2}(F_0)} \mathrm{Orb}(\gamma, f'_v, s) \cdot \prod_{u \neq v} \mathrm{Orb}(\gamma, f'_u),$$

so that, with $J_v(f') := J_v(f', 0)$, we have

$$J(f') = \sum_v J_v(f'). \tag{7.10}$$

We set

$$\begin{aligned} \partial J(f') &:= \frac{d}{ds}\Big|_{s=0} J(f', s), \\ \partial J_v(f') &:= \frac{d}{ds}\Big|_{s=0} J_v(f', s), \\ \partial \mathrm{Orb}(\gamma, f'_v) &:= \frac{d}{ds}\Big|_{s=0} \mathrm{Orb}(\gamma, f'_v, s). \end{aligned} \tag{7.11}$$

Note that

$$\partial J_v(f') = \sum_{\gamma \in G'(F_0)_{\mathrm{rs}}/H'_{1,2}(F_0)} \partial \mathrm{Orb}(\gamma, f'_v) \cdot \prod_{u \neq v} \mathrm{Orb}(\gamma, f'_u). \tag{7.12}$$

Then we may decompose

$$\partial J(f') = \sum_v \partial J_v(f'). \tag{7.13}$$

Without the regularity assumption on f', the integral (7.3) may diverge in general. For all $f' \in \mathscr{H}(G'(\mathbb{A}_{F_0}))$, the truncation process of Zydor [59] allows us to define a meromorphic distribution $J(\cdot, s)$ ([59, Th. 0.1]) which is holomorphic away from $s = \pm 1$. This allows us to define (7.9) and (7.11). We will use these distributions to formulate our conjectures in the next section. Zydor also obtains a coarse decomposition of $J(\cdot, s)$ into a sum of global orbital integrals, although for a non-regular-semisimple orbit there is no natural decomposition into a product of local orbital integrals.

7.3. **Smooth transfer.** The notion of smooth transfer between functions on unitary groups and on linear groups or their symmetric spaces is based on the concept of matching, cf. [44, §2] and [55, 57]. Using the results of Section 2.2, we can transpose this concept to our situation at hand.

Our definitions below depend on the choice of the transfer factor ω and on the choices of Haar measures. For definiteness, we will always take the transfer factor from [44, §2.4] (this is a slight variant of [57, §2.4]), which works for all places v,

$$\omega(\gamma) = \prod\nolimits_v \omega_v(\gamma_v), \quad \gamma = (\gamma_v) \in G'(\mathbb{A}_{F_0}). \tag{7.14}$$

The transfer factor has the properties

(1) (η-invariance) For $h_1 \in H_1'(\mathbb{A}_{F_0})$ and $h_2 \in H_2'(\mathbb{A}_{F_0})$, we have $\omega(h_1^{-1}\gamma h_2) = \eta(h_2)\omega(\gamma)$.

(2) (product formula) For $\gamma \in G'(F_0)$ we have

$$\prod\nolimits_v \omega_v(\gamma) = 1. \tag{7.15}$$

Now let p be a rational prime. We fix a Haar measure on $Z^{\mathbb{Q}}(\mathbb{Q}_p)$, and we define the orbital integral for $f_p \in C_c^\infty(\widetilde{HG}(\mathbb{Q}_p))$ and $g \in \widetilde{H}(\mathbb{Q}_p)\backslash\widetilde{HG}(\mathbb{Q}_p)_{\mathrm{rs}}/\widetilde{H}(\mathbb{Q}_p)$,

$$\mathrm{Orb}(g, f_p) := \int_{(\widetilde{H}(\mathbb{Q}_p)\times\widetilde{H}(\mathbb{Q}_p))/\Delta(Z^{\mathbb{Q}}(\mathbb{Q}_p))} f_p(h_1^{-1} g h_2)\, dh_1\, dh_2. \tag{7.16}$$

Definition 7.5. A function $f_p \in C_c^\infty(\widetilde{HG}(\mathbb{Q}_p))$ and a collection $(f_v') \in \prod_{v|p} C_c^\infty(G'(F_{0,v}))$ of functions are *transfers* of each other if for any element $\gamma = (\gamma_v) \in \prod_{v|p} G'(F_{0,v})_{\mathrm{rs}}$, the following identity holds:

$$\omega(\gamma)\prod_{v|p}\mathrm{Orb}(\gamma_v, f_v') = \begin{cases}\mathrm{Orb}(g, f_p), & \text{whenever } g \text{ matches } \gamma;\\ 0, & \text{no } g \in \widetilde{HG}(\mathbb{Q}_p) \text{ matches } \gamma.\end{cases}$$

We make the same definition for a function $f_\infty \in C_c^\infty(\widetilde{HG}(\mathbb{R}))$ and a collection of functions $(f_v') \in \prod_{v|\infty} C_c^\infty(G'(F_{0,v}))$.

Recall from (6.6) the product decomposition $\widetilde{HG}(\mathbb{Q}_p) = Z^{\mathbb{Q}}(\mathbb{Q}_p) \times \prod_{v|p} G_W(F_{0,v})$. We will always choose the Haar measures compatible with this product decomposition.

Let us explain the relation to smooth transfer between functions in $C_c^\infty(G'(F_{0,v}))$ and functions in $C_c^\infty(G_W(F_{0,v}))$, as W varies through the isomorphism classes of hermitian spaces of dimension n over F_v. This definition is based on the concept of matching between elements of $G'(F_{0,v})_{\mathrm{rs}}$ and elements of $G_W(F_{0,v})_{\mathrm{rs}}$ (see [43, §2] for non-archimedean places v of F_0 that are non-split in F; the definition extends in an obvious way to the archimedean places and to the split non-archimedean places).

Definition 7.6. A function $f_p \in C_c^\infty(\widetilde{HG}(\mathbb{Q}_p))$ is *completely decomposed* if it is of the form

$$f_p = \phi_p \otimes \bigotimes_{v|p} f_v, \tag{7.17}$$

where $\phi_p \in C_c^\infty(Z^{\mathbb{Q}}(\mathbb{Q}_p))$ and $f_v \in C_c^\infty(G_W(F_{0,v}))$. A pure tensor $f = \bigotimes_p f_p \in \mathscr{H}(\widetilde{HG}(\mathbb{A}_f))$ is *completely decomposed* if f_p is completely decomposed for all p.

Note that an arbitrary element in $\mathscr{H}(\widetilde{HG}(\mathbb{A}_f))$ is a linear combination of completely decomposed pure tensors.

Remark 7.7. Let $f_p = \phi_p \otimes \bigotimes_{v|p} f_v$ be completely decomposed. Set

$$c(\phi_p) := \int_{Z^{\mathbb{Q}}(\mathbb{Q}_p)} \phi_p(z)\, dz.$$

By Lemma 2.1, we have, for $g \in \widetilde{HG}(\mathbb{Q}_p)$ corresponding to the collection $g_v \in (H \times G)(F_{0,v})$,

$$\mathrm{Orb}(g, f_p) = c(\phi_p)\prod_{v|p}\mathrm{Orb}(g_v, f_v),$$

where

$$\mathrm{Orb}(g_v, f_v) = \int_{H(F_{0,v})\times H(F_{0,v})} f_v(h_1^{-1} g_v h_2)\, dh_1\, dh_2$$

is the orbital integral in [44, §2]. If the orbital integrals of f_p do not vanish identically, then f_p and $(f'_v)_{v|p}$ are transfers of each other in the sense of Definition 7.5 if and only if for some non-zero constants c_v such that $c(\phi_p) = \prod_{v|p} c_v$, the functions f_v and $c_v f'_v$ are transfers of each other for each v in the sense of [44, §2].

Definition 7.8. Let v be an archimedean place of F_0. A function $f'_v \in C_c^\infty(G'(F_{0,v}))$ is a *Gaussian test function* if it transfers to the constant function $\mathbf{1}$ on $G_{W_0}(F_{0,v})$, where W_0 denotes the negative-definite hermitian space, and transfers to the zero function on $G_W(F_{0,v})$ for any other hermitian space W (i.e., in the terminology[11] of [55, Def. 3.5], f'_v is pure of type W_0 and a transfer of $\mathbf{1}_{G_{W_0}(F_{0,v})}$).

A pure tensor $f' = \otimes_v f'_v \in \mathscr{H}(G'(\mathbb{A}_{F_0}))$ is a *Gaussian test function* if the archimedean components f'_v for $v \mid \infty$ are all (up to scalar factor) Gaussian test functions.

Implicitly we have fixed a Haar measure on each (compact) $H_{W_0}(F_{0,v})$. We will assume that the volume of $H_{W_0}(F_{0,v})$ is one. Then f'_v is a Gaussian test function if and only if for all $\gamma \in G'(F_{0,v})_{\mathrm{rs}}$,

$$\omega_v(\gamma)\,\mathrm{Orb}(\gamma, f'_v) = \begin{cases} 1, & \text{there exists } g \in G_{W_0}(F_{0,v}) \text{ matching } \gamma; \\ 0, & \text{no } g \in G_{W_0}(F_{0,v}) \text{ matches } \gamma. \end{cases} \tag{7.18}$$

The existence of Gaussian test functions is still conjectural. A Gaussian test function does not have regular support, in the sense of Definition 7.4.

Definition 7.9. A pure tensor $f = \otimes_p f_p \in \mathscr{H}(\widetilde{HG}(\mathbb{A}_f))$ and a pure tensor $f' = \otimes_v f'_v \in \mathscr{H}(G'(\mathbb{A}_{F_0,f}))$ are *smooth transfers* of each other if they are expressible in a way that f_p and $(f'_v)_{v|p}$ are transfers of each other for each prime p. Here we choose a product measure on $\widetilde{HG}(\mathbb{A}_f)$, resp. $G'(\mathbb{A}_{F_0,f})$; also, the adelic transfer factor (7.14) is simply the product of the local transfer factors.

Remark 7.10. The existence of local smooth transfer is known for non-archimedean places [57]; hence for any $f \in \mathscr{H}(\widetilde{HG}(\mathbb{A}_f))$ as above, there exists a smooth transfer $f' \in \mathscr{H}(G'(\mathbb{A}_{F_0,f}))$ as above, and conversely.

Lemma 7.11. *Let $f' = \otimes_v f'_v \in \mathscr{H}(G'(\mathbb{A}_{F_0}))$ be a Gaussian test function. Assume that f' has regular support at some place λ of F_0. Then for any place v_0 of F_0 split in F,*

$$\partial J_{v_0}(f') = 0.$$

Proof. This is [55, Prop. 3.6(ii)]. Note that implicitly our test function f' is pure of an incoherent type [55, §3.1], so that for every regular semisimple γ, the local orbital integral $\mathrm{Orb}(\gamma, f'_v, s)$ vanishes at $s = 0$ for at least one non-split place v. □

8. The conjectures for the arithmetic intersection pairing

In this section we formulate a conjectural formula for the Gillet–Soulé arithmetic intersection pairing for cycles on the integral models of $M_K(\widetilde{HG})$ we introduced earlier. This formula uses the distributions introduced in Section 7.

8.1. **The global conjecture, trivial level structure.** Let $\Lambda^\flat \subset W^\flat$ and $\Lambda \subset W$ be a pair of O_F-lattices related as in Section 5.1, and recall the models $\mathcal{M}_{K^\circ_{\widetilde{H}}}(\widetilde{H})$, $\mathcal{M}_{K^\circ_{\widetilde{G}}}(\widetilde{G})$, and $\mathcal{M}_{K^\circ_{\widetilde{HG}}}(\widetilde{HG})$ over $\operatorname{Spec} O_E$ defined in loc. cit. for the Shimura varieties of Section 3. In the case $F_0 = \mathbb{Q}$ and $\mathcal{M}_{K^\circ_{\widetilde{HG}}}(\widetilde{HG})$ is non-compact, we are implicitly replacing $\mathcal{M}_{K^\circ_{\widetilde{HG}}}(\widetilde{HG})$ by its toroidal compactification. Then the model $\mathcal{M}_{K^\circ_{\widetilde{HG}}}(\widetilde{HG})$ is proper and flat over $\operatorname{Spec} O_E$. Furthermore, it is regular provided that there are no places ν of E for which E_ν is ramified over $\mathbb{Q}_p$ and (v_0, Λ_{v_0}) is of AT type (1), where v_0 denotes the place of F_0 induced by ν via φ_0. Throughout this section, we

[11]But note that here G_W is the product of two unitary groups.

assume that there are no places v_0 for which (v_0, Λ_{v_0}) is of AT type (4) (the justification for this assumption is given by Remark 4.12).

The compact open subgroup $K^\circ_{\widetilde H} \subset \widetilde H(\mathbb{A}_f)$ contains $K^\circ_{\widetilde{HG}} \cap \widetilde H(\mathbb{A}_f)$, with equality when there are no places v of F_0 for which (v, Λ_v) is of AT type (2). In this case, there is a closed embedding

$$\mathcal{M}_{K^\circ_{\widetilde H}}(\widetilde H) \hookrightarrow \mathcal{M}_{K^\circ_{\widetilde{HG}}}(\widetilde{HG}).$$

Like in (6.15), we obtain a cycle (with $\mathbb{Q}$-coefficients) $z_{K^\circ_{\widetilde{HG}}} = \mathrm{vol}(K^\circ_{\widetilde H})[\mathcal{M}_{K^\circ_{\widetilde H}}(\widetilde H)]$ on $\mathcal{M}_{K^\circ_{\widetilde{HG}}}(\widetilde{HG})$. We denote by the same symbol its class in the rational Chow group,

$$z_{K^\circ_{\widetilde{HG}}} \in \mathrm{Ch}^{n-1}\big(\mathcal{M}_{K^\circ_{\widetilde{HG}}}(\widetilde{HG})\big)_{\mathbb{Q}}. \tag{8.1}$$

In general, let $v_1, \ldots, v_m \in S^W_{\mathrm{AT}}$ be the places for which (v_i, Λ_{v_i}) is of AT type (2). We use the closed embedding (5.5) to define the cycle $z_{K^\circ_{\widetilde{HG}}} = \mathrm{vol}(K^{\circ\prime}_{\widetilde H})[\mathcal{M}_{K^{\circ\prime}_{\widetilde H}}(\widetilde H)]$ and its class in the Chow group.

Remark 8.1. When defining the cycle class (8.1), we use a Haar measure on $\widetilde H(\mathbb{A}_f)$. We will always choose the product of the measures used to define the local orbital integrals (7.16).

We denote by $\widehat{\mathrm{Ch}}^{n-1}(\mathcal{M}_{K^\circ_{\widetilde{HG}}}(\widetilde{HG}))$ the arithmetic Chow group. Elements are represented by pairs (Z, g_Z), where Z is a cycle and g_Z is a Green's current (cf. [14, §3.3]). We are going to use the Gillet–Soulé arithmetic intersection pairing, cf. [14],

$$(\ ,\ )_{\mathrm{GS}} \colon \widehat{\mathrm{Ch}}^{n-1}\big(\mathcal{M}_{K^\circ_{\widetilde{HG}}}(\widetilde{HG})\big) \times \widehat{\mathrm{Ch}}^{n-1}\big(\mathcal{M}_{K^\circ_{\widetilde{HG}}}(\widetilde{HG})\big) \longrightarrow \mathbb{R}.$$

We extend this from a symmetric pairing to a hermitian pairing on the corresponding $\mathbb{C}$-vector space ($\mathbb{C}$-linear combinations of (Z, g_Z)),

$$(\ ,\ )_{\mathrm{GS}} \colon \widehat{\mathrm{Ch}}^{n-1}\big(\mathcal{M}_{K^\circ_{\widetilde{HG}}}(\widetilde{HG})\big)_{\mathbb{C}} \times \widehat{\mathrm{Ch}}^{n-1}\big(\mathcal{M}_{K^\circ_{\widetilde{HG}}}(\widetilde{HG})\big)_{\mathbb{C}} \longrightarrow \mathbb{C}. \tag{8.2}$$

We choose a Green's current $g_{z_{K^\circ_{\widetilde{HG}}}}$ of the cycle (with $\mathbb{Q}$-coefficients) $z_{K^\circ_{\widetilde{HG}}}$ to get an element in the rational arithmetic Chow group,

$$\widehat z_{K^\circ_{\widetilde{HG}}} = (z_{K^\circ_{\widetilde{HG}}}, g_{z_{K^\circ_{\widetilde{HG}}}}) \in \widehat{\mathrm{Ch}}^{n-1}\big(\mathcal{M}_{K^\circ_{\widetilde{HG}}}(\widetilde{HG})\big)_{\mathbb{Q}}. \tag{8.3}$$

The Green's current is not unique. We shall work in the following with an arbitrary but fixed choice.

Let

$$\mathscr{H}^{\mathrm{spl},\Phi}_{K^\circ_{\widetilde{HG}}} \subset \mathscr{H}_{K^\circ_{\widetilde{HG}}} = \mathscr{H}\big(\widetilde{HG}(\mathbb{A}_f), K^\circ_{\widetilde{HG}}\big) \tag{8.4}$$

be the partial Hecke algebra spanned by completely decomposed pure tensors of the form $f = \otimes_p f_p \in \mathscr{H}_{K^\circ_{\widetilde{HG}}}$, where $f_p = \phi_p \otimes \bigotimes_{v|p} f_v$, as in Definition 7.6, with $\phi_p = \mathbf{1}_{K_{Z^{\mathbb{Q}},p}}$ for all p and where $f_v = \mathbf{1}_{K^\circ_{H\times G,v}}$ unless $v \in \Sigma^{\mathrm{spl},\Phi}$. Here $\Sigma^{\mathrm{spl},\Phi}$ is as in (5.6). We have

$$\mathscr{H}^{\mathrm{spl},\Phi}_{K^\circ_{\widetilde{HG}}} \simeq \bigotimes_{v\in\Sigma^{\mathrm{spl},\Phi}} \mathscr{H}_{HG,K^\circ_{HG,v}}.$$

By Lemma 4.5(ii), we have $\Sigma^{\mathrm{spl},\Phi} \supset \Sigma^{S,\deg=1}$ and

$$\mathscr{H}^{\mathrm{spl},\Phi}_{K^\circ_{\widetilde{HG}}} \supset \mathscr{H}^{S,\deg=1}_{K^\circ_{\widetilde{HG}}},$$

for any finite set S of places. Here $\mathscr{H}^{S,\deg=1}_{K^\circ_{\widetilde{HG}}}$ is as in (6.8).

For $f \in \mathscr{H}^{\mathrm{spl},\Phi}_{K^\circ_{\widetilde{HG}}}$, we introduce via (5.10) a Hecke correspondence, hence an induced endomorphism $\widehat R(f)$ on the arithmetic Chow group $\widehat{\mathrm{Ch}}^{n-1}(\mathcal{M}_{K^\circ_{\widetilde{HG}}}(\widetilde{HG}))_{\mathbb{C}}$, cf. [14, 5.2.1]. Using the arithmetic intersection pairing (8.2), we define

$$\begin{aligned} \mathrm{Int}^\natural(f) &:= \big(\widehat R(f)\widehat z_{K^\circ_{\widetilde{HG}}}, \widehat z_{K^\circ_{\widetilde{HG}}}\big)_{\mathrm{GS}}, \\ \mathrm{Int}(f) &:= \frac{1}{\tau(Z^{\mathbb{Q}})[E:F]} \cdot \mathrm{Int}^\natural(f). \end{aligned} \tag{8.5}$$

Here

$$\tau(Z^{\mathbb{Q}}) := \mathrm{vol}\big(Z^{\mathbb{Q}}(\mathbb{A}_f)/Z^{\mathbb{Q}}(\mathbb{Q})\big).$$

Conjecture 8.2 (Global conjecture, trivial level structure)**.** *Let* $f = \otimes_p f_p \in \mathscr{H}^{\mathrm{spl},\Phi}_{K^\circ_{\widetilde{HG}}}$*, and let* $f' = \otimes_v f'_v \in \mathscr{H}(G'(\mathbb{A}_{F_0}))$ *be a Gaussian test function such that* $\otimes_{v<\infty} f'_v$ *is a smooth transfer of* f*. Then*

$$\mathrm{Int}(f) = -\partial J(f') - J(f'_{\mathrm{corr}}),$$

where $f'_{\mathrm{corr}} \in C_c^\infty(G'(\mathbb{A}_{F_0}))$ *is a correction function. Furthermore, we may choose* f' *such that* $f'_{\mathrm{corr}} = 0$.

Remark 8.3. The notion of smooth transfer at each individual place v depends on the choice of transfer factors, and of Haar measures on various groups. However, the validity of the conjecture does not depend on these choices (use the product formula (7.15)).

This conjecture has the following drawback. Since we cannot impose any regular support assumptions on functions in $\mathscr{H}^{\mathrm{spl},\Phi}_{K^\circ_{\widetilde{HG}}}$, the left-hand side of the asserted equality may involve self-intersection numbers, and these are difficult to calculate explicitly. Analogously, on the right-hand side, the terms in Zydor's truncation that are not regular-semisimple orbital integrals are more delicate. Nevertheless, assuming a spectral decomposition of $J(f', s)$ that generalizes the case of special test functions in Proposition 7.2 and (7.6), Conjecture 8.2 relates the intersection number $\mathrm{Int}(f)$ to the first derivative of L-functions in the Arithmetic Gan–Gross–Prasad conjecture 6.12 and 6.14.

8.2. **The global conjecture, non-trivial level structure.** In this subsection, we use the integral models of the Shimura varieties with deeper level structures depending on the choice of a function $\mathbf{m}$ as in (5.7). Note that the models $\mathcal{M}_{K^{\mathbf{m}}_{\widetilde{HG}}}(\widetilde{HG})$ are not regular in the fibers over places lying above the support of $\mathbf{m}$. Therefore, the Gillet–Soulé pairing (8.2) is not defined for them. However, under certain hypotheses that assure that our physical cycle z_K and its physical transform under a Hecke correspondence do not intersect in the generic fiber, we can define a naive intersection number for them.

Similarly to the case with trivial level structure, we obtain a cycle (with $\mathbb{Q}$-coefficients) $z_{K^{\mathbf{m}}_{\widetilde{HG}}} = \mathrm{vol}(K^{\mathbf{m}'}_{\widetilde{H}})[\mathcal{M}_{K^{\mathbf{m}'}_{\widetilde{H}}}(\widetilde{H})]$ on $\mathcal{M}_{K^{\mathbf{m}}_{\widetilde{HG}}}(\widetilde{HG})$, cf. (8.1) and, again, we denote by the same symbol its class in the Chow group. We choose a Green's current $g_{z_{K^{\mathbf{m}}_{\widetilde{HG}}}}$ of the cycle (with $\mathbb{Q}$-coefficients) $z_{K^{\mathbf{m}}_{\widetilde{HG}}}$ to get an element in the arithmetic Chow group,

$$\widehat{z}_{K^{\mathbf{m}}_{\widetilde{HG}}} = (z_{K^{\mathbf{m}}_{\widetilde{HG}}}, g_{z_{K^{\mathbf{m}}_{\widetilde{HG}}}}) \in \widehat{\mathrm{Ch}}^{n-1}\big(\mathcal{M}_{K^{\mathbf{m}}_{\widetilde{HG}}}(\widetilde{HG})\big)_{\mathbb{Q}}. \tag{8.6}$$

Let

$$\mathscr{H}^{\mathrm{spl},\Phi}_{K^{\mathbf{m}}_{\widetilde{HG}}} := \mathscr{H}\big(\widetilde{HG}(\mathbb{A}_f), K^{\mathbf{m}}_{\widetilde{HG}}\big)^{\mathrm{spl},\Phi} \tag{8.7}$$

be the partial Hecke algebra spanned by completely decomposed pure tensors of the form $f = \otimes_p f_p \in \mathscr{H}_{K^{\mathbf{m}}_{\widetilde{HG}}}$, where $f_p = \phi_p \otimes \bigotimes_{v|p} f_v$, as in Definition 7.6, with $\phi_p = \mathbf{1}_{K_{Z^{\mathbb{Q}},p}}$ for all p, and where $f_v = \mathbf{1}_{K^\circ_{H\times G,v}}$ unless $v \in \Sigma^{\mathrm{spl},\Phi}$. We have

$$\mathscr{H}^{\mathrm{spl},\Phi}_{K^{\mathbf{m}}_{\widetilde{HG}}} \simeq \bigotimes_{v\in\Sigma^{\mathrm{spl},\Phi}} \mathscr{H}_{HG,K^{\mathbf{m}}_{HG,v}}.$$

By Lemma 4.5(ii), we have

$$\mathscr{H}^{\mathrm{spl},\Phi}_{K^{\mathbf{m}}_{\widetilde{HG}}} \supset \mathscr{H}^{S,\deg=1}_{K^{\mathbf{m}}_{\widetilde{HG}}}$$

for any finite set S of places of $\mathbb{Q}$ such that the places of F_0 above S contain the support of $\mathbf{m}$.

To any $f \in \mathscr{H}^{\mathrm{spl},\Phi}_{K^{\mathbf{m}}_{\widetilde{HG}}}$ we associate via (5.10) a Hecke correspondence on $\mathcal{M}_{K^{\mathbf{m}}_{\widetilde{HG}}}(\widetilde{HG})$.

Definition 8.4. Let λ be a non-archimedean place of F_0, of residue characteristic ℓ. Let $f_\ell \in C_c^\infty(\widetilde{HG}(\mathbb{Q}_\ell))$ be completely decomposed, i.e., $f_\ell = \phi_\ell \otimes \bigotimes_{v|\ell} f_v$, cf. Definition 7.6. Then f_ℓ *has regular support at* λ if $\operatorname{supp} f_\lambda \subset G_W(F_{0,\lambda})_{\mathrm{rs}}$. If $f = \otimes_p f_p \in \mathscr{H}^{\mathrm{spl},\Phi}_{K^{\mathbf{m}}_{\widetilde{HG}}}$ is a completely decomposed pure tensor, then f *has regular support at* λ if f_ℓ has regular support at λ.

Theorem 8.5. *Let* $f = \otimes_p f_p \in \mathscr{H}^{\mathrm{spl},\Phi}_{K^{\mathbf{m}}_{\widetilde{HG}}}$ *be a completely decomposed pure tensor. Assume that* f *has regular support at some place* λ *of* F_0*. Then the following statements on the support of the intersection of the cycles* $z_{K^{\mathbf{m}}_{\widetilde{HG}}}$ *and* $R(f)z_{K^{\mathbf{m}}_{\widetilde{HG}}}$ *of* $\mathcal{M}_{K^{\mathbf{m}}_{\widetilde{HG}}}(\widetilde{HG})$ *hold.*

(i) The support does not meet the generic fiber.

(ii) Let ν be a place of E lying over a place of F_0 which splits in F. Then the support does not meet the special fiber $\mathcal{M}_{K^{\mathbf{m}}_{\widetilde{HG}}}(\widetilde{HG}) \otimes_{O_E} \kappa_\nu$.

(iii) Let ν be a place of E lying over a place of F_0 which does not split in F. Then the support meets the special fiber $\mathcal{M}_{K^{\mathbf{m}}_{\widetilde{HG}}}(\widetilde{HG}) \otimes_{O_E} \kappa_\nu$ only in its basic locus.[12]

Remark 8.6. Let $F_0 = \mathbb{Q}$ and assume that $\mathcal{M}_{K^{\mathbf{m}}_{\widetilde{HG}}}(\widetilde{HG})$ is non-compact. Then the closure in the toroidal compactification [19, §2] of $\mathcal{M}_{K^{\mathbf{m}}_{\widetilde{HG}}}(\widetilde{HG})$ of the support of the intersection of the cycles $z_{K^{\mathbf{m}}_{\widetilde{HG}}}$ and $R(f)z_{K^{\mathbf{m}}_{\widetilde{HG}}}$ does not meet the boundary. Indeed, this follows from Theorem 8.5(iii) because the basic locus of $\mathcal{M}_{K^{\mathbf{m}}_{\widetilde{HG}}}(\widetilde{HG}) \otimes_{O_E} \kappa_\nu$ does not meet the boundary.

The proof of Theorem 8.5 will be based on the following lemma.

Lemma 8.7. *Let k be an algebraically closed field which is an O_E-algebra, with corresponding place ν of E. Let (A_0, ι_0) be an abelian variety with O_F-action with Kottwitz condition of signature $((0,1)_{\varphi\in\Phi})$, cf. (3.8), and let (A, ι) be an abelian variety with O_F-action with Kottwitz condition of type r as in Remark 3.4. Assume there exists an F-linear isogeny*

$$A_0^n \longrightarrow A.$$

Then k is of positive characteristic p, where p is the residue characteristic of ν. Furthermore, the place v_0 of F_0 induced by ν is non-split in F, and the isogeny classes of A_0 and A only depend on the CM type Φ. If v_0 is the only place of F_0 above p, then A_0 and A are supersingular.

Proof. We prove the first statement by contradiction. Assume that k is of characteristic zero. Then an isogeny as above induces an $F \otimes k$-linear isomorphism

$$\mathrm{Lie}(A_0)^n \xrightarrow{\sim} \mathrm{Lie}\, A.$$

Such an isomorphism cannot exist due to the different Kottwitz conditions on A_0^n and A.

Now let the place v_0 of F_0 induced by ν be split. Then there is a splitting of the p-divisible group of A_0^n, resp. A, according to the two places of F above F_0,

$$A_0^n[p^\infty] = X_0^{(1)} \times X_0^{(2)}, \quad A[p^\infty] = X^{(1)} \times X^{(2)},$$

An F-linear isogeny as above induces isogenies of p-divisible groups,

$$X_0^{(1)} \longrightarrow X^{(1)}, \quad X_0^{(2)} \longrightarrow X^{(2)}.$$

However, by the Kottwitz conditions, the dimension of $X_0^{(1)}$, resp. $X_0^{(2)}$, is divisible by n, whereas the dimension of $X^{(1)}$, resp. $X^{(2)}$, is $\equiv \pm 1 \bmod n$. Hence such an isogeny cannot exist.

Now assume that v_0 is non-split in F. The rational Dieudonné module M_0 of $A_0[p^\infty]$ is a free $F \otimes W(k)$-module of rank one. Consider its decomposition according to the places w of F above p,

$$M_0 = \bigoplus_w M_{0,w}.$$

Then each summand is isoclinic. More precisely, if w lies above a non-split place of F_0 (such as the place w_0 induced by ν), then $M_{0,w}$ is isoclinic of slope $1/2$; and if $\overline{w} \neq w$, then the slope of $M_{0,w}$ is equal to a_w/d_w, where

$$a_w := \#\{\, \varphi \in \Phi \mid w_\varphi = w \,\}, \quad d_w := [F_w : \mathbb{Q}_p].$$

Here we chose an embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_p$ so that any $\varphi \in \Phi$ induces a place w_φ of F. □

[12] In this special case, the basic locus is characterized as follows. Let $(A_0, \iota_0, \lambda_0, A^\flat, \iota^\flat, \lambda^\flat, \overline{\eta}^\flat, \varphi^\flat, A, \iota, \lambda, \overline{\eta}, \varphi)$ correspond to a point of $\mathcal{M}_{K^{\mathbf{m}}_{\widetilde{HG}}}(\widetilde{HG})$ with values in an algebraically closed extension of $\kappa(\nu)$. Consider the decomposition $A[p^\infty] = \prod_{w|p} A[w^\infty]$, resp. $A^\flat[p^\infty] = \prod_{w|p} A^\flat[w^\infty]$, of the p-divisible group of A, resp. $A^\flat$, under the action of $O_F \otimes \mathbb{Z}_p$; then all factors $A[w^\infty]$, resp. $A^\flat[w^\infty]$, are isoclinic.

Proof of Theorem 8.5. (i): Suppose that

$$(A_0, \iota_0, \lambda_0, A^\flat, \iota^\flat, \lambda^\flat, \overline{\eta}^\flat, \varphi^\flat, A, \iota, \lambda, \overline{\eta}, \varphi) \in \mathcal{M}_{K^{\mathbf{m}}_{\widetilde{HG}}}(\widetilde{HG})(k)$$

is a point in the support, where k is an algebraically closed field of characteristic zero. Then $(A, \iota, \lambda) = (A^\flat \times A_0, \iota^\flat \times \iota_0, \lambda^\flat \times \lambda_0(u))$, and there exists $g \in (H \times G)(\mathbb{A}_{F_0,f})_{\mathrm{rs}}$ such that there is an isogeny

$$\phi\colon A = A^\flat \times A_0 \longrightarrow A$$

which makes the diagram

$$\begin{CD} \widehat{V}(A_0, A) @>{\eta}>> -W \otimes_F \mathbb{A}_{F,f} \\ @V{\phi}VV @VV{g}V \\ \widehat{V}(A_0, A) @>{\eta}>> -W \otimes_F \mathbb{A}_{F,f} \end{CD} \tag{8.8}$$

commute. From the splitting $A = A^\flat \times A_0$, we also have

$$u\colon A_0 \longrightarrow A,$$

which makes the diagram

$$\begin{CD} \widehat{V}(A_0, A_0) @>{\eta_0}>> \mathbb{A}_{F,f} \\ @V{u}VV @VV{u}V \\ \widehat{V}(A_0, A) @>{\eta}>> -W \otimes_F \mathbb{A}_{F,f} \end{CD} \tag{8.9}$$

commute, where η_0 is defined in the the obvious way. We consider the homomorphism

$$(\phi^i u)_{0 \le i \le n-1} : A_0^n \longrightarrow A$$

whose ith component is $\phi^i u : A_0 \to A$. We claim that this is an isogeny. It suffices to show that its induced map on any rational Tate module is an isomorphism. This follows by the commutativity of the diagram (8.8) from the regular semi-simplicity of g. This conclusion contradicts Lemma 8.7.

(ii), (iii): Now let $(A_0, \iota_0, \lambda_0, A^\flat, \iota^\flat, \lambda^\flat, \overline{\eta}^\flat, \varphi^\flat, A, \iota, \lambda, \overline{\eta}, \varphi) \in \mathcal{M}_{K^{\mathbf{m}}_{\widetilde{HG}}}(\widetilde{HG})(k)$ be a point in the support, where k is an algebraically closed field of positive characteristic p. When there exists $g \in (H \times G)(\mathbb{A}^p_{F_0,f})_{\mathrm{rs}}$ such that there exists a commutative diagram (8.8) (with an upper index ^{p} added everywhere), the argument is as before, by reduction to Lemma 8.7.

Now assume that the place v_0 of F_0 induced by ν is split in F and that there exists $g \in G(F_{0,v_0})_{\mathrm{rs}}$ such that there is an isogeny

$$\phi\colon A = A^\flat \times A_0 \longrightarrow A$$

which makes a diagram analogous to (8.8) commute. To explain this diagram, we use a compatible system of Drinfeld level structures,

$$\widetilde{\varphi}\colon \Lambda_{w_0}[\pi_{w_0}^{-1}]/\Lambda_{w_0} \longrightarrow T_{w_0}(A_0, A),$$

which induces for every m a Drinfeld level w_0^m-structure on the w_0^m-torsion subgroup,

$$\varphi\colon \pi_{w_0}^{-m}\Lambda_{w_0}/\Lambda_{w_0} \longrightarrow T_{w_0}(A_0, A)[w_0^m].$$

Then the analog of (8.8) is the commutative diagram for sufficiently large m',

$$\begin{CD} \Lambda_{w_0}[\pi_{w_0}^{-1}]/\Lambda_{w_0} @>{\widetilde{\varphi}}>> T_{w_0}(A_0, A) \\ @V{\pi_{w_0}^{m'} g}VV @VV{\pi_{w_0}^{m'} \phi}V \\ \Lambda_{w_0}[\pi_{w_0}^{-1}]/\Lambda_{w_0} @>{\widetilde{\varphi}}>> T_{w_0}(A_0, A). \end{CD} \tag{8.10}$$

From the splitting $A = A^\flat \times A_0$, we also have

$$u\colon A_0 \longrightarrow A,$$

which makes the diagram

$$\begin{tikzcd} O_{F,w_0}[\pi_{w_0}^{-1}]/O_{F,w_0} \arrow[r, "\widetilde{\varphi}_0"] \arrow[d, "u"'] & T_{w_0}(A_0, A_0) \arrow[d, "u"] \\ \Lambda_{w_0}[\pi_{w_0}^{-1}]/\Lambda_{w_0} \arrow[r, "\widetilde{\varphi}"] & T_{w_0}(A_0, A) \end{tikzcd} \tag{8.11}$$

commute, where $\widetilde{\varphi}_0$ is the limit over m of the homomorphisms (4.23). We consider the homomorphism

$$(\phi^i u)_{0\leq i\leq n-1} : A_0^n \longrightarrow A$$

whose ith component is $\phi^i u : A_0 \to A$. We again claim that this is an isogeny, which would contradict Lemma 8.7. It suffices to show that its induced map on the p-divisible group is an isogeny. This follows by the commutativity of diagram (8.10) from the regular semi-simplicity of g. $\square$

Let us assume that $f \in \mathscr{H}^{\mathrm{spl},\Phi}_{K^{\mathbf{m}}_{\widetilde{HG}}}$ is a completely decomposed pure tensor which has regular support at some place λ of F_0. Then by Theorem 8.5(i), the generic fibers of the cycles $z_{K^{\mathbf{m}}_{\widetilde{HG}}}$ and $R(f)z_{K^{\mathbf{m}}_{\widetilde{HG}}}$ do not intersect, and we may define

$$\begin{aligned} \mathrm{Int}^{\natural}_{\nu}(f) &:= \big\langle \widehat{R}(f)\widehat{z}_{K^{\mathbf{m}}_{\widetilde{HG}}}, \widehat{z}_{K^{\mathbf{m}}_{\widetilde{HG}}} \big\rangle_{\nu} \log q_{\nu}, \\ \mathrm{Int}(f) &:= \frac{1}{\tau(Z^{\mathbb{Q}})[E:F]} \sum_{\nu} \mathrm{Int}^{\natural}_{\nu}(f). \end{aligned} \tag{8.12}$$

Here the first quantity is defined for a non-archimedean place ν through the Euler–Poincaré characteristic of a derived tensor product on $\mathcal{M}_{K^{\mathbf{m}}_{\widetilde{HG}}}(\widetilde{HG}) \otimes_{O_E} O_{E,(\nu)}$, comp. [14, 4.3.8(iv)]. Note that the intersection numbers are indeed defined for a non-archimedean place because the intersection of the cycles $z_{K^{\mathbf{m}}_{\widetilde{HG}}}$ and $R(f)z_{K^{\mathbf{m}}_{\widetilde{HG}}}$ avoids all fibers of $\mathcal{M}_{K^{\mathbf{m}}_{\widetilde{HG}}}(\widetilde{HG})$ over places ν lying over the support of $\mathbf{m}$, as follows from Theorem 8.5(ii). Therefore the intersection of these cycles takes place in the regular locus of $\mathcal{M}_{K^{\mathbf{m}}_{\widetilde{HG}}}(\widetilde{HG})$, cf. Theorem 8.5, and hence the Euler–Poincaré characteristic is finite. For an archimedean place ν, the last quantity is defined by the archimedean component of the arithmetic intersection theory and we have set $\log q_\nu := [E_\nu : \mathbb{R}] = 2$, cf. [14].

Conjecture 8.8 (Global conjecture, nontrivial level structure)**.** *Let $f = \otimes_p f_p \in \mathscr{H}^{\mathrm{spl},\Phi}_{K^{\mathbf{m}}_{\widetilde{HG}}}$ be a completely decomposed pure tensor and let $f' = \otimes_v f'_v \in \mathscr{H}(G'(\mathbb{A}))$ be a Gaussian test function such that $\otimes_{v<\infty} f'_v$ is a smooth transfer of f. Assume that f has regular support at some place λ of F_0. Then*

$$\mathrm{Int}(f) = -\partial J(f') - J(f'_{\mathrm{corr}}),$$

where $f'_{\mathrm{corr}} \in C_c^\infty(G'(\mathbb{A}))$ is a correction function. Furthermore, $f' = \otimes_v f'_v$ may be chosen such that f' has regular support at λ and that $f'_{\mathrm{corr}} = 0$.

Remark 8.9. Part of the conjecture asserts that a change of the Green's current is compensated by a change of the correction function f'_{corr}.

Remark 8.10. Similar to the case of trivial level structure, when there exists a split place v such that f'_v is essentially the matrix coefficient of a supercuspidal representation (cf. Proposition 7.2), by (7.6), Conjecture 8.8 relates the intersection number $\mathrm{Int}(f)$ to the first derivative of L-functions in the Arithmetic Gan–Gross–Prasad conjecture 6.12 and 6.14. The hypothesis on the existence of such a split place v could be dropped once a full spectral decomposition of $J(f', s)$ for all test functions is available.

Note that here the right-hand side is well-defined (cf. Section 7.1). We also point out that if $f \neq 0$ has regular support at some place λ of F_0, then λ must be in $\operatorname{supp} \mathbf{m}$ (in particular λ is split in F). Since we will not need this statement, we omit the proof.

Lemma 8.11. *(i) Let $u \in \operatorname{supp} \mathbf{m}$ be a place above p (in particular u is split in F). There exists a non-zero function $f_p \in \mathscr{H}_{K^{\mathbf{m}}_{\widetilde{HG}},p}$ that has regular support at u.*

(ii) For any $f_p \in \mathscr{H}_{K^{\mathbf{m}}_{\widetilde{HG}},p}$ with regular support at the place u above p, there exists a transfer $(f'_v)_{v|p}$ such that f'_u has regular support at u.

Proof. Let m be a positive integer. At a place u split in F, by choosing a basis of a self-dual lattice in $W^\flat \otimes F_{0,u}$, resp. $W \otimes F_{0,u}$, we may identify $H(F_{0,u}) = \mathrm{GL}_{n-1}(F_{0,u})$ and $G(F_{0,u}) = \mathrm{GL}_n(F_{0,u})$, and $K^m_{H,u}$, resp. $K^m_{G,u}$, with the principal congruence subgroups of level m. We may choose the basis of the lattice in $W \otimes F_{0,u}$ by adding the special vector u to the basis of the lattice in $W^\flat \otimes F_{0,u}$. Therefore we may further assume that the embedding $H(F_{0,u}) \hookrightarrow G(F_{0,u})$ has the property that $K^m_{H,u} = K^m_{G,u} \cap H(F_{0,u})$. Moreover, the stabilizers of the lattices are identified with $\mathrm{GL}_{n-1}(O_{F_0,u})$ and $\mathrm{GL}_n(O_{F_0,u})$ respectively, and $\mathrm{GL}_{n-1} \hookrightarrow \mathrm{GL}_n$ sends h to $\mathrm{diag}(h, 1)$.

Now let $f_p = \phi_p \otimes \bigotimes_{v|p} f_v$ be completely decomposed. It suffices to show that there exists a non-zero $f_u = f_{n-1,u} \otimes f_{n,u} \in \mathscr{H}((H \times G)(F_{0,u}), K^m_{H,u} \times K^m_{G,u})$ with regular support. We construct such a function by first setting $f_{n-1,u} = \mathbf{1}_{K^m_{H,u}}$. Note that the pair of functions $(f_{n-1,u}, f_{n,u})$ defines the function $\widetilde{f}_u \in \mathscr{H}(G(F_{0,u}), K^m_{G,u})$ by “contraction” under the map

$$\begin{aligned} H \times G &\longrightarrow G \\ (h, g) &\longmapsto h^{-1}g, \end{aligned} \tag{8.13}$$

namely,

$$\widetilde{f}_u(g) = \int_{H(F_{0,u})} f_{n-1,u}(h) f_{n,u}(hg)\, dh.$$

We then have $\widetilde{f}_u = \mathrm{vol}(K^m_{H,u}) f_{n,u}$. Then the function $f_u = f_{n-1,u} \otimes f_{n,u}$ has regular support (with respect to the $H \times H$-action) if (and only if) $\widetilde{f}_u$ or, equivalently, $f_{n,u}$ has regular support (with respect to the conjugation action of H). This holds because the inverse image of $G(F_{0,u})_{\mathrm{rs}}$ under the contraction map (8.13) is exactly $(H \times G)(F_{0,u})_{\mathrm{rs}}$.

We now choose $f_{n,u}$ supported in $\mathrm{GL}_n(O_{F_0,u})$. Recall that $G(F_{0,u})_{\mathrm{rs}}$ is defined by the equation $\Delta \neq 0$ where Δ is a polynomial in the entries of GL_n with coefficients in $\mathbb{Z}$, and it is easy to exhibit an element $g \in \mathrm{GL}_n(O_{F_0,u})$ such that $\Delta(g) \in \mathbb{Z}^\times = \{\pm 1\}$. Then the function $f_{n,u} = \mathbf{1}_{K^m_{G,u} g K^m_{G,u}}$ has regular support. Indeed, consider the reduction map $\mathrm{GL}_n(O_{F_0,u}) \to \mathrm{GL}_n(O_{F_0,u}/\varpi_u^m)$ where ϖ_u is a uniformizer. It is easy to see that for $k, k' \in 1 + \varpi_u^m M_n(O_{F_0,u})$ we have $\Delta(kgk') \equiv \Delta(g) \equiv \pm 1 \mod \varpi_u^m$. In particular we have $K^m_{G,u} g K^m_{G,u} \subset \mathrm{GL}_n(O_{F_0,u})_{\mathrm{rs}}$. This completes the proof of the first part.

To show the second part, by a reduction process similar to the first part (cf. [57, Prop. 2.5]), it suffices to work with the inhomogeneous version, i.e., to show there exists a smooth transfer f'_u with support in $S_n(F_{0,u})_{\mathrm{rs}}$. We may identify $S_n(F_{0,u}) = \mathrm{GL}_n(F_{0,u})$ and then the notions of regular semi-simplicity on $S_n(F_{0,u})$ and on $G(F_{0,u})$ coincide. This completes the proof. □

The left-hand side of (8.12) can be localized, i.e., we can write it as a sum over all non-archimedean places,

$$\mathrm{Int}(f) = \sum_v \mathrm{Int}_v(f), \tag{8.14}$$

where

$$\mathrm{Int}_v(f) := \frac{1}{\tau(Z^{\mathbb{Q}}) \cdot [E : F]} \sum_{\nu | v} \mathrm{Int}^\natural_\nu(f). \tag{8.15}$$

By Lemma 8.11, the smooth transfer f' of f can be chosen such that f' has regular support at λ, which we assume from now on. Then also the right-hand side of Conjecture 8.8 can be written as a sum of local contributions of each place of F_0, cf. (7.13) for $\partial J(f')$ and (7.9) for $J(f'_{\mathrm{corr}})$.

Proposition 8.12. *In the situation of Conjecture 8.8, let v_0 be a place of F_0 that is split in F. Then*

$$\mathrm{Int}_{v_0}(f) = \partial J_{v_0}(f') = 0.$$

Proof. In Lemma 7.11 we have proved $\partial J_{v_0}(f') = 0$. Now $\mathrm{Int}_{v_0}(f) = 0$ follows from Theorem 8.5(ii). □

In the next subsection, we are going to formulate a semi-global conjecture for each non-split place v (including archimedean ones) which refines Conjecture 8.8.

8.3. **The semi-global conjecture.** Let v_0 be a place of F_0 above the place $p \leq \infty$ of $\mathbb{Q}$. By Proposition 8.12, from now on we may and do assume that v_0 is non-split in F.

Now assume that v_0 is non-archimedean. We assume that v_0 is either of hyperspecial level type or of AT parahoric level type, in the sense of Section 4. We also take up the notation of Subsections 4.1 and 4.4 and denote, for any place ν of E lying over v_0, by $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})_{(\nu)}$, $\mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H})_{(\nu)}$, and $\mathcal{M}_{K_{\widetilde{HG}}}(\widetilde{HG})_{(\nu)}$ the corresponding semi-global moduli stacks over $\operatorname{Spec} O_{E,(\nu)}$. Let $K_{\widetilde{HG}} = K^p_{\widetilde{HG}} \times K_{\widetilde{HG},p} \subset \widetilde{HG}(\mathbb{A}_f)$ be as in Section 4.1, resp. Section 4.4. Let

$$\mathscr{H}^p_{K_{\widetilde{HG}}} \subset \mathscr{H}_{K_{\widetilde{HG}}} = \mathscr{H}\big(\widetilde{HG}(\mathbb{A}_f), K_{\widetilde{HG}}\big) \tag{8.16}$$

be the partial Hecke algebra spanned by completely decomposed pure tensors of the form $f = \otimes_\ell f_\ell \in \mathscr{H}_{K_{\widetilde{HG}}}$, where $f_\ell = \phi_\ell \otimes \bigotimes_{v|\ell} f_v$, as in Definition 7.6, with $\phi_\ell = \mathbf{1}_{K_{Z^{\mathbb{Q}},\ell}}$ for all ℓ, and where

$$f_p = \mathbf{1}_{K_{\widetilde{HG},p}}.$$

We note that this defines a bigger Hecke algebra than (8.7) when $K_{\widetilde{HG}} = K^{\mathbf{m}}_{\widetilde{HG}}$,

$$\mathscr{H}^p_{K_{\widetilde{HG}}} \supset \mathscr{H}^{\mathrm{spl},\Phi}_{K^{\mathbf{m}}_{\widetilde{HG}}}. \tag{8.17}$$

Let $f = \otimes_\ell f_\ell \in \mathscr{H}^p_{K_{\widetilde{HG}}}$ be completely decomposed with regular support at some place λ. We define as before in (8.15)

$$\begin{aligned} \operatorname{Int}^\natural_\nu(f) &:= \big\langle \widehat{R}(f)\widehat{z}_{K_{\widetilde{HG}}}, \widehat{z}_{K_{\widetilde{HG}}} \big\rangle_\nu \log q_\nu, \\ \operatorname{Int}_{v_0}(f) &:= \frac{1}{\tau(Z^{\mathbb{Q}})[E:F]} \sum_{\nu|v_0} \operatorname{Int}^\natural_\nu(f), \end{aligned} \tag{8.18}$$

where again the contribution of the place ν is defined through the Euler–Poincaré characteristic of a derived tensor product on $\mathcal{M}_{K_{\widetilde{HG}}}(\widetilde{HG})_{(\nu)}$. This extends the definition (8.15) to the bigger Hecke algebra $\mathscr{H}^p_{K_{\widetilde{HG}}}$.

We proceed similarly for an archimedean place $v_0 \in \operatorname{Hom}(F_0, \mathbb{R})$. Denote, for any place ν of E lying over v_0, by $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})_{(\nu)}$, $\mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H})_{(\nu)}$, and $\mathcal{M}_{K_{\widetilde{HG}}}(\widetilde{HG})_{(\nu)}$ the corresponding complex analytic spaces (in fact, orbifolds). Note that the Green's current $g_{z_{K_{\widetilde{HG}}}}$ is the multi-set $g_{z_{K_{\widetilde{HG}}},\nu}$ indexed by $\nu \in \operatorname{Hom}(E, \mathbb{C})$. We define

$$\begin{aligned} \operatorname{Int}^\natural_\nu(f) &:= \big\langle \widehat{R}(f)\widehat{z}_{K_{\widetilde{HG}}}, \widehat{z}_{K_{\widetilde{HG}}} \big\rangle_\nu \log q_\nu, \\ \operatorname{Int}_{v_0}(f) &:= \frac{1}{\tau(Z^{\mathbb{Q}})[E:F]} \sum_{\nu|v_0} \operatorname{Int}^\natural_\nu(f), \end{aligned} \tag{8.19}$$

where the first quantity is defined before Conjecture 8.8.

A refinement of Conjecture 8.8 is now given by the following statement.

Conjecture 8.13 (Semi-global conjecture). *Fix a place v_0 of F_0 above a place $p \leq \infty$ of $\mathbb{Q}$. Let $f = \otimes_\ell f_\ell \in \mathscr{H}^p_{K_{\widetilde{HG}}}$ ($\mathscr{H}_{K_{\widetilde{HG}}}$ if p is archimedean) be completely decomposed, and let $f' = \otimes_v f'_v \in \mathscr{H}(G'(\mathbb{A}_{F_0}))$ be a Gaussian test function such that $\otimes_{v<\infty} f'_v$ is a smooth transfer of f. Assume that for some ℓ prime to v_0 and some place λ above ℓ, the function f has regular support at λ in the sense of Definition 8.4 and that f' has regular support at λ in the sense of Definition 7.4.*

(i) Assume that v_0 is non-archimedean of hyperspecial type, cf. Section 4.1, and that $f'_{v_0} = \mathbf{1}_{G'(O_{F_0,v_0})}$. Then

$$\operatorname{Int}_{v_0}(f) = -\partial J_{v_0}(f').$$

(ii) Assume that v_0 is archimedean, or non-archimedean of AT type, cf. Section 4.4. Then

$$\operatorname{Int}_{v_0}(f) = -\partial J_{v_0}(f') - J(f'_{\mathrm{corr}}[v_0]),$$

where $f'_{\mathrm{corr}}[v_0] = \otimes_v f'_{\mathrm{corr},v}$*, with* $f'_{\mathrm{corr},v} = f'_v$ *for* $v \neq v_0$*, is a correction function. Furthermore,* f' *may be chosen such that* $f'_{\mathrm{corr}}[v_0]$ *is zero.*

Theorem 8.14. *The semi-global conjecture Conjecture 8.13 for all places* v_0 *implies the global conjecture Conjecture 8.8.*

Proof. By (7.10), (7.13), and (8.14), it follows from Proposition 8.12 for split places v_0 of F_0 and the semi-global conjecture Conjecture 8.13 for non-split places v_0 that

$$\mathrm{Int}(f) = -\partial J(f') - \sum_{v_0 \text{ bad}} J(f'_{\mathrm{corr}}[v_0]),$$

where the sum runs over a finite set of places v_0 in Conjecture 8.13(ii). Here we note that by (8.17) we may apply the semi-global conjecture Conjecture 8.13 for the given test function f in the global conjecture Conjecture 8.8. This proves Conjecture 8.8 by taking

$$f'_{\mathrm{corr}} = \sum_{v_0 \text{ bad}} f'_{\mathrm{corr}}[v_0]. \qquad \square$$

In the direction towards the semi-global conjecture, there are the following results.

Theorem 8.15. *Conjecture 8.13(i) holds when* $n \leq 3$.

Proof. Let p denote the residue characteristic of v_0, and as previously in the paper, let S_p denote the set of places of F_0 lying above p. It suffices to show that the result holds if we assume the Arithmetic Fundamental Lemma (AFL) conjecture, which is known under these circumstances; see [55, Th. 5.5], comp. also [35].

We imitate the proof of [55, Th. 3.11]. More precisely, we consider the non-archimedean uniformization along the basic locus,

$$\big(\mathcal{M}_{(\nu)} \otimes_{O_{E,(\nu)}} O_{\breve{E}_\nu}\big)^\wedge = \widetilde{HG}'(\mathbb{Q}) \Big\backslash \Big[\mathcal{N}' \times \widetilde{HG}(\mathbb{A}_f^p)/K^p_{\widetilde{HG}}\Big]. \tag{8.20}$$

Here the hat on the left-hand side denotes the completion along the basic locus in the special fiber of $\mathcal{M}_{(\nu)} := \mathcal{M}_{K_{\widetilde{HG}}}(\widetilde{HG})_{(\nu)}$. The group $\widetilde{HG}'$ is an inner twist of $\widetilde{HG}$. More precisely, the group $\widetilde{HG}'$ is associated to the pair of hermitian spaces $(W'^\flat, W')$, where $W'^\flat$ and W' are negative definite at all archimedean places, and isomorphic to $W^\flat$, resp. W, locally at all non-archimedean places except at v_0. Furthermore, $\mathcal{N}'$ is the RZ space relevant in this situation. Using Lemma 8.16 below, can write

$$\mathcal{N}' \simeq \big(Z^{\mathbb{Q}}(\mathbb{Q}_p)/K_{Z^{\mathbb{Q}},p}\big) \times \mathcal{N}_{O_{\breve{E}_\nu}} \times \prod_{v \in S_p \smallsetminus \{v_0\}} (H \times G)(F_{0,v})/(K_{H,v} \times K_{G,v}). \tag{8.21}$$

Here $\mathcal{N}_{O_{\breve{E}_\nu}} = \mathcal{N} \,\widehat{\otimes}_{O_{\breve{F}_{w_0}}} O_{\breve{E}_\nu}$, where $\mathcal{N} = \mathcal{N}_{n-1} \times_{O_{\breve{F}_{w_0}}} \mathcal{N}_n$ is the *relative* RZ space of [55]. More precisely, the formal scheme in the uniformization theorem is the RZ space of polarized p-divisible groups with action by O_{F,w_0} satisfying the Kottwitz condition (3.9) of signature $((1, n-1)_{\varphi_0}, (0, n)_{\varphi \in \Phi_{v_0} \smallsetminus \{\varphi_0\}})$. That this coincides with the relative RZ space of [55, §2.2] follows from [36, Th. 4.1], where in the notation of loc. cit. we take $E_0 = F_{0,v_0}$, $E = F_{w_0}$, $(r, s) = (1, n-1)$, and $K = \mathbb{Q}_p$.

Therefore we may rewrite (8.20) as

$$\big(\mathcal{M}_{(\nu)} \otimes_{O_{E,(\nu)}} O_{\breve{E}_\nu}\big)^\wedge = \widetilde{HG}'(\mathbb{Q}) \Big\backslash \Big[\mathcal{N}_{O_{\breve{E}_\nu}} \times \widetilde{HG}(\mathbb{A}_f^{v_0})/K^{v_0}_{\widetilde{HG}}\Big]. \tag{8.22}$$

Here we denote for simplicity (even though $\widetilde{HG}$ is not an algebraic group over F_0)

$$\widetilde{HG}(\mathbb{A}_f^{v_0})/K^{v_0}_{\widetilde{HG}} = \widetilde{HG}(\mathbb{A}_f^p)/K^p_{\widetilde{HG}} \times \big(Z^{\mathbb{Q}}(\mathbb{Q}_p)/K_{Z^{\mathbb{Q}},p}\big) \times \prod_{v \in S_p \smallsetminus \{v_0\}} (H \times G)(F_{0,v})/(K_{H,v} \times K_{G,v}).$$

There is also a similar uniformization of the basic locus of $\mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H})_{(\nu)}$ involving the twisted form $\widetilde{H}'$ of H.

By Theorem 8.5(iii), the intersection of the cycles is supported in the basic locus, and hence we can imitate the proof of [55, Th. 3.9]. The difference is that here we have an extra central subgroup $Z^{\mathbb{Q}}$. By the same procedure as in loc. cit., we see that (8.19) can be written as a sum

$$\mathrm{Int}_{v_0}(f) = \frac{1}{[E:F]} \sum_{g \in \mathbb{O}(\widetilde{HG}(\mathbb{Q}))_{\mathrm{rs}}} \mathrm{Orb}(g, f^p) \cdot \Big(c(\phi_p) \prod_{v \in S_p \smallsetminus \{v_0\}} \mathrm{Orb}(g, f_v) \Big) \cdot \mathrm{Int}^\natural_{v_0}(g), \tag{8.23}$$

cf. Remark 7.7. Here the volume factor $\tau(Z^{\mathbb{Q}}) = \mathrm{vol}(Z^{\mathbb{Q}}(\mathbb{A}_f)/Z^{\mathbb{Q}}(\mathbb{Q}))$ is canceled with the one in the definition of $\mathrm{Int}_{v_0}(f)$, and

$$\mathrm{Orb}(g, f^p) = \prod_{\ell \nmid p} \mathrm{Orb}(g, f_\ell).$$

Also, we have set

$$\mathbb{O}(\widetilde{HG}(\mathbb{Q}))_{\mathrm{rs}} := \widetilde{H}'(\mathbb{Q}) \backslash \widetilde{HG}'(\mathbb{Q})_{\mathrm{rs}} / \widetilde{H}'(\mathbb{Q}),$$

and, for $g \in (H \times G)(F_{0,v_0})$, in analogy with (8.18),

$$\begin{aligned} \mathrm{Int}^\natural_{v_0}(g) &:= \sum_{\nu | v_0} \mathrm{Int}^\natural_\nu(g), \\ \mathrm{Int}^\natural_\nu(g) &:= \big\langle \Delta(\mathcal{N}_{n-1, O_{\breve{E}_\nu}}), g\Delta(\mathcal{N}_{n-1, O_{\breve{E}_\nu}}) \big\rangle_{\mathcal{N}_{O_{\breve{E}_\nu}}} \log q_\nu. \end{aligned} \tag{8.24}$$

Now note that

$$\begin{aligned} \sum_{\nu | v_0} \mathrm{Int}^\natural_\nu(g) &= \sum_{\nu | v_0} \big\langle \Delta(\mathcal{N}_{n-1, O_{\breve{E}_\nu}}), g\Delta(\mathcal{N}_{n-1, O_{\breve{E}_\nu}}) \big\rangle_{\mathcal{N}_{O_{\breve{E}_\nu}}} \log q_\nu \\ &= [E:F] \big\langle \Delta(\mathcal{N}_{n-1}), g\Delta(\mathcal{N}_{n-1}) \big\rangle_{\mathcal{N}} \log q_{w_0}. \end{aligned} \tag{8.25}$$

Here we use the equality $\sum_{\nu | w_0} e_{\nu/w_0} f_{\nu/w_0} = \sum_{\nu|w_0} d_{\nu/w_0} = [E:F]$. By the AFL identity (cf. [44, §4]), we have

$$2\big\langle \Delta(\mathcal{N}_{n-1}), g\Delta(\mathcal{N}_{n-1}) \big\rangle_{\mathcal{N}} \log q_{v_0} = -\omega_{v_0}(\gamma)\, \partial\mathrm{Orb}(\gamma, f'_{v_0}), \tag{8.26}$$

for any $\gamma \in G'(F_{0,v_0})_{\mathrm{rs}}$ matching g. Since v_0 is inert in F, we have $\log q_{w_0} = 2 \log q_{v_0}$, and hence

$$\mathrm{Int}^\natural_{v_0}(g) = -[E:F] \omega_{v_0}(\gamma)\, \partial\mathrm{Orb}(\gamma, f'_{v_0}). \tag{8.27}$$

Since f and f' are smooth transfers of each other, we have for $\gamma \in G'(F_0)$ matching $g \in \mathbb{O}(\widetilde{HG}(\mathbb{Q}))_{\mathrm{rs}}$,

$$\mathrm{Orb}(g, f^p) = \prod_{v < \infty,\, v \nmid p} \omega_v(\gamma)\, \mathrm{Orb}(\gamma, f'_v). \tag{8.28}$$

By Remark 7.7, since the orbital integrals of f_{v_0} and f'_{v_0} do not vanish identically, one of the following holds.

(1) $c(\phi_p) = 0$ or one of f_v, for $v \in S_p \smallsetminus \{v_0\}$, has identically vanishing orbital integrals.

(2) the orbital integrals of f_p do not vanish identically.

In the first case, one of f'_v, for $v \in S_p \smallsetminus \{v_0\}$, has identically vanishing orbital integrals, and we have

$$\mathrm{Int}_{v_0}(f) = 0 = \partial J_{v_0}(f'),$$

where the second equality follows from (7.12).

Therefore we only need to consider the second case. By Remark 7.7, there are non-zero constants c_v such that $\prod_{v \in S_p} c_v = c(\phi_p)$, and for every $v \in S_p$, $c_v f'_v$ is a transfer of f_v. By computing orbital integrals at some special g and γ (e.g., [53]) we conclude that $c_{v_0} = 1$ (this follows in any case if the FL conjecture holds, which is known when the residue characteristic is big enough). It follows that

$$c(\phi_p) \prod_{v \in S_p \smallsetminus \{v_0\}} \mathrm{Orb}(g, f_v) = \prod_{v \in S_p \smallsetminus \{v_0\}} \omega_v(\gamma)\, \mathrm{Orb}(\gamma, f'_v). \tag{8.29}$$

Note that for v archimedean, f'_v is a Gaussian test function so that we have for regular semisimple $\gamma \in G'(F_{0,v})_{\mathrm{rs}}$ (cf. (7.18))

$$\omega_v(\gamma)\operatorname{Orb}(\gamma, f'_v) = \begin{cases} 1, & \text{there exists } g \in (H \times G)(F_{0,v}) \text{ matching } \gamma; \\ 0, & \text{no } g \in (H \times G)(F_{0,v}) \text{ matches } \gamma. \end{cases}$$

By the last equality for archimedean v, by (8.28) for non-archimedean v with $v \nmid p$, by (8.29) for $v \in S_p \smallsetminus \{v_0\}$, and by (8.27) for $v = v_0$, we have (cf. (8.23))

$$\operatorname{Int}_{v_0}(f) = -\sum_{\gamma \in G'(F_0)_{\mathrm{rs}}/H'_{1,2}(F_0)} \omega_{v_0}(\gamma)\,\partial\operatorname{Orb}(\gamma, f'_{v_0}) \cdot \prod_{v \neq v_0} \omega_v(\gamma)\operatorname{Orb}(\gamma, f'_v).$$

Here the sum runs a priori only over those γ which match some $g \in G_W(F_0)$. However, those γ which do not match any $g \in G_W(F_0)$ have vanishing orbital integrals away from v_0, cf. [55, Prop. 3.6, eq. (3.4)].

By the product formula for transfer factors (7.15), we have $\prod_v \omega_v(\gamma) = 1$, and hence

$$\operatorname{Int}_{v_0}(f) = -\sum_{\gamma \in G'(F_0)_{\mathrm{rs}}/H'_{1,2}(F_0)} \partial\operatorname{Orb}(\gamma, f'_{v_0}) \cdot \prod_{v \neq v_0} \operatorname{Orb}(\gamma, f'_v).$$

By (7.12), the right-hand side equals $-\partial J_{v_0}(f')$, and this completes the proof. □

In the preceding proof, we used the following lemma.

Lemma 8.16. *Let $L_0/\mathbb{Q}_p$ be a finite extension and L/L_0 an étale L_0-algebra of rank 2. Assume $p \neq 2$ if L is a field. Let $\Phi_L \subset \operatorname{Hom}_{\mathbb{Q}_p}(L, \overline{\mathbb{Q}}_p)$ be a local CM type for L/L_0, and let $r_L\colon \operatorname{Hom}_{\mathbb{Q}_p}(L, \overline{\mathbb{Q}}_p) \to \{0, n\}$ be a* banal *generalized CM type of rank n. Let E' be the join of the reflex fields of Φ_L and r_L. Let $\overline{k}$ be an algebraic closure of the residue field of $O_{E'}$. Let $(\mathbb{X}_0, \iota_{\mathbb{X}_0}, \lambda_{\mathbb{X}_0})$ be a local CM triple of type $\overline{\Phi}_L$ over $\overline{k}$, where $\lambda_{\mathbb{X}_0}$ is principal. Let $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ be a local CM triple of type r over $\overline{k}$ which satisfies the Eisenstein condition, cf.* (B.5). *Assume that* $\ker \lambda_{\mathbb{X}} \subset \mathbb{X}[\pi]$, *where π denotes a uniformizer*[13] *of L. Let $\mathcal{N}_{\overline{\Phi}_L, r_L}$ be the formal scheme over* $\operatorname{Spf} O_{\breve{E}'}$ *that represents the functor which associates to $S \in \operatorname{Nilp}_{O_{\breve{E}'}}$ the set of isomorphism classes of tuples $(X_0, \iota_0, \lambda_0, X, \iota, \lambda, \varrho_0, \varrho)$, where $(X_0, \iota_0, \lambda_0)$ is a local CM triple of type $\overline{\Phi}_L$, and (X, ι, λ), is a local CM triple of type r_L, over S, and where*

$$\rho_0\colon (X_0, \iota_0, \lambda_0) \times_S \overline{S} \longrightarrow \mathbb{X}_0 \times_{\operatorname{Spec} \overline{k}} \overline{S}, \quad \rho\colon (X, \iota, \lambda) \times_S \overline{S} \longrightarrow \mathbb{X} \times_{\operatorname{Spec} \overline{k}} \overline{S}$$

are quasi-isogenies respecting the O_L-actions and the polarizations. Here (X, ι, λ) is supposed to satisfy the Eisenstein condition. Then

$$\mathcal{N}_{\overline{\Phi}_L, r_L} \simeq G(L_0)/K,$$

where G is the unitary group of an L/L_0-hermitian vector space of invariant $\operatorname{inv}^{r_L}(\mathbb{X}_0, \mathbb{X})$, *and where K is the stabilizer of a vertex lattice of type $t := \log_q |\ker \lambda_{\mathbb{X}}|$, $q := \# O_L/\pi O_L$.*

Here the invariant $\operatorname{inv}^{r_L}(\mathbb{X}_0, \mathbb{X})$ is defined in Remark A.3.

Proof. By Lemmas B.1 and B.4, $\mathcal{N}_{\overline{\Phi}_L, r_L}$ is formally étale over $\operatorname{Spf} O_{\breve{E}'}$. So it remains to determine the point set $\mathcal{N}_{\overline{\Phi}_L, r_L}(\overline{k})$ or, equivalently, the point set $\mathcal{N}^{\mathrm{rig}}_{\overline{\Phi}_L, r_L}(\breve{E}')$, where $\mathcal{N}^{\mathrm{rig}}_{\overline{\Phi}_L, r_L}$ denotes the generic fiber. Consider the crystalline period map $\pi\colon \mathcal{N}^{\mathrm{rig}}_{\overline{\Phi}_L, r_L} \to \mathcal{F} \otimes_{E'} \breve{E}'$, cf. [47]. However, by the banality of r_L, the Grassmannian $\mathcal{F}$ consists of a single point. Furthermore, the fiber over this point is identified with $G(L_0)/K$, since $\mathcal{N}^{\mathrm{rig}}_{\overline{\Phi}_L, r_L}$ corresponds in the RZ tower to the level K. □

Remark 8.17. We have considered local CM triples of type $\overline{\Phi}_L$ in the lemma because these are what naturally arise from the Kottwitz condition (3.8) in the moduli problem for $\mathcal{M}_0$. Of course, one could just as well consider the moduli space $\mathcal{N}_{\Phi_L, r_L}$; then one replaces Φ_L with $\overline{\Phi}_L$ in the definition of $\operatorname{inv}^{r_L}(\mathbb{X}_0, \mathbb{X})$, cf. Remark A.2.

[13] If $L = L_0 \oplus L_0$, this means, as elsewhere in the paper, an ordered pair of uniformizers in the usual sense.

Remark 8.18. Consider the J-group in the sense of Kottwitz [47] associated to the situation of Lemma 8.16, when L is a field. Let $M(\mathbb{X}_0)$, resp. $M(\mathbb{X})$, be the rational Dieudonné module of $\mathbb{X}_0$, resp. $\mathbb{X}$. Let $M = \mathrm{Hom}_L(M(\mathbb{X}_0), M(\mathbb{X}))$. Then M is a rational Dieudonné module free of rank n over $L \otimes_{\mathbb{Z}_p} W(\overline{k})$ which, by our assumption, is isoclinic of slope 0. It is equipped with a hermitian form $h\colon M \times M \to L \otimes_{\mathbb{Z}_p} W(\overline{k})$. Let C be the space of Frobenius invariants in M. Then the restriction of h makes C into an L/L_0-hermitian vector space of dimension n. The J-group is the unitary group $J = \mathrm{U}(C)$. We claim that $J \simeq G$. Indeed, the difference between $\mathrm{inv}^{r_L}(\mathbb{X}_0, \mathbb{X})$ and $\mathrm{inv}(C)$ is equal to $\mathrm{sgn}(r_L)$, where

$$\mathrm{sgn}(r_L) = (-1)^{\sum_{\varphi \in \Phi_L} (r_L)_\varphi},$$

cf. (A.7), (A.11). Since r_L is banal, the exponent is a multiple of n. Therefore the assertion follows in the case when n is even. When n is odd, any two unitary groups of L/L_0-hermitian vector spaces of dimension n are isomorphic.

Theorem 8.19. *Conjecture 8.13(ii) holds when $n \leq 3$ and v_0 is non-archimedean.*

Proof. This follows in the same way as in the proof of the previous Theorem 8.15, from the AT conjecture proved for $n \leq 3$ in [43] and [44].

We indicate where we need to modify the proof. To simplify, we only prove the "Furthermore" part of Conjecture 8.13(ii). We first consider the ramified case in Section 4.4. In (8.21), we have $\mathcal{N}_{O_{\breve{E}_\nu}} = \mathcal{N} \widehat{\otimes}_{O_{\breve{F}_{w_0}}} O_{\breve{E}_\nu}$, where $\mathcal{N} = \mathcal{N}_{n-1} \times_{O_{\breve{F}_{w_0}}} \mathcal{N}_n$ is the relative RZ space in [43, 44]. Then we replace the AFL identity (8.26) by the AT identity (cf. [43, §5] when n is odd, and [44, §12] when n is even),

$$\big\langle \Delta(\mathcal{N}_{n-1}), g\Delta(\mathcal{N}_{n-1}) \big\rangle_{\mathcal{N}} \log q_{v_0} = -\omega_{v_0}(\gamma)\, \partial\mathrm{Orb}(\gamma, f'_{v_0}), \tag{8.30}$$

where f'_{v_0} is a function in the AT conjecture. Now since v_0 is ramified in F, we have $\log q_{w_0} = \log q_{v_0}$ and hence the equation (8.27) remains true by (8.25).

In the case of unramified AT type, we replace the corresponding space $\mathcal{N}$ after (8.21) by the RZ space in [44].

In both cases, the rest of the proof is the same. □

Remark 8.20. The proof in the ramified case explains the discrepancy of the factor 2 in the ramified ATC [43, 44] and the AFL. In these identities, it would be most natural to normalize the intersection number by the factor $\log q_{w_0}$.

Appendix A. Sign invariants

In this appendix we adapt the sign invariants of [31] to the setting of the moduli problems introduced in Sections 4 and 5; see also [32]. We continue with the notation in the main body of the paper, with $F/\mathbb{Q}$ a CM field, F_0 its maximal totally real subfield of index 2, and Φ a CM type for F. However, we will allow for more general functions r than in (3.11). Set $d := [F_0 : \mathbb{Q}]$, and let v be a finite place of F_0 which is non-split in F. (In the case v splits in F, the analog of the theory in this appendix is trivial, insofar as the value group $F_{0,v}^\times / \mathrm{Nm}\, F_v^\times$ is trivial.) Let k be an arbitrary field. We are first going to define an invariant

$$\mathrm{inv}_v(A_0, \iota_0, \lambda_0, A, \iota, \lambda)^\natural \in F_{0,v}^\times / \mathrm{Nm}\, F_v^\times \tag{A.1}$$

attached to the following objects over k:

- abelian varieties A_0 and A over k of respective dimensions d and nd;
- rational actions $\iota_0\colon F \to \mathrm{End}^\circ(A_0)$ and $\iota\colon F \to \mathrm{End}^\circ(A)$; and
- quasi-polarizations $\lambda_0 \in \mathrm{Hom}^\circ(A_0, A_0^\vee)$ and $\lambda \in \mathrm{Hom}^\circ(A, A^\vee)$ whose corresponding Rosati involutions induce the non-trivial automorphism on F/F_0.

We will then write

$$\mathrm{inv}_v(A_0, \iota_0, \lambda_0, A, \iota, \lambda) \in \{\pm 1\} \tag{A.2}$$

for the image of $\mathrm{inv}_v(A_0, \iota_0, \lambda_0, A, \iota, \lambda)^\natural$ under the identification $F_{0,v}^\times / \mathrm{Nm}\, F_v^\times \cong \{\pm 1\}$. We note in advance that this invariant will depend on A_0 and A only up to isogeny. We give the definition separately in the cases that the place v does not and does divide the characteristic of k.

(a) v *does not divide* $\operatorname{char} k$. Let ℓ denote the residue characteristic of v, and let $V_\ell(A_0)$ and $V_\ell(A)$ denote the respective rational ℓ-adic Tate modules of A_0 and A. Set

$$\widehat{V}_\ell(A_0, A) := \operatorname{Hom}_F\bigl(V_\ell(A_0), V_\ell(A)\bigr).$$

Then $\widehat{V}_\ell(A_0, A)$ is a free $F \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$-module of rank n. The polarizations λ_0 and λ endow $\widehat{V}_\ell(A_0, A)$ with a nondegenerate hermitian form h, defined by

$$h(\phi_1, \phi_2) := \lambda_0^{-1} \circ \phi_2^\vee \circ \lambda \circ \phi_1 \in \operatorname{End}_{F\otimes\mathbb{Q}_\ell}\bigl(\widehat{V}_\ell(A_0, A)\bigr) \cong F \otimes \mathbb{Q}_\ell.$$

The decomposition $F_0 \otimes \mathbb{Q}_\ell = \prod_{v'|\ell} F_{0,v'}$, where v' runs through the places of F_0 over ℓ, induces a decomposition

$$\widehat{V}_\ell(A_0, A) = \bigoplus_{v'|\ell} \widehat{V}_{v'}(A_0, A),$$

and each $\widehat{V}_{v'}(A_0, A)$ is a free $F_{v'}$-module of rank n. By assumption F_v is a field, and we define the invariant at v as for any n-dimensional $F_v/F_{0,v}$-hermitian space in the main body of the paper (1.4),

$$\operatorname{inv}_v(A_0, \iota_0, \lambda_0, A, \iota, \lambda)^\natural := (-1)^{n(n-1)/2} \det \widehat{V}_v(A_0, A) \in F_{0,v}^\times / \operatorname{Nm} F_v^\times,$$

where $\det \widehat{V}_v(A_0, A)$ is the class mod $\operatorname{Nm} F_v^\times$ of any hermitian matrix representing the component h_v. We note that $\operatorname{inv}_v(A_0, \iota_0, \lambda_0, A, \iota, \lambda)^\natural$ is unchanged after any base change $k \to k'$.

(b) v *divides* $\operatorname{char} k = p$. Let $\overline{k}$ be an algebraic closure of k. Let $W = W(\overline{k})$ denote the ring of Witt vectors of $\overline{k}$, and let σ denote the Frobenius operator on W. The decomposition $O_{F_0} \otimes \mathbb{Z}_\ell = \prod_{v'|\ell} O_{F_0,v'}$ induces decompositions of p-divisible groups $A_0[p^\infty] = \prod_{v'|\ell} A_0[v'^\infty]$ and $A[p^\infty] = \prod_{v'|\ell} A[v'^\infty]$, and we denote by $M(A_0)_{\mathbb{Q},v}$ and $M(A)_{\mathbb{Q},v}$ the respective rational Dieudonné modules of $A_0[v^\infty]$ and $A[v^\infty]$ over $W_{\mathbb{Q}}$. Let $\underline{F}_0$ and $\underline{F}$ denote the respective Frobenius operators of $M(A_0)_{\mathbb{Q},v}$ and $M(A)_{\mathbb{Q},v}$. Then $M(A_0)_{\mathbb{Q},v}$ and $M(A)_{\mathbb{Q},v}$ carry actions of F_v which commute with the Frobenius operators and make $M(A_0)_{\mathbb{Q},v}$ and $M(A)_{\mathbb{Q},v}$ into free $F_v \otimes_{\mathbb{Q}_p} W_{\mathbb{Q}}$-modules of respective ranks 1 and n. Furthermore $M(A_0)_{\mathbb{Q},v}$ is isoclinic of slope $1/2$. We consider the internal Hom in the category of isocrystals with F_v-action,

$$M_{\mathbb{Q},v} := \operatorname{Hom}_{F_v\otimes W_{\mathbb{Q}}}\bigl(M(A_0)_{\mathbb{Q},v}, M(A)_{\mathbb{Q},v}\bigr),$$

which is a free $F_v \otimes_{\mathbb{Q}_p} W_{\mathbb{Q}}$-module of rank n. Here, as for any internal Hom object, the Frobenius operator $\underline{F}_{M_{\mathbb{Q},v}}$ on $M_{\mathbb{Q},v}$ sends $\phi \mapsto \underline{F} \circ \phi \circ \underline{F}_0^{-1}$. The polarizations λ_0 and λ endow $M_{\mathbb{Q},v}$ with an $F_v \otimes_{\mathbb{Q}_p} W_{\mathbb{Q}}/F_{0,v} \otimes_{\mathbb{Q}_p} W_{\mathbb{Q}}$-hermitian form h, defined by

$$h(\phi_1, \phi_2) := \lambda_0^{-1} \circ \phi_2^\vee \circ \lambda \circ \phi_1 \in \operatorname{End}_{F_v\otimes W_{\mathbb{Q}}}\bigl(M(A_0)_{\mathbb{Q},v}\bigr) \cong F_v \otimes_{\mathbb{Q}_p} W_{\mathbb{Q}}.$$

Let

$$N_{\mathbb{Q},v} := \bigwedge\nolimits^n_{F_v\otimes W_{\mathbb{Q}}} M_{\mathbb{Q},v}.$$

Then $N_{\mathbb{Q},v}$ is isoclinic of slope zero, and h induces a hermitian form $(\ ,\ )$ on it satisfying

$$(\underline{F}_{N_{\mathbb{Q},v}} x, \underline{F}_{N_{\mathbb{Q},v}} y) = (x, y)^\sigma, \quad x, y \in N_{\mathbb{Q},v}.$$

For any element $x_0 \in N_{\mathbb{Q},v}$ fixed by $\underline{F}_{N_{\mathbb{Q},v}}$, the class $(x_0, x_0) \in F_{0,v}^\times / \operatorname{Nm} F_v^\times$ is independent of the choice of x_0, and we define

$$\operatorname{inv}_v(A_0, \iota_0, \lambda_0, A, \iota, \lambda)^\natural := (-1)^{n(n-1)/2}(x_0, x_0) \in F_{0,v}^\times / \operatorname{Nm} F_v^\times$$

for such x_0.

Now let $r\colon \operatorname{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}) \to \mathbb{Z}_{\geq 0}$ be a generalized CM type for F of rank n, i.e., a function $\varphi \mapsto r_\varphi$ satisfying $r_\varphi + r_{\overline{\varphi}} = n$ for all $\varphi \in \operatorname{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}})$, cf. [31, Def. 2.1]. Also, let r_0 be the *opposite* of the canonical generalized CM type for F of rank one attached to the CM type Φ,

$$r_{0,\varphi} = \begin{cases} 0, & \varphi \in \Phi; \\ 1, & \varphi \notin \Phi, \end{cases} \quad \varphi \in \operatorname{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}). \tag{A.3}$$

Let E be the subfield of $\overline{\mathbb{Q}}$ characterized by

$$\operatorname{Gal}(\overline{\mathbb{Q}}/E) = \bigl\{\, \sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \bigm| \sigma \circ \Phi = \Phi \text{ and } r_{\sigma\varphi} = r_\varphi \text{ for all } \varphi \in \Phi \,\bigr\}. \tag{A.4}$$

Thus E is the join of the reflex fields of r and of r_0, in the sense of [31, §2]. Note that, in the situation of the main body of the paper, this definition of E agrees with (3.1); but in constrast to the main body of the paper, in general F need not admit an embedding into E.

Recall from loc. cit. that a *triple of CM type* r over an O_E-scheme S is a triple (A, ι, λ) consisting of an abelian scheme A over S, an action $\iota\colon O_F \to \mathrm{End}_S(A)$ satisfying the Kottwitz condition of type r, and a polarization $\lambda\colon A \to A^\vee$ such that Ros_λ induces on O_F, via ι, the nontrivial Galois automorphism of F/F_0. We denote by $\mathcal{M}_{r_0,r}$ the stack over $\operatorname{Spec} O_E$ of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda)$, where $(A_0, \iota_0, \lambda_0)$ is a CM triple of type r_0 and (A, ι, λ) is a CM triple of type r.

Let k be a field which is an O_E-algebra, and let $(A_0, \iota_0, \lambda_0, A, \iota, \lambda) \in \mathcal{M}_{r_0,r}(k)$. Again let v be a finite place of F_0 which is non-split in F. We are going to define the *r-adjusted invariant* $\mathrm{inv}_v^r(A_0, \iota_0, \lambda_0, A, \iota, \lambda)$ (it depends on both r_0 and r). If the residue characteristic of v is different from the characteristic of k, then we set

$$\mathrm{inv}_v^r(A_0, \iota_0, \lambda_0, A, \iota, \lambda) := \mathrm{inv}_v(A_0, \iota_0, \lambda_0, A, \iota, \lambda). \tag{A.5}$$

Now suppose that the residue characteristic of v is equal to the characteristic p of k. Let ν be the place of E determined by the structure map $O_E \to k$, and let $\widetilde{\nu}\colon \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ be an embedding which induces ν. Let

$$\Phi_{\nu,v} := \bigl\{\, \varphi \in \Phi \bigm| \widetilde{\nu} \circ \varphi|_{F_0} \text{ induces } v \,\bigr\}.$$

Then the set $\Phi_{\nu,v}$ is independent of the choice of $\widetilde{\nu}$ inducing ν, and using $\widetilde{\nu}$ we may identify

$$\mathrm{Hom}_{\mathbb{Q}_p}(F_v, \overline{\mathbb{Q}}_p) \simeq \Phi_{\nu,v} \sqcup \overline{\Phi}_{\nu,v}. \tag{A.6}$$

Let

$$r_{\nu,v} := r|_{\Phi_{\nu,v} \sqcup \overline{\Phi}_{\nu,v}}.$$

Then we define

$$\mathrm{sgn}(r_{\nu,v}) := (-1)^{\sum_{\varphi \in \Phi_{\nu,v}} r_\varphi} \tag{A.7}$$

and

$$\mathrm{inv}_v^r(A_0, \iota_0, \lambda_0, A, \iota, \lambda) := \mathrm{sgn}(r_{\nu,v}) \cdot \mathrm{inv}_v(A_0, \iota_0, \lambda_0, A, \iota, \lambda). \tag{A.8}$$

The analog of [32, App.] (which corrects [31, Prop. 3.2]) is now as follows.

Proposition A.1. *Let $(A_0, \iota_0, \lambda_0, A, \iota, \lambda) \in \mathcal{M}_{r_0,r}(S)$, where S is a connected scheme over $\operatorname{Spec} O_E$. Then for every non-archimedean place v of F_0 which is non-split in F, the function*

$$s \longmapsto \mathrm{inv}_v^r(A_{0,s}, \iota_{0,s}, \lambda_{0,s}, A_s, \iota_s, \lambda_s)$$

is constant on S.

Proof. The proof is easy when the residue characteristic of v is invertible in $\mathcal{O}_S$, in which case $\widehat{V}_\ell(A_0, A)$ is a lisse étale sheaf on S, comp. the proof of [31, Prop. 3.2]. A similar argument works when S is a scheme over $\mathbb{F}_p$, where p is the residue characteristic of v, cf. loc. cit. The remaining cases are reduced to the case $S = \operatorname{Spec} O_L$, where L is the completion of a subfield of $\overline{\mathbb{Q}}_p$ which contains E and such that its ring of integers O_L is a discrete valuation ring with residue field $k = \overline{\mathbb{F}}_p$. To compare the invariants at the generic and closed points of S, as in loc. cit. we are going to use p-adic Hodge theory. Let A_L and $A_{0,L}$ denote the respective generic fibers of A and A_0, and let A_k and $A_{0,k}$ denote the respective special fibers. Let ν denote the induced place of E.

We decompose the homomorphism module of the rational p-adic Tate modules of $A_{0,L}$ and A_L, resp. the homomorphism module of the rational Dieudonné modules of $A_{0,k}$ and A_k, with respect to the actions of $F \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \prod_{w|p} F_w$,

$$\widehat{V}_p := \mathrm{Hom}_{F \otimes \mathbb{Q}_p}\bigl(\widehat{V}_p(A_{0,L}), \widehat{V}_p(A_L)\bigr) = \bigoplus\nolimits_{w|p} \widehat{V}_w,$$

$$M_{\mathbb{Q}} := \mathrm{Hom}_{F \otimes \breve{\mathbb{Q}}_p}\bigl(M(A_{0,k})_{\mathbb{Q}}, M(A_k)_{\mathbb{Q}}\bigr) = \bigoplus\nolimits_{w|p} M_{\mathbb{Q},w}.$$

Here w runs through the places of F, and we recall the notation $\breve{\mathbb{Q}}_p = W(k)_{\mathbb{Q}}$. Furthermore, for each place $w \mid p$, $\widehat{V}_w$ is a free F_w-vector space of rank n, and $M_{\mathbb{Q},w}$ is a free $F_w \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p$-module of rank n. At our distinguished place v, we set

$$S_v := \bigwedge\nolimits_{F_v}^n \widehat{V}_v \quad \text{and} \quad N_{\mathbb{Q},v} := \bigwedge\nolimits_{F_v \otimes \breve{\mathbb{Q}}_p}^n M_{\mathbb{Q},v}.$$

Then S_v and $N_{\mathbb{Q},v}^{\mathrm{Frob}=1}$ are one-dimensional F_v-vector spaces (the latter because $N_{\mathbb{Q},v}$ is isoclinic of slope zero) equipped with natural $F_v/F_{0,v}$-hermitian forms. Our problem is to compare these hermitian spaces, which we will do via [47, Prop. 1.20].

To explain the group-theoretic setup in our application of loc. cit., let T be the torus over $\mathbb{Q}_p$ which is the kernel in the exact sequence

$$1 \longrightarrow T \longrightarrow \mathrm{Res}_{F_v/\mathbb{Q}_p} \mathbb{G}_{m,F_v} \xrightarrow{\mathrm{Nm}} \mathrm{Res}_{F_{0,v}/\mathbb{Q}_p} \mathbb{G}_{m,F_{0,v}} \longrightarrow 1.$$

Then $H^1(\mathbb{Q}_p, T) = F_{0,v}^\times / \mathrm{Nm}\, F_v^\times$. The spaces S_v and $N_{\mathbb{Q},v}^{\mathrm{Frob}=1}$ are natural $\mathbb{Q}_p$-rational representations of T, and we may regard the isomorphisms of hermitian vector spaces $\mathrm{Isom}(N_{\mathbb{Q},v}^{\mathrm{Frob}=1}, S_v)$ as an étale sheaf on $\mathrm{Spec}\, \mathbb{Q}_p$. This is a T-torsor. To calculate its class $\mathrm{cl}(N_{\mathbb{Q},v}^{\mathrm{Frob}=1}, S_v)$ via loc. cit., we seek to express the filtered isocrystal $N_{\mathbb{Q},v}$ in the form $\mathcal{I}(N_{\mathbb{Q},v}^{\mathrm{Frob}=1})$ for an admissible pair (μ, b) in T, in the notation of [47, 1.17].

Since $N_{\mathbb{Q},v}$ is isoclinic of slope zero, we take $b \in T(\breve{\mathbb{Q}}_p)$ to be the identity. To determine the cocharacter μ, we need to identify the filtration on $N_{\mathbb{Q},v} \otimes_{\breve{\mathbb{Q}}_p} \overline{\breve{\mathbb{Q}}}_p$. Choose any embedding $\overline{\mathbb{Q}}_p \to \overline{\breve{\mathbb{Q}}}_p$, and identify $\mathrm{Hom}_{\mathbb{Q}_p}(F_v, \overline{\mathbb{Q}}_p) \simeq \Phi_{\nu,v} \sqcup \overline{\Phi}_{\nu,v}$ as in (A.6). By the Kottwitz condition, the filtration on $M(A_k)_{\mathbb{Q},v} \otimes_{\breve{\mathbb{Q}}_p} \overline{\breve{\mathbb{Q}}}_p \cong \bigoplus_{\varphi \in \Phi_{\nu,v} \sqcup \overline{\Phi}_{\nu,v}} M(A_k)_{\mathbb{Q},\varphi}$ is given by, for each φ,

$$M(A_k)_{\mathbb{Q},\varphi} \supset^{r_\varphi} \mathrm{Fil}^1_\varphi \supset 0,$$

where the displayed terms are in respective degrees 0, 1, and 2, and the upper index on the first containment means that the cokernel is of dimension r_φ. The filtration on $M(A_{0,k})_{\mathbb{Q},v} \otimes_{\breve{\mathbb{Q}}_p} \overline{\breve{\mathbb{Q}}}_p \cong \bigoplus_{\varphi \in \Phi_{\nu,v} \sqcup \overline{\Phi}_{\nu,v}} M(A_{0,k})_{\mathbb{Q},\varphi}$ is similarly given by, for each φ,

$$M(A_{0,k})_{\mathbb{Q},\varphi} \supset^{r_{0,\varphi}} \mathrm{Fil}^1_{0,\varphi} \supset 0,$$

where $r_{0,\varphi}$ is given in (A.3). The unique jump in this filtration occurs in degree $1 - r_{0,\varphi}$. The filtration on the dual space $M(A_{0,k})_{\mathbb{Q},\varphi}^\vee$ therefore has unique jump in degree $r_{0,\varphi} - 1$. Therefore in the filtration on the one-dimensional space

$$N_{\mathbb{Q},\varphi} = \bigwedge\nolimits_{\overline{\breve{\mathbb{Q}}}_p}^n M_{\mathbb{Q},\varphi} \cong \big(M(A_{0,k})_{\mathbb{Q},\varphi}^\vee\big)^{\otimes n} \otimes \bigwedge\nolimits_{\overline{\breve{\mathbb{Q}}}_p}^n M(A_k)_{\mathbb{Q},\varphi},$$

the unique jump occurs in degree $n - r_\varphi + n(r_{0,\varphi} - 1) = nr_{0,\varphi} - r_\varphi$.

Now consider the natural identification

$$X_*(T) \cong \ker\big[\mathrm{Ind}_{F_{0,v}}^{F_v}\big(\mathrm{Ind}_{\mathbb{Q}_p}^{F_{0,v}} \mathbb{Z}\big) \to \mathrm{Ind}_{\mathbb{Q}_p}^{F_{0,v}} \mathbb{Z}\big].$$

The corresponding filtration on $N_{\mathbb{Q},\varphi}$ is then given by the cocharacter $\mu \in X_*(T)$ with

$$\mu_\varphi = nr_{0,\varphi} - r_\varphi, \quad \varphi \in \Phi_{\nu,v} \sqcup \overline{\Phi}_{\nu,v}.$$

Then $N_{\mathbb{Q},\varphi} = \mathcal{I}(N_{\mathbb{Q},v}^{\mathrm{Frob}=1})$ for the above choices of μ and b.

Now, by p-adic Hodge theory, in the case of the abelian scheme A, there is a canonical isomorphism

$$\widehat{V}_p(A_L) \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}} \cong M(A_k)_{\mathbb{Q}} \otimes_{\breve{\mathbb{Q}}_p} B_{\mathrm{crys}} \tag{A.9}$$

compatible with all structures on both sides (e.g. the Frobenii, the F-actions, and the polarization forms), cf. [10, 50]. Here B_{crys} is the crystalline period ring of Fontaine [11]. Moreover, after extension of scalars under the inclusion $B_{\mathrm{crys}} \subset B_{\mathrm{dR}}$, this isomorphism is compatible with the filtrations on both sides. Furthermore, there is an analogous isomorphism with A_0 in place of A. Taking homomorphism modules on both sides between the v-components of (A.9) and its analog for A_0, and then passing to top exterior powers, we obtain an isomorphism between free $F_v \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}}$-modules of rank one,

$$S_v \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}} \cong N_{\mathbb{Q},v} \otimes_{\breve{\mathbb{Q}}_p} B_{\mathrm{crys}},$$

again compatible with all structures on both sides, and in particular with the hermitian forms. Taking Frobenius-fixed elements in the zeroth filtration modules, we obtain an isometry of $F_v/F_{0,v}$-hermitian spaces,

$$S_v \cong \mathcal{F}\big(\mathcal{I}(N_{\mathbb{Q},v}^{\mathrm{Frob}=1})\big),$$

where $\mathcal{F}$ denotes Fontaine's functor from admissible filtered isocrystals to Galois representations.

We conclude that the class $\mathrm{cl}(N_{\mathbb{Q},v}^{\mathrm{Frob}=1}, S_v)$ is computed by the formula $\kappa(b) - \mu^\sharp$ in [47, Prop. 1.20]. Here $\mu^\sharp$ denotes the image of μ in the coinvariants $X_*(T)_\Gamma$, where $\Gamma := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Since b is trivial, under the identification $X_*(T)_\Gamma \cong H^1(\mathbb{Q}_p, T) \cong \mathbb{Z}/2\mathbb{Z}$, we obtain

$$\mathrm{cl}(N_{\mathbb{Q},v}^{\mathrm{Frob}=1}, S_v) = \mu^\sharp = \sum_{\varphi\in\Xi_v} \mu_\varphi, \tag{A.10}$$

where Ξ_v is any half-system, i.e., $\Xi_v \sqcup \overline{\Xi}_v \cong \mathrm{Hom}_{\mathbb{Q}_p}(F_v, \overline{\mathbb{Q}}_p)$. In the formula for $\mathrm{sgn}(r_{\nu,v})$, we took $\Xi_v = \Phi_{\nu,v}$. ☐

Remark A.2. In the definition of the r-adjusted invariant above, we took the function r_0 in (A.3) to be the opposite of the canonical rank one function for Φ because this is what occurs in the moduli problem for $\mathcal{M}_0^{\mathfrak{a}}$ in the main body of the paper, cf. (3.8). Of course, we could have instead worked with respect to the canonical function (sending $\varphi \mapsto 1$ for $\varphi \in \Phi$ and $\varphi \mapsto 0$ for $\varphi \notin \Phi$), which is tantamount to replacing Φ by $\overline{\Phi}$. In this case one defines the r-adjusted invariant at a place v dividing $\mathrm{char}\, k$ to be $(-1)^{\sum_{\varphi\in\overline{\Phi}_v} r_\varphi} \cdot \mathrm{inv}_v(A_0, \iota_0, \lambda_0, A, \iota, \lambda)$, and the statement and proof of Proposition A.1 for this version go through virtually without change.

Remark A.3. There is an obvious variant for p-divisible groups. More precisely, let L_0 be a finite extension of $\mathbb{Q}_p$, and let L/L_0 be a quadratic extension. Fix a local CM type $\Phi_L \subset \mathrm{Hom}_{\mathbb{Q}_p}(L, \overline{\mathbb{Q}}_p)$ for L/L_0, and let $r_L\colon \mathrm{Hom}_{\mathbb{Q}_p}(L, \overline{\mathbb{Q}}_p) \to \mathbb{Z}_{\geq 0}$ be a local generalized CM type of rank n for L/L_0, cf. [31, §5]. Let k be a field of characteristic p which is an $O_{E_{\Phi_L,r_L}}$-algebra, where $E_{\Phi_L,r_L} \subset \overline{\mathbb{Q}}_p$ denotes the join of the reflex fields for Φ_L and for r_L. Let $(X_0, \iota_0, \lambda_0)$ and (X, ι, λ) be p-divisible groups over k with actions by O_L and quasi-polarizations whose associated Rosati involutions induce on F the Galois conjugation over L_0. Assume that (X_0, ι_0) is of CM type $\overline{\Phi}_L$ and that (X, ι) is of generalized CM type r_L, cf. [31, §5]. Then there is associated a sign invariant

$$\mathrm{inv}^{r_L}(X_0, \iota_0, \lambda_0, X, \iota, \lambda) := \mathrm{sgn}(r_L) \cdot \mathrm{inv}(X_0, \iota_0, \lambda_0, X, \iota, \lambda), \tag{A.11}$$

with properties analogous to the case of abelian varieties. In fact, returning to all of our notation from the global setting, suppose that k is an O_E-algebra, and let $(A_0, \iota_0, \lambda_0, A, \iota, \lambda) \in \mathcal{M}_{r_0,r}(k)$. Consider the decomposition of the corresponding p-divisible groups induced by the decomposition $O_{F_0} \otimes \mathbb{Z}_p = \prod_{v|p} O_{F_0,v}$,

$$A_0[p^\infty] = \prod_{v|p} A_0[v^\infty] \quad \text{and} \quad A[p^\infty] = \prod_{v|p} A[v^\infty].$$

Let ν denote the place of E determined by k, and for a place $v \mid p$ which is non-split in F, take $L := F_v$, $\Phi_L := \Phi_{\nu,v}$, and $r_L := r_{\nu,v}$ (where we implicitly choose an identification $\mathrm{Hom}_{\mathbb{Q}_p}(F_v, \overline{\mathbb{Q}}_p) \simeq \Phi_{\nu,v} \sqcup \overline{\Phi}_{\nu,v}$ as above). Then

$$\mathrm{inv}_v^r(A_0, \iota_0, \lambda_0, A, \iota, \lambda) = \mathrm{inv}^{r_{\nu,v}}\big(A_0[v^\infty], \iota_0[v^\infty], \lambda_0[v^\infty], A[v^\infty], \iota[v^\infty], \lambda[v^\infty]\big).$$

Appendix B. Local models in the case of banal signature

In this appendix, we prove that local models attached to Weil restrictions of GL_n and GU_n, defined using an analog of the Eisenstein condition of [48], are trivial in the case of *banal signature*. Let L/K be a finite separable extension of discretely valued henselian fields, with respective valuation rings O_L and O_K. Let π be a uniformizer for L, and fix an algebraic closure $\overline{K}$ of K. The material in this appendix applies to the main body of the paper in the case $K = \mathbb{Q}_p$ and $L = F_w$ for w a p-adic place of the number field F.

B.1. **The GL_n case.** Let n be a positive integer, and fix a function

$$\begin{aligned} r\colon\ \mathrm{Hom}_K(L,\overline{K}) &\longrightarrow \{0,n\} \\ \varphi &\longmapsto r_\varphi. \end{aligned} \tag{B.1}$$

The *reflex field* attached to r is the fixed field $E_r \subset \overline{K}$ of the subgroup of the Galois group,

$$\big\{\, \sigma \in \mathrm{Gal}(\overline{K}/K) \bigm| r_{\sigma\circ\varphi} = r_\varphi \text{ for all } \varphi \in \mathrm{Hom}_K(L,\overline{K}) \,\big\}.$$

Then E_r is a finite extension of K, contained in the normal closure of L in $\overline{K}$. Note that in contrast to the analogous global situation considered in the main body of the paper (with the particular choice of r in (3.11)), L need not admit an embedding into E_r. Let $\mathcal{L}$ be a periodic O_L-lattice chain in L^n.

The *local model* attached to the group $\mathrm{Res}_{L/K}\,\mathrm{GL}_n$, the function r, and the lattice chain $\mathcal{L}$ is the scheme $M = M_{\mathrm{Res}_{L/K}\mathrm{GL}_n, r, \mathcal{L}}$ over $\mathrm{Spec}\, O_{E_r}$ representing the following functor. To each O_{E_r}-scheme S, the functor associates the set of isomorphism classes of families $(\Lambda \otimes_{O_K} \mathcal{O}_S \twoheadrightarrow \mathcal{P}_\Lambda)_{\Lambda\in\mathcal{L}}$ such that

- for each Λ, $\mathcal{P}_\Lambda$ is an $O_L \otimes_{O_K} \mathcal{O}_S$-linear quotient of $\Lambda \otimes_{O_K} \mathcal{O}_S$, locally free on S as an $\mathcal{O}_S$-module;
- for each inclusion $\Lambda \subset \Lambda'$ in $\mathcal{L}$, the arrow $\Lambda \otimes_{O_K} \mathcal{O}_S \to \Lambda' \otimes_{O_K} \mathcal{O}_S$ induces an arrow $\mathcal{P}_\Lambda \to \mathcal{P}_{\Lambda'}$;
- for each Λ, the isomorphism $\Lambda \otimes_{O_K} \mathcal{O}_S \xrightarrow[\sim]{\pi\otimes 1} (\pi\Lambda) \otimes_{O_K} \mathcal{O}_S$ identifies $\mathcal{P}_\Lambda \xrightarrow{\sim} \mathcal{P}_{\pi\Lambda}$; and
- for each Λ, $\mathcal{P}_\Lambda$ satisfies the *Kottwitz condition*

$$\mathrm{char}_{\mathcal{O}_S}(a \otimes 1 \mid \mathcal{P}_\Lambda) = \prod_{\varphi\in\mathrm{Hom}_K(L,\overline{K})} \big(T - \varphi(a)\big)^{r_\varphi} \quad \text{for all} \quad a \in O_L. \tag{B.2}$$

We further require that the family $(\Lambda \otimes_{O_K} \mathcal{O}_S \twoheadrightarrow \mathcal{P}_\Lambda)_{\Lambda\in\mathcal{L}}$ satisfies the (analog of the) *Eisenstein condition* of [48, §8], which in our case takes the following form. Let L^t denote the maximal unramified extension of K in L. We first formulate the condition when S is an $O_{E_r\widetilde{L}^t}$-scheme, where $E_r\widetilde{L}^t$ is the compositum in $\overline{K}$ of E_r and the normal closure $\widetilde{L}^t$ of L^t; by Lemma B.1 below, the condition will descend over O_{E_r} (and yield $M \xrightarrow{\sim} \mathrm{Spec}\, O_{E_r}$). For each $\psi \in \mathrm{Hom}_K(L^t,\overline{K})$, set

$$\begin{aligned} A_\psi &:= \big\{\, \varphi \in \mathrm{Hom}_K(L,\overline{K}) \bigm| \varphi|_{F^t} = \psi \text{ and } r_\varphi = n \,\big\}, \\ B_\psi &:= \big\{\, \varphi \in \mathrm{Hom}_K(L,\overline{K}) \bigm| \varphi|_{F^t} = \psi \text{ and } r_\varphi = 0 \,\big\}. \end{aligned}$$

Further set

$$Q_{A_\psi}(T) := \prod_{\varphi\in A_\psi} \big(T - \varphi(\pi)\big) \quad \text{and} \quad Q_{B_\psi}(T) := \prod_{\varphi\in B_\psi} \big(T - \varphi(\pi)\big).$$

Then Q_{A_ψ} and Q_{B_ψ} are polynomials with coefficients in $O_{E_r\widetilde{L}^t}$. Since we assume that S is an $O_{E_r\widetilde{L}^t}$-scheme, there is a natural isomorphism

$$O_{L^t} \otimes_{O_K} \mathcal{O}_S \xrightarrow{\sim} \prod_{\psi\in\mathrm{Hom}_K(L^t,\overline{K})} \mathcal{O}_S, \tag{B.3}$$

whose ψ-component is $\psi \otimes \mathrm{id}$. This induces a decomposition, for each Λ,

$$\mathcal{P}_\Lambda \xrightarrow{\sim} \bigoplus_{\psi\in\mathrm{Hom}_K(L^t,\overline{K})} (\mathcal{P}_\Lambda)_\psi. \tag{B.4}$$

The Eisenstein condition is that, for each Λ,

$$Q_{A_\psi}(\pi \otimes 1)|_{(\mathcal{P}_\Lambda)_\psi} = 0 \quad \text{for all} \quad \psi \in \mathrm{Hom}_K(L^t,\overline{K}). \tag{B.5}$$

To complete the definition of the moduli problem, an isomorphism from $(\Lambda \otimes_{O_K} \mathcal{O}_S \twoheadrightarrow \mathcal{P}_\Lambda)_{\Lambda\in\mathcal{L}}$ to $(\Lambda \otimes_{O_K} \mathcal{O}_S \twoheadrightarrow \mathcal{P}'_\Lambda)_{\Lambda\in\mathcal{L}}$ consists of an isomorphism $\mathcal{P}_\Lambda \xrightarrow{\sim} \mathcal{P}'_\Lambda$ for each Λ, compatible with the given epimorphisms in the obvious way. Note that such an isomorphism is unique if it exists.

The main result is that the moduli scheme M we have defined is trivial, in the following sense.

Lemma B.1. *Let S be an $O_{E_r\widetilde{L}^t}$-scheme. Then $M(S)$ consists of a single point.*

Proof. It suffices to consider the case that $\mathcal{L}$ consists of the homothety class of a single lattice Λ; the general case then follows immediately. Let $\Lambda \otimes_{O_K} \mathcal{O}_S \cong \bigoplus_{\psi \in \mathrm{Hom}_K(L^t,\overline{K})} (\Lambda \otimes_{O_K} \mathcal{O}_S)_\psi$ denote the decomposition induced by (B.3). Then the Eisenstein condition forces

$$(\mathcal{P}_\Lambda)_\psi = (\Lambda \otimes_{O_K} \mathcal{O}_S)_\psi / Q_{A_\psi}(\pi \otimes 1) \cdot (\Lambda \otimes_{O_K} \mathcal{O}_S)_\psi, \tag{B.6}$$

which completes the proof. □

As we have already noted, it follows by descent that $M \cong \operatorname{Spec} O_{E_r}$. This also shows that the Eisenstein condition is independent of the choice of uniformizer π.

Remark B.2. In the special case $n = 1$, we may take $\Lambda = O_L$ in the proof of Lemma B.1, and then the Kottwitz condition already implies (B.6). Thus the Eisenstein condition is redundant in this case.

We also note that the Eisenstein condition is redundant in the unramified case, comp. [48, Prop. 2.2].

Lemma B.3. *Suppose that L/K is unramified. Then the Kottwitz condition* (B.2) *on $\mathcal{P}_\Lambda$ implies the Eisenstein condition* (B.5).

Proof. When $L = L^t$ is unramified over K, then all sets A_ψ have at most one element. If A_ψ is empty, then $(\mathcal{P}_\Lambda)_\psi = 0$ and the condition (B.5) is empty. If A_ψ is non-empty, then the condition (B.5) is equivalent to the definition of the ψ-eigenspace in the decomposition (B.4). □

B.2. **The unitary case.** In this subsection we assume that the residue characteristic of K is not 2. We retain the setup of the previous subsection, and we assume in addition that L is a quadratic extension of a field L_0/K. Let $a \mapsto \overline{a}$ denote the nontrivial automorphism of L/L_0, and for each $\varphi \in \mathrm{Hom}_K(L, \overline{K})$, define $\overline{\varphi}(a) := \varphi(\overline{a})$. We assume that the function r in (B.1) satisfies $r_\varphi + r_{\overline{\varphi}} = n$ for all $\varphi \in \mathrm{Hom}_K(L, \overline{K})$. Furthermore, we endow L^n with a nondegenerate L/L_0-hermitian form h, and we assume that the lattice chain $\mathcal{L}$ is self-dual for h. We define the alternating K-bilinear form $\langle\ ,\ \rangle\colon L^n \times L^n \to K$ as follows. Let $\vartheta_{L_0/K}^{-1}$ be a generator of the inverse different $\mathfrak{d}_{L_0/K}^{-1}$. If L/L_0 is unramified, then choose an element $\zeta \in O_L^\times$ such that $\overline{\zeta} = -\zeta$ (since $p \neq 2$, such a ζ always exists), and set

$$\langle x, y\rangle := \mathrm{tr}_{L/K}\bigl(\vartheta_{L_0/K}^{-1} \zeta h(x,y)\bigr), \quad x, y \in L^n.$$

If L/L_0 is ramified, then choose the uniformizer π to satisfy $\pi^2 \in L_0$ (since $p \neq 2$, such a π always exists), and set

$$\langle x, y\rangle := \mathrm{tr}_{L/K}\bigl(\vartheta_{L_0/K}^{-1} \pi^{-1} h(x,y)\bigr), \quad x, y \in L^n.$$

Then in both cases, the dual $\Lambda^\vee$ of an O_L-lattice Λ in L^n is the same with respect h as it is with respect to $\langle\ ,\ \rangle$.

The *local model* attached to the group $\mathrm{Res}_{L_0/K} \mathrm{GU}(h)$, the function r, and the lattice chain $\mathcal{L}$ is by definition the closed subscheme $M_{\mathrm{Res}_{L_0/K} \mathrm{GU}(h), r, \mathcal{L}}$ of $M_{\mathrm{Res}_{L/K} \mathrm{GL}_n, r, \mathcal{L}}$ defined by the additional condition

- for each Λ, the perfect pairing $(\Lambda \otimes_{O_K} \mathcal{O}_S) \times (\Lambda^\vee \otimes_{O_K} \mathcal{O}_S) \xrightarrow{\langle\ ,\ \rangle \otimes \mathcal{O}_S} \mathcal{O}_S$ identifies $\ker[\Lambda \otimes_{O_K} \mathcal{O}_S \twoheadrightarrow \mathcal{P}_\Lambda]^\perp$ with $\ker[\Lambda^\vee \otimes_{O_K} \mathcal{O}_S \twoheadrightarrow \mathcal{P}_{\Lambda^\vee}]$.

It is a trivial consequence of Lemma B.1 that this additional condition is redundant, and that we again have the following.

Lemma B.4. $M_{\mathrm{Res}_{L_0/K} \mathrm{GU}(h), r, \mathcal{L}} \cong \operatorname{Spec} O_{E_r}$. □

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