ON THE EXISTENCE OF F-CRYSTALS

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By an F-isocrystal we mean a pair (N, F), consisting of a finite-dimensional vector space over the fraction field L of the ring $W(\overline{\mathbf{F}}_p)$ of Witt vectors of $\overline{\mathbf{F}}_p$ and a Frobenius-linear bijective endomorphism of N. Isocrystals form a category in an obvious way. By Dieudonné, F-isocrystals are classified up to isomorphism by their Newton slope sequence. More precisely, let

$$(\mathbf{Q}^n)_+ = \{(\nu_1, \dots, \nu_n) \in \mathbf{Q}^n;\ \nu_1 \geq \nu_2 \geq \dots \geq \nu_n\} \quad .$$

Then we obtain an injective map (the Newton map)

$$\{\text{isocrystals of dimension } n\}/\simeq \longrightarrow (\mathbf{Q}^n)_+ \ ,\ (N, F) \longmapsto \boldsymbol{\nu}(N, F) \quad .$$

Its image is characterized by the following integrality condition. Let us write $\nu \in (\mathbf{Q}^n)_+$ in the form

$$\nu = (\nu(1)^{m_1}, \dots, \nu(r)^{m_r}) \quad , \quad \text{where } \nu(1) > \nu(2) > \dots > \nu(r) \quad .$$

Then the integrality condition states that $m_i \nu(i) \in \mathbf{Z}$, $\forall i = 1, \dots r$.

Let now (N, F) be an isocrystal of dimension n. Let M be a $W(\overline{\mathbf{F}}_p)$-lattice in N. Then the relative position of M and FM is measured by the Hodge slope sequence $\boldsymbol{\mu} = \boldsymbol{\mu}(M) = \mathrm{inv}(M, FM) \in (\mathbf{Z}^n)_+$. Here $(\mathbf{Z}^n)_+ = \mathbf{Z}^n \cap (\mathbf{Q}^n)_+$, and $(\mu_1, \dots, \mu_n) \in (\mathbf{Z}^n)_+$ equals $\boldsymbol{\mu}(M)$ iff there exists a $W(\overline{\mathbf{F}}_p)$-basis $e_1, \dots, e_n$ of M such that $p^{\mu_1} e_1, \dots, p^{\mu_n} e_n$ is a $W(\overline{\mathbf{F}}_p)$-basis of $F(M)$. Mazur's inequality states that

$$\boldsymbol{\mu}(M) \geq \boldsymbol{\nu}(N, F) \quad ,$$

where the partial order relation on $(\mathbf{Q}^n)_+$ is the usual dominance order, comp. section 1.

One result in this paper is a converse to this statement.

Theorem A: *Let (N, F) be an isocrystal of dimension n. Let $\mu \in (\mathbf{Z}^n)_+$ be such that $\mu \geq \boldsymbol{\nu}(N, F)$. Then there exists a $W(\overline{\mathbf{F}}_p)$-lattice M in N with $\mu = \boldsymbol{\mu}(M)$.*

This is the content of Theorem 4.11 below which also gives the corresponding statement for the group of symplectic similitudes. As a matter of fact, one can formulate a corresponding statement for any quasi-split group over a p-adic field F

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which splits over an unramified extension of F ([R], section 4), and we conjecture that Theorem A is true in this generality. Much of our argument in section 4 below is formulated in the context of a split group with simply connected derived group, but we have not carried out the proof in this generality. Also note that if $\boldsymbol{\nu}(N,F) \in (\mathbf{Z}^n)_+$, the general case was handled in [R] as an application of the positivity property of the Satake isomorphism. This positivity property also plays a crucial role in our proof of Theorem A. We also note that when $\boldsymbol{\nu}(N,F)$ is of the form $\boldsymbol{\nu}(N,F) = (\nu,\dots,\nu)$, one can write down explicitly a lattice M as in Theorem A, and similarly in the more general case when μ is decomposable with respect to $\boldsymbol{\nu}(N,F)$ (i.e., the Hodge polygon passes through all break points of the Newton polygon). The general case is reduced to this decomposable case, but then it does not seem so easy to produce explicitly a lattice M with the required properties.

Theorem A may be considered as a statement on generalized affine Deligne-Lusztig varieties. Let $\bar{I} \subset \mathbf{Z}/n\mathbf{Z}$ be a non-empty subset and let $M_\bullet$ be a periodic lattice chain of type $\bar{I}$. Then the relative position of $M_\bullet$ and $FM_\bullet$ is an element $\boldsymbol{\mu}(M_\bullet) \in \tilde{W}^{\bar{I}} \setminus \tilde{W}/\tilde{W}^{\bar{I}}$. Here $\tilde{W} = \mathbf{Z}^n \rtimes S_n$ is the extended affine Weyl group of GL_n and $\tilde{W}^{\bar{I}}$ is the parabolic subgroup of $\tilde{W}$ corresponding to $\bar{I}$. The *generalized affine Deligne-Lusztig variety of type $\bar{I}$ corresponding to $w \in \tilde{W}^{\bar{I}} \setminus \tilde{W}/\tilde{W}^{\bar{I}}$* is the set of all periodic lattice chains $M_\bullet$ of type $\bar{I}$ with $\boldsymbol{\mu}(M_\bullet) = w$ (comp. [R], section 4). It seems a difficult question to determine for which w this set is non-empty. Theorem A gives an answer to this question in case $\bar{I} = \{0\}$, in which case a periodic lattice chain of type $\bar{I}$ is simply a lattice and $\tilde{W}^{\bar{I}} \setminus \tilde{W}/\tilde{W}^{\bar{I}}$ can be identified with $(\mathbf{Z}^n)_+$.

The question raised above becomes more tractable in case we form a certain finite union of Deligne-Lusztig varieties. Let $\mu \in (\mathbf{Z}^n)_+$ be a minuscule element (i.e. $\mu_1 - \mu_n \leq 1$) and consider $\mathbf{Z}^n$ as a subgroup of $\tilde{W}$. Let

$$\mathrm{Adm}(\mu) = \{w \in \tilde{W};\ w \leq \mu' \ \text{ for some } \mu' \in S_n\mu\}$$

be the *μ-admissible set* ([KR]). For a non-empty subset $\bar{I}$ let $\mathrm{Adm}_{\bar{I}}(\mu)$ be the image of $\mathrm{Adm}(\mu)$ in $\tilde{W}^{\bar{I}} \setminus \tilde{W}/\tilde{W}^{\bar{I}}$. We note that by [KR] this coincides with the *μ-permissible subset* of $\tilde{W}^{\bar{I}} \setminus \tilde{W}/\tilde{W}^{\bar{I}}$. Let $X(\mu,F)_{\bar{I}}$ be the union of the generalized Deligne-Lusztig varieties of type $\bar{I}$ corresponding to elements in $\mathrm{Adm}_{\bar{I}}(\mu)$.

Our second main result in this paper is the following theorem.

Theorem B: *Let $\mu = \omega_r = (1^r, 0^{n-r})$ for some $0 \leq r \leq n$.*

(i) For any non-empty subset $\bar{I} \subset \mathbf{Z}/n\mathbf{Z}$,

$$X(\mu,F)_{\bar{I}} \neq \emptyset \text{ if and only if } \mu \geq \boldsymbol{\nu}(N,F)$$

(ii) For any non-empty subsets $\bar{I}$ and $\bar{J}$ of $\mathbf{Z}/n\mathbf{Z}$ with $\bar{J} \subset \bar{I}$ the forgetful map

$$X(\mu,F)_{\bar{I}} \longrightarrow X(\mu,F)_{\bar{J}}$$

is surjective.

This is the content of Proposition 1.1, which concerns the group GL_n. Proposition 2.1 is the analogous statement for the group GSp_{2n}, i.e. for isocrystals with a symplectic structure. In section 3 we formulate the general problem. Section 4 is devoted to the proof of Theorem A, for GL_n and GSp_{2n}. In section 5 we treat the groups $R_{F'/F}\mathrm{GL}_n$ and $R_{F'/F}\mathrm{GSp}_{2n}$ (restriction of scalars from a finite unramified extension). If we had proved Theorem A for all unramified reductive groups, this section could be eliminated. In section 6 we prove an auxiliary result which is then used in section 7 to extend Theorem B to the groups $R_{F'/F}\mathrm{GL}_n$ and $R_{F'/F}\mathrm{GSp}_{2n}$.

Our motivation for the results proved in this paper comes from the fact that they make it possible to reformulate in many cases the conjecture in [LR] on the reduction of Shimura varieties. Whereas in loc.cit. the concept of *admissible morphisms of Galois gerbs* was defined using the Bruhat-Tits building, it is possible to replace that condition by imposing on the corresponding element $b \in B(G)$ that it lie in the subset $B(G,\mu)$. Here $B(G,\mu)$ is the finite subset of $B(G)$ defined by the group-theoretic version of Mazur's theorem [K II], [RR]. The possibility of such a reformulation is implicitly behind the considerations in section 6 of [K II].

When we presented these results at the Raynaud conference in Paris, Fontaine pointed out to us that Theorem A was known to him earlier in a different guise (in the case of GL_n). Namely, he had established the existence of a weakly admissible filtration of type μ on the isocrystal N, provided that $\mu \geq \boldsymbol{\nu}(N,F)$. From this the existence of the lattice M follows by appealing to the theorem of Laffaille.

M. R. wishes to thank the department of mathematics of the University of Chicago for its hospitality during his visit in the fall of 2000, when the results of this paper were obtained. He also thanks the department of mathematics of the University of Minnesota for the possibility of presenting these results.

Notation.

F a finite extension of $\mathbf{Q}_p$

L the completion of the maximal unramified extension of F

O_F resp. O_L the rings of integers of F resp. L

$\pi \in O_F$ a uniformizer

$\mathbf{F}$ the residue field of O_L

σ the relative Frobenius automorphism of L/F.

We follow the tradition of denoting a σ-linear automorphism of an L-vector space by F (from "Frobenius"); there should be no danger of confusing this with the notation for the ground field F.

1. The result for GL_n

Let (N,F) be an F-isocrystal, i.e. a finite-dimensional L- vector space with a σ-linear bijective endomorphism. Let n denote the dimension of N. To the F-isocrystal (N,F) is associated its slope vector $\boldsymbol{\nu} = \boldsymbol{\nu}(F) = (\nu_1,\dots,\nu_n) \in (\mathbf{Q}^n)_+$. Here $(\mathbf{Q}^n)_+ = \{(\nu_1,\dots,\nu_n) \in \mathbf{Q}^n;\ \nu_1 \geq \nu_2 \geq \dots \geq \nu_n\}$.

Fix an integer r with $0 \leq r \leq n$. We call the F-isocrystal (N,F) *minuscule of weight* r if the slope vector $\boldsymbol{\nu} = \boldsymbol{\nu}(F)$ of (N,F) satisfies the following condition

$$(1.1) \qquad 0 \leq \nu_n \leq \dots \leq \nu_1 \leq 1 \quad , \quad \sum_{i=1}^{n} \nu_i = r \quad .$$

An equivalent condition is the following. Let ω_r be the vector $(1,\dots,1,0,\dots,0)$, where 1 is repeated r times and 0 is repeated $n-r$ times. On $(\mathbf{Q}^n)_+$ we have the usual *dominance order,* for which $\nu \leq \mu$ if and only if

$$
\begin{aligned}
\nu_1 &\leq \mu_1 \\
\nu_1 + \nu_2 &\leq \mu_1 + \mu_2 \\
&\vdots \\
\nu_1 + \ldots + \nu_{n-1} &\leq \mu_1 + \ldots + \mu_{n-1} \\
\nu_1 + \ldots + \nu_n &= \mu_1 + \ldots + \mu_n \quad .
\end{aligned} \tag{1.2}
$$

Then it is easy to see that the condition (1.1) is equivalent to the condition

$$
\boldsymbol{\nu}(F) \leq \omega_r \quad . \tag{1.3}
$$

Let $\bar{I} \subset \mathbf{Z}/n\mathbf{Z}$ be a non-empty subset and let $I \subset \mathbf{Z}$ the inverse image of $\bar{I}$ under the canonical surjection $\mathbf{Z} \to \mathbf{Z}/n\mathbf{Z}$. A *periodic lattice chain of type* $\bar{I}$ in the L-vector space N is a set $\mathbf{M}$ of O_L-lattices M_i $(i \in I)$ for which

(1.4.1) if $i < j$ in I, then $M_i \subset M_j$ with length $(M_j/M_i) = j - i$

(1.4.2) $M_{i+n} = \pi^{-1} M_i$.

In case $\bar{I} = \mathbf{Z}/n\mathbf{Z}$ we also speak of a *full periodic lattice chain.* If $\bar{I}$ consists of a single element, then a periodic lattice chain of type $\bar{I}$ is simply given by a lattice (namely M_i for the unique $i \in I$ with $0 \leq i < n$). We denote by $X_{\bar{I}}$ the set of periodic lattice chains of type $\bar{I}$.

We now fix an isocrystal (N, F) of dimension n. We denote by $X(\omega_r, F)_{\bar{I}}$ the set of periodic lattice chains of type $\bar{I}$ in N which satisfy the following condition,

$$
\text{for all } i \in I \text{ we have } \pi M_i \subset F M_i \subset M_i \text{ and } \dim_{\mathbf{F}} M_i/F M_i = r. \tag{1.5}
$$

An equivalent condition is the following. By the elementary divisor theorem we can associate to any pair of O_L-lattices M, M' in N their relative position $\mathrm{inv}(M, M') = \boldsymbol{\mu} = (\mu_1, \ldots, \mu_n) \in (\mathbf{Z}^n)_+$. Here $(\mathbf{Z}^n)_+ = \mathbf{Z}^n \cap (\mathbf{Q})_+$. Then the condition (1.5) is equivalent to

$$
\text{for all } i \in I \text{ we have } \mathrm{inv}(M_i, F M_i) = \omega_r. \tag{1.6}
$$

Also it is clear that it suffices to check the conditions (1.5) and (1.6) on a set of representatives of I mod n.

For a non-empty subset $\bar{J}$ of $\bar{I}$ there is an obvious forgetful map

$$
X(\omega_r, F)_{\bar{I}} \longrightarrow X(\omega_r, F)_{\bar{J}} \quad . \tag{1.7}
$$

We may now formulate the main result of this section.

Proposition 1.1. *(i) For any non-empty* $\bar{I} \subset \mathbf{Z}/n\mathbf{Z}$ *we have*

$$
X(\omega_r, F)_{\bar{I}} \neq \emptyset \text{ if and only if } F \text{ is minuscule of weight } r.
$$

(ii) For any non-empty subsets $\bar{I}$ *and* $\bar{J}$ *of* $\mathbf{Z}/n\mathbf{Z}$ *with* $\bar{J} \subset \bar{I}$*, the natural map* (1.7) *is surjective.*

To prove this proposition we make the following preliminary remarks.

a) Let $\bar{I}$ consist of a single element. Then the statement "$X(\omega_r, F)_I \neq \emptyset \Longrightarrow F$ is minuscule of weight r" is exactly the content of Mazur's theorem that the Hodge polygon of an F-crystal lies below the Newton polygon of its associated F-isocrystal and has the same endpoint (use the reformulations (1.3) resp. (1.6) of the relevant conditions).

b) If $X(\omega_r, F)_{\bar{I}} \neq \emptyset$ and $\bar{J}$ is a non-empty subset of $\bar{I}$, then obviously $X(\omega_r, F)_{\bar{J}} \neq \emptyset$.

Taking into account a) and b) we see that (i) in Proposition 1.1 follows from (ii) and the following lemma.

Lemma 1.2. *Let F be minuscule of weight r. Then there exists an O_L-lattice M in N with*

$$\mathrm{inv}(M, FM) = \omega_r \quad .$$

Proof. Let us first assume that F is isoclinic, i.e. $\boldsymbol{\nu}(F) = (\nu, \ldots, \nu)$ with $0 \leq \nu \leq 1$ and $n\nu = r$. In this case the F-isocrystal is uniquely determined up to isomorphism and there exists a basis $e_1, \ldots, e_n$ of N such that

$$Fe_1 = e_2,\ Fe_2 = e_3, \ldots, Fe_{n-1} = e_n,\ Fe_n = \pi^r e_1 \quad . \tag{1.8}$$

Then the following lattice is as required,

$$M = O_L \cdot \pi^{r-1} \cdot e_1 \oplus O_L \pi^{r-2} \cdot e_2 \oplus \ldots \oplus O_L \pi \cdot e_{r-1} \oplus O_L \cdot e_r \oplus \ldots \oplus O_L \cdot e_{n-1} \oplus O_L \cdot e_n \ .$$

The general case follows since the isocrystal (N, F) is the direct sum of isoclinic isocrystals (N_i, F_i) $(i = 1, \ldots, s)$ which are minuscule of weight r_i, with $\sum_{i=1}^{s} r_i = r$. □

To prove (ii) of Proposition 1.1, we may assume that $\bar{I} = \mathbf{Z}/n\mathbf{Z}$. Hence starting from $\bar{J}$ we may enlarge $\bar{J}$ by one element at a time. We are then reduced to proving the following statement.

Lemma 1.3. *Consider O_L-lattices M, M' such that*

$$M \underset{\neq}{\supset} M' \supset \pi M \quad ,$$

with $\mathrm{inv}(M, FM) = \omega_r$, $\mathrm{inv}(M', FM') = \omega_r$. *Then there exists an O_L-lattice $\tilde{M}$ such that*

$$M \supset \tilde{M} \supset M' \quad ,$$

with $\dim_{\mathbf{F}} \tilde{M}/M' = 1$ *and* $\mathrm{inv}(\tilde{M}, F\tilde{M}) = \omega_r$.

Proof. We introduce the σ^{-1}-linear operator V defined by the identity

$$VF = FV = \pi \quad . \tag{1.9}$$

Then, since F is minuscule of weight r, the condition $\mathrm{inv}(M, FM) = \omega_r$ on a lattice M is equivalent to the condition

$$FM \subset M \quad \text{and} \quad VM \subset M \quad . \tag{1.10}$$

Consider the $\mathbf{F}$-vector space $W = M/M'$ with the induced $\sigma^{\pm 1}$-linear operators $\overline{F}, \overline{V}$ which satisfy $\overline{F}\,\overline{V} = \overline{V}\,\overline{F} = 0$. By the previous remarks it suffices to find a line ℓ in W which is stable under $\overline{F}$ and $\overline{V}$. We distinguish cases.

Case 1. $\overline{F}$ is bijective.
In this case there exists an $\mathbf{F}$-basis of W consisting of $\overline{F}$-invariant vectors (Dieudonné). Let ℓ be the line generated by one of these basis vectors. Since $\overline{V} = 0$ in this case, this line is stable under $\overline{F}$ and $\overline{V}$.

Case 2. Ker $\overline{F} \neq (0)$.
The map $\overline{V}$ induces a map from Ker $\overline{F}$ to itself. If this induced map fails to be bijective, we take ℓ to be any line in its kernel. If the induced map is bijective, so that there exists a basis of Ker $\overline{F}$ consisting of $\overline{V}$-invariant vectors, then we take ℓ to be the line generated by one of the basis vectors. □

2. The result for GSp_{2n}

Let $(N, \langle\ ,\ \rangle)$ be a symplectic vector space of dimension $2n$ over L. Let F be a σ-linear bijective endomorphism of N satisfying

$$\langle Fx, Fy\rangle = c \cdot \langle x, y\rangle^{\sigma} \quad , \quad x, y \in N \tag{2.1}$$

for some fixed $c \in L^{\times}$. We call $(N, \langle\ ,\ \rangle, F)$ a *symplectic F-isocrystal.* The slope vector $\boldsymbol{\nu}(F)$ of the isocrystal (N, F) then satisfies

$$\nu_1 + \nu_{2n} = \nu_2 + \nu_{2n-1} = \ldots = \nu_n + \nu_{n+1} = d \quad , \tag{2.2}$$

where $d = \mathrm{val}(c)$ is the π-adic valuation of c. We call the symplectic F-isocrystal *minuscule of weight* r for some r with $0 \leq r \leq 2n$ if the underlying F-isocrystal is minuscule of weight r in the sense of (1.1). Note that only $r = 0$, $r = n$ and $r = 2n$ are possible and that then $d = 0, 1$ or 2 respectively.

Let $\bar{I}$ be a non-empty *symmetric* subset of $\mathbf{Z}/2n\mathbf{Z}$, i.e., invariant under multiplication by -1. Let I be the inverse image of $\bar{I}$ under the surjection $\mathbf{Z} \to \mathbf{Z}/2n\mathbf{Z}$. A periodic lattice chain M_i $(i \in I)$ of type $\bar{I}$ is called *selfdual* if there exists $d \in \mathbf{Z}$ such that

$$M_i^{\perp} = M_{-i+d\cdot 2n} \quad , \quad i \in I \quad . \tag{2.3}$$

Here for any O_L-lattice M in N we put

$$M^{\perp} = \{x \in N;\ \langle x, M\rangle \subset O_L\} \quad . \tag{2.4}$$

We denote by $X_{\bar{I}}^{G}$ the set of selfdual periodic lattice chains of type $\bar{I}$ in N. Let now $(N, \langle\ ,\ \rangle, F)$ be a symplectic F-isocrystal. For a non-empty symmetric subset $\bar{I}$ in $\mathbf{Z}/2n\mathbf{Z}$, let $X^{G}(\omega_r, F)_{\bar{I}}$ denote the set of periodic selfdual lattice chains of type $\bar{I}$ in N which lie in the set $X(\omega_r, F)_{\bar{I}}$ in the sense of (1.5) for GL_{2n}.

The following result is the analogue of Proposition 1.1 in the present context.

Proposition 2.1. *(i) For any non-empty symmetric subset $\bar{I}$ of $\mathbf{Z}/2n\mathbf{Z}$ we have*

$X^{G}(\omega_r, F)_{\bar{I}} \neq \emptyset$ if and only if the symplectic F-isocrystal $(N, \langle\ ,\ \rangle, F)$ is minuscule of weight r.

(ii) For any non-empty symmetric subsets $\bar{I}$ and $\bar{J}$ of $\mathbf{Z}/2n\mathbf{Z}$ with $\bar{J} \subset \bar{I}$, the natural map

$$X^G(\omega_r, F)_{\bar{I}} \longrightarrow X^G(\omega_r, F)_{\bar{J}}$$

is surjective.

Again, by Mazur's theorem, we infer that if $X^G(\omega_r, F)_{\bar{I}} \neq \emptyset$, then F is minuscule of weight r (in particular $r = 0$, or n, or $2n$). Conversely, assume that F is minuscule of weight r. If $r = 0$, then N admits a symplectic basis of F- invariant vectors (Dieudonné), hence defines an F-form $(N_0, \langle\ ,\ \rangle_0)$ of $(N, \langle\ ,\ \rangle)$. Any self-dual O_F-lattice in N_0 defines an element of $X^G(\omega_0, F)_{\bar{I}}$, where $\bar{I} = \{0\}$. Furthermore, the assertion (ii) of Proposition 2.1 just amounts to the fact that any selfdual periodic lattice chain may be completed to a full selfdual periodic lattice chain. This is well-known, comp. [KR], section 10. The case $r = 2n$ reduces to the previous one by replacing F by $\pi^{-1} \cdot F$. Hence from now on we may assume that F is minuscule of weight n.

Lemma 2.2. *Let $(N, \langle\ ,\ \rangle, F)$ be a symplectic F-isocrystal which is minuscule of weight n. Then there exists a selfdual O_F-lattice M such that*

$$M \supset FM \supset \pi M \quad and \quad (FM)^\perp = \pi^{-1}FM \quad .$$

In other words $M \in X^G(\omega_n, F)_{\bar{I}}$ with $\bar{I} = \{0\}$.

Proof. By hypothesis $0 \leq \nu_{2n} \leq \nu_{2n-1} \leq \ldots \leq \nu_1 \leq 1$ and

$$\nu_1 + \nu_{2n} = \nu_2 + \nu_{2n-1} = \ldots = \nu_n + \nu_{n+1} = 1 \quad .$$

From the slope decomposition of N we deduce a direct sum decomposition

$$N = N' \oplus \tilde{N} \oplus N'' \quad ,$$

where N' resp. N'' includes all slope components of slope $< 1/2$ resp. $> 1/2$ and where $\tilde{N}$ is the sum of all slope components of slope $1/2$. Then N' and N'' are totally isotropic subspaces which are in duality by $\langle\ ,\ \rangle$ and $\tilde{N}$ is orthogonal to $(N' \oplus N'')$. A selfdual lattice in $N' \oplus N''$ may be obtained by taking any O_F-lattice M' in N' and then forming $M' \oplus M''$ where

$$M'' = {M'}^\perp = \{x \in N'';\ \langle x, M' \rangle \subset O_L\} \quad .$$

Using the result of section 1 for $\mathrm{GL}_{n'}$, where $n' = \dim N'$, we are reduced to considering $\tilde{N}$, i.e., we may assume from the start that all slopes of N are equal to $1/2$. In this case the symplectic F- isocrystal is uniquely determined up to isomorphism and there exists a basis of N such that

$$\begin{aligned} &Fe_i = -e_{2n-i+1},\ Fe_{2n-i+1} = \pi e_i \quad i = 1, \ldots, n \text{ and} \\ &\langle e_i, e_j \rangle = 0,\ \langle e_{2n-i+1}, e_{2n-j+1} \rangle = 0,\ \langle e_i, e_{2n-j+1} \rangle = \delta_{ij}, \quad i, j = 1, \ldots, n \quad . \end{aligned}$$

Then the O_L-lattice M generated by $e_1, \ldots, e_{2n}$ satisfies the required conditions. $\square$

To complete the proof of Proposition 2.1, it suffices now to prove assertion (ii) in the case where $\bar{I} = \mathbf{Z}/2n\mathbf{Z}$. Enlarging $\bar{J}$ one step at a time we then reduce to the case in which $\bar{J} \subset \bar{I}$ is as in [KR], 10.2. In other words, we fix $k \in J$ such that $k+1 \notin J$ and obtain $\bar{I}$ by adding to $\bar{J}$ one or two elements, namely the class(es) of $k+1$ and $-(k+1)$ modulo $2n\mathbf{Z}$.

Let ℓ be the smallest integer in J such that $\ell > k$; thus $k < \ell \leq k + 2n$. Since $\bar{J} \subset \bar{I}$, there is a natural map

$$f : X_{\bar{I}}^{G} \longrightarrow X_{\bar{J}}^{G} \quad . \tag{2.5}$$

We are interested in the fiber $f^{-1}(\mathbf{M})$ over an element $\mathbf{M} = (M_i)_{i \in J}$ of $X_{\bar{J}}^{G}$. We associate to an element $\tilde{\mathbf{M}} = (M_i)_{i \in I}$ of $f^{-1}(\mathbf{M})$ the lattice $M := M_{k+1}$. Clearly this lattice satisfies

$$M_k \subset M \subset M_\ell \tag{2.6.1}$$

$$\dim_{\mathbf{F}} M/M_k = 1 \quad . \tag{2.6.2}$$

Lemma 2.3. *The map $\tilde{\mathbf{M}} \mapsto M$ is a bijection from $f^{-1}(\mathbf{M})$ to the set of lattices M in N satisfying* (2.6.1) *and* (2.6.2).

Proof. One way to prove the lemma would be to appeal to general results of Bruhat-Tits. In the special case at hand it is also easy to give an elementary proof, as we now do.

The map is obviously injective since $\tilde{\mathbf{M}}$ contains with M also $M^{\perp}$ and all multiples of these two lattices. To prove surjectivity we start with M satisfying (2.6.1) and (2.6.2) and have to construct $\tilde{\mathbf{M}} \in f^{-1}(\mathbf{M})$ which gives M. We imitate the proof of [KR], Lemma 10.3.

Let $\overline{P} = \bar{J} \cup \{\overline{k+1}\}$ (for $m \in \mathbf{Z}$, we write $\overline{m}$ for its class modulo $2n$). Let $\overline{Q} = -\overline{P}$. Then $\bar{I} = \overline{P} \cup \overline{Q}$; and for the inverse images P and Q of $\overline{P}$ and $\overline{Q}$ in $\mathbf{Z}$, we have $P = -Q$.

There is a unique periodic lattice chain $\mathbf{X}$ of type $\overline{P}$ such that $X_{k+1} = M$ and $X_j = M_j$ for $j \in J$. Let $d \in \mathbf{Z}$ be the unique integer such that $M_j^{\perp} = M_{-j+d\cdot 2n}$ for $j \in J$. There is a unique periodic lattice chain $\mathbf{Y}$ of type $\overline{Q}$ such that $Y_{-(k+1)} = \pi^d M^{\perp}$ and $Y_j = M_j$ for $j \in J$.

We claim that

$$p \in P \ , \ q \in Q \ , \ p \leq q \Longrightarrow X_p \subset Y_q \quad . \tag{2.7}$$

This is obvious if there exists $j \in J$ such that $p \leq j \leq q$, so we now assume the contrary. It is harmless to suppose that $p = k+1$. Then necessarily $q = \ell - 1$ and $\overline{\ell} = -\overline{k}$. Consider the $\mathbf{F}$-vector space $V = M_\ell / M_k$. Then the lattices X_p resp. Y_q correspond to subspaces U_1 resp. U_2 of V, where U_1 is of dimension one and U_2 is of codimension one. We have to show that $U_1 \subset U_2$. But on V we have the symplectic form defined by the fact that $M_\ell = \pi^r \cdot M_k^{\perp}$, where r is defined by $\ell = -k + r \cdot 2n$. Furthermore, we have $Y_{\ell-1} = \pi^r \cdot X_{k+1}^{\perp}$. Equivalently, we have $U_2 = U_1^{\perp}$ for the symplectic form on V. The claim now follows from the fact that any line in a symplectic vector space is isotropic.

We also claim that

$$p \in P \ , \ q \in Q \quad q < p \Longrightarrow Y_q \subset X_p \quad . \tag{2.8}$$

This is clear since there always exists $j \in J$ such that $q \leq j \leq p$.

Now suppose $p \in P$, $q \in Q$ and $p = q$. Then from (2.7) we have $X_p \subset Y_q$. But both are lattices which contain $M_{k-r\cdot 2n}$ for sufficiently large r and with the same index, hence $X_p = X_q$. Thus, without ambiguity, we may define the periodic lattice chain $\tilde{\mathbf{M}} = (M_i)_{i\in I}$ of type $\bar{I}$ by putting $M_i = X_i$ if $i \in P$ and $M_i = Y_i$ if $i \in Q$. It is obvious that this is indeed a selfdual lattice chain contained in $f^{-1}(\mathbf{M})$ and that $\tilde{\mathbf{M}} \mapsto M$. □

Using this lemma, the surjectivity assertion (ii) in Proposition 2.1 is reduced to the corresponding statement for GL_{2n}, which is Lemma 1.3. □

3. The general problem

Let G be a connected reductive group over F. For simplicity we assume that G splits over L. (The problem addressed in this section can be formulated without this hypothesis, but then becomes more technical and even more speculative). Let $\tilde{T}$ be a maximal split torus over L. Let $\mathcal{B} = \mathcal{B}(G_{ad}, L)$ be the Bruhat-Tits building of the adjoint group over L. To $\tilde{T}$ corresponds an apartment in $\mathcal{B}$. Let $\tilde{K}_0$ be an Iwahori subgroup of $G(L)$ corresponding to an alcove in the apartment for $\tilde{T}$. Let $\tilde{W}$ be the *Iwahori Weyl group of* $\tilde{T}$,

$$\tilde{W} = \tilde{N}(L)/\tilde{T}(L)_1 \quad . \tag{3.1}$$

Here $\tilde{N}$ denotes the normalizer of $\tilde{T}$ and $\tilde{T}(L)_1$ the maximal bounded subgroup of $\tilde{T}(L)$. Then $\tilde{T}(L)_1 = \tilde{T}(L) \cap \tilde{K}_0$. Let $\tilde{K}$ be the parahoric subgroup of $G(L)$ corresponding to a facet of the base alcove. Let

$$\tilde{W}^{\tilde{K}} = \tilde{N}(L) \cap \tilde{K}/\tilde{T}(L) \cap \tilde{K}_0 \quad . \tag{3.2}$$

Then there is a canonical bijection

$$\tilde{K} \setminus G(L)/\tilde{K} = \tilde{W}^{\tilde{K}} \setminus \tilde{W}/\tilde{W}^{\tilde{K}} \quad . \tag{3.3}$$

We therefore obtain a succession of maps whose composition will be denoted by inv,

(3.4)
$$\mathrm{inv}: G(L)/\tilde{K}\times G(L)/\tilde{K}\to G(L)\backslash(G(L)/\tilde{K}\times G(L)/\tilde{K})=\tilde{K}\backslash G(L)/\tilde{K}=\tilde{W}^{\tilde{K}}\backslash\tilde{W}/\tilde{W}^{\tilde{K}}.$$

We now fix a conjugacy class of minuscule one-parameter subgroups μ of G defined over L. We may assume that μ factors through $\tilde{T}$ and determines an orbit in $X_*(\tilde{T})$ under the conjugation action of the finite Weyl group $W = \tilde{N}(L)/\tilde{T}(L)$. Let $\mathrm{Adm}(\mu) \subset \tilde{W}$ be the *admissible subset corresponding to* μ ([KR], Introduction),

$$\mathrm{Adm}(\mu) = \{w \in \tilde{W};\ w \leq t_{\mu'},\ \text{some } \mu'\} \quad . \tag{3.5}$$

Here μ' denotes an element of the W-orbit in $X_*(\tilde{T})$ defined by μ, and $t_{\mu'}$ the corresponding element of $\tilde{W}$. In (3.5) appears the Bruhat order on $\tilde{W}$ determined by the base alcove. We denote by

$$\mathrm{Adm}_{\tilde{K}}(\mu) \subset \tilde{W}^{\tilde{K}} \setminus \tilde{W} / \tilde{W}^{\tilde{K}} \tag{3.6}$$

the image of $\mathrm{Adm}(\mu)$ under the natural projection. It is independent of the choice of $\tilde{K}_0$ contained in $\tilde{K}$. We will assume that $\tilde{K}$ is σ-invariant, or equivalently that the corresponding facet in the building is σ-invariant. Then $K = \tilde{K}^{\langle\sigma\rangle}$ is a parahoric subgroup of $G(F)$. We note that, conversely, K determines $\tilde{K}$ and the corresponding σ-invariant facet in $\mathcal{B}$.

Our final choice is an element $b \in G(L)$. We then define

$$X(\mu, b)_K = \{g \in G(L)/\tilde{K};\ \mathrm{inv}(g, b\sigma(g)) \in \mathrm{Adm}_{\tilde{K}}(\mu)\} \ . \tag{3.7}$$

Let $\tilde{K}'$ be a σ-invariant parahoric subgroup of $G(L)$ containing $\tilde{K}$. Then $K' = \tilde{K}'^{\langle\sigma\rangle}$ is a parahoric subgroup of $G(F)$ and there is a canonical projection map

$$X(\mu, b)_K \longrightarrow X(\mu, b)_{K'} \ . \tag{3.8}$$

Let $B(G)$ be the set of σ-conjugacy classes in $G(L)$ and let $[b] \in B(G)$ be the σ-conjugacy class of b. We denote by $B(G, \mu)$ the finite subset of $B(G)$ defined by the group theoretic version of Mazur's theorem ([K II], §6).

Conjecture 3.1. *(i) For any parahoric subgroup K of $G(F)$ we have*

$$X(\mu, b)_K \neq \emptyset \Longleftrightarrow [b] \in B(G, \mu) \ .$$

(ii) For any pair of parahoric subgroups $K \subset K'$ of $G(F)$, the map (3.8) is surjective.

It is not clear whether the hypothesis that μ is minuscule is indeed necessary for the statements in this Conjecture.

Let $G = \mathrm{GL}_n$. A conjugacy class of minuscule one-parameter subgroups of G is of the form $\mu = \omega_r + k \cdot \omega_n = \omega_r + k \cdot \mathbf{1}$ for a unique r with $0 \leq r < n$ and some $k \in \mathbf{Z}$. Here $\mathbf{1} = (1, 1, \dots, 1)$. The validity of Conjecture 3.1 is unchanged if μ is replaced by ω_r, so we assume this now.

The conjugacy classes of parahoric subgroups correspond in a one-to-one way to the set of non-empty subsets $\bar{I} \subset \mathbf{Z}/n\mathbf{Z}$ and the corresponding coset space may be identified with the space $X_{\bar{I}}$ of periodic lattice chains of type $\bar{I}$. Let $\mathbf{M} = (M_i)_{i \in I}$ and $\mathbf{M}' = (M'_i)_{i \in I}$ be two elements of $X_{\bar{I}}$. Then

$$\mathrm{inv}(\mathbf{M}, \mathbf{M}') \in \mathrm{Adm}_{\tilde{K}_I}(\mu) \Leftrightarrow M_i \supset M'_i \supset \pi M_i \text{ and } \dim_{\mathbf{F}} M_i/M'_i = r,\ \forall i \in I \ . \tag{3.9}$$

Indeed, this follows from the identification of $\mathrm{Adm}_{\tilde{K}_I}(\mu)$ with the μ-permissible set inside $\tilde{W}^{K_I} \setminus \tilde{W} / \tilde{W}^{K_I}$, [KR], [HN]. In fact, for any dominant coweight μ we have (comp. [HN], 9.7)

$$\mathrm{inv}(\mathbf{M}, \mathbf{M}') \in \mathrm{Perm}_{\tilde{K}_I}(\mu) \Longleftrightarrow \mathrm{inv}(M_i, M'_i) \leq \mu \ , \quad \forall i \in I \ . \tag{3.10}$$

If μ is minuscule, the inequality on the right hand side is necessarily an equality which yields the condition appearing in (3.9). These remarks imply that the results of section 1 prove Conjecture 3.1 in the case of GL_n.

Similarly, the results of section 2 prove Conjecture 3.1 in the case of $G = \mathrm{GSp}_{2n}$. In fact, in this case the μ-admissible set is the intersection of the μ-permissible set for GL_{2n} with the extended affine Weyl group of GSp_{2n}, cf. [KR], see also [HN], Prop. 9.7.

4. A converse to Mazur's inequality

In this section we let $G = \mathrm{GL}_n$ or $G = \mathrm{GSp}_{2n}$. Our aim is to prove a converse to Mazur's theorem, strengthening for these groups Prop. 4.2. of [R]. Much of our argument remains valid for an arbitrary split group with simply connected derived group.

We start with a lemma which is the group-theoretic interpretation of the first half of the proof of Lemma 1.2. Let A be a maximal split torus in G. We denote by $\pi_1(G)$ the algebraic fundamental group of G. Since G_{der} is simply connected, $\pi_1(G)$ is the factor group of $X_*(A)$ by the lattice generated by the coroots, and is a free abelian group. We denote by

$$\kappa_G : G(L) \longrightarrow \pi_1(G) \tag{4.1}$$

the homomorphism introduced in [K II]. We denote by $\tilde{K} = G(O_L)$ the special maximal bounded subgroup determined by a Chevalley form of G adapted to A.

Proposition 4.1. *Let $g \in G(L)$ and let $b \in G(L)$ be a basic element. Then the σ-conjugacy class of b meets $\tilde{K} g \tilde{K}$ if and only if $\kappa_G(g) = \kappa_G(b)$.*

Proof. One direction is trivial, since $\kappa_G(\tilde{K}) = \{0\}$ and since σ-conjugate elements have identical images under κ_G. For the converse direction we may use the Cartan decomposition of $G(L)$ to assume that $g \in A(L)$ and even $g = a \in A(F)$. Let $w \in W$ be an elliptic element, i.e. $X_*(A)^w_{\mathbf{R}} = X_*(Z^\circ_G)_{\mathbf{R}}$. Here Z°_G denotes the connected center of G. Equivalently, any w-invariant element of $A(F)$ has finite order modulo the center. Let $\dot{w} \in N_G(A)(F) \cap G(O_F)$ be a representative of w in $G(O_F)$. We claim that $a\dot{w}$ is a basic element in $G(L)$. Once this is established, we conclude from $\kappa_G(a\dot{w}) = \kappa_G(a) = \kappa_G(b)$ that $a\dot{w}$ and b are σ-conjugate ([K I], 5.6), which finishes the proof since $a\dot{w} \in \tilde{K} a \tilde{K}$.

To see that $a\dot{w}$ is basic it suffices to show that its norm under a sufficiently large finite extension F' of F contained in L is central ([K I], 4.3.). Since $a\dot{w} \in G(F)$, we have to see that a sufficiently high power of $a\dot{w}$ is central. But

$$(a\dot{w})^r = a \cdot w(a) \cdot \ldots \cdot w^{r-1}(a) \cdot \dot{w}^r \quad . \tag{4.2}$$

If r is divisible by the order of w in W we have that $aw(a) \cdot \ldots \cdot w^{r-1}(a)$ is w-invariant and hence is of finite order modulo the center. The same applies to $\dot{w}^r$ and hence our claim is proved. $\square$

In the sequel we fix a Borel subgroup $B = AU$. We denote by $X_*(A)_{\mathrm{dom}}$ resp. $X_*(A)_{\mathbf{Q},\mathrm{dom}}$ the set of dominant elements in $X_*(A)$ resp. $X_*(A) \otimes \mathbf{Q}$. Recall ([K II], 4.2) that to $b \in G(L)$ is associated its Newton point $\overline{\nu}(b) \in X_*(A)_{\mathbf{Q},\mathrm{dom}}$. We will denote by $\leq$ the usual partial order on $X_*(A)_{\mathbf{Q},\mathrm{dom}}$, i.e. $\nu \leq \nu'$ iff $\nu' - \nu$ is a non-negative linear combination of positive coroots. Note that, since the derived group of G is simply connected, the partial order induced on $X_*(A)_{\mathrm{dom}}$ is that denoted in [R] by $\overset{!}{\leq}$, i.e. $\nu \overset{!}{\leq} \nu'$ iff $\nu' - \nu$ is a non-negative *integral* linear combination of positive coroots.

Corollary 4.2. *Let $b \in G(L)$ be basic with associated Newton point $\overline{\nu} = \overline{\nu}(b) \in X_*(A)_{\mathbf{Q},\mathrm{dom}}$. Let $\mu \in X_*(A)_{\mathrm{dom}}$ with $\overline{\nu} \leq \mu$. Then there exists $h \in G(L)/\tilde{K}$ with*

$$\mathrm{inv}(h, b\sigma(h)) = \mu \quad .$$

Proof. Let $g = \pi^\mu \in A(F)$. Then $\kappa_G(g) = \kappa_G(b)$ and applying the previous proposition we find $h \in G(L)$ with $h^{-1}b\sigma(h) \in \tilde{K}\pi^\mu\tilde{K}$, as desired. $\square$

Remark 4.3. In the case of GL_n the previous construction can be made totally explicit. In this case $\pi_1(G) = \mathbf{Z}$ and any basic $b \in G(L)$ with $\kappa_G(b) = r \in \mathbf{Z}$ is σ-conjugate to the element F described by (1.8). Let $\mu \in (\mathbf{Z}^n)_+$ with $\sum_{i=1}^n \mu_i = r$. Then the lattice M spanned by the vectors

$$\pi^{\sum_{i=2}^n \mu_i} e_1, \pi^{\sum_{i=3}^n \mu_i} e_2, \ldots, \pi^{\mu_n} e_{n-1}, e_n \tag{4.3}$$

satisfies $\mathrm{inv}(M, FM) = \mu$.

Let now $P = MN$ be a parabolic subgroup containing B, where M is the unique Levi subgroup containing A. We sometimes consider M as a factor group of P. For $\mu \in X_*(A)$ we denote by $M(\mu)$ the image of $\tilde{K}\pi^\mu\tilde{K} \cap P(L)$ in $M(L)$.

Lemma 4.4. *Let $b \in M(L)$ and let $\mu \in X_*(A)$. Then the σ-conjugacy class of b in $G(L)$ meets $\tilde{K}\pi^\mu\tilde{K}$ if and only if the σ-conjugacy class of b in $M(L)$ meets $M(\mu)$.*

Proof. Assume that the σ-conjugacy class of b meets $\tilde{K}\pi^\mu\tilde{K}$. By the Iwasawa decomposition, there then exists $p \in P(L)$ with $pb\sigma(p)^{-1} \in \tilde{K}\pi^\mu\tilde{K}$. Writing $p = m.n \in M(L).N(L)$ we conclude that $mb\sigma(m)^{-1} \in M(\mu)$.

Conversely, assume there exists $m \in M(L)$ with $mb\sigma(m)^{-1} \in M(\mu)$. Hence there exists $n \in N(L)$ with $mb\sigma(m)^{-1}n \in \tilde{K}\pi^\mu\tilde{K}$. But by [K II], 3.6, the two elements $mb\sigma(m)^{-1}n$ and $mb\sigma(m)^{-1}$ are σ-conjugate by an element in $P(L)$. Hence b is σ-conjugate in $G(L)$ to an element in $\tilde{K}\pi^\mu\tilde{K}$. $\square$

Let $\mu \in X_*(A)$ and let

$$\mathcal{P}_\mu = \{\nu \in X_*(A);\ \kappa_G(\nu) = \kappa_G(\mu),\ \nu \in \mathrm{Conv}(W\mu)\} \quad . \tag{4.4}$$

Here we have denoted by $\kappa_G : X_*(A) \to \pi_1(G)$ the map which sends μ to $\kappa_G(\pi^\mu)$. Also $\mathrm{Conv}(W\mu)$ denotes the convex hull of $W\mu$ in $X_*(A) \otimes \mathbf{R}$. Note that since the derived group of G is simply connected, the first condition in (4.4) is implied by the second.

Lemma 4.5. *We have*

$$\kappa_M(M(\mu)) = \kappa_M(\mathcal{P}_\mu) \quad .$$

Proof. Let $m \in M(\mu)$ and let us prove that $\kappa_M(m) \in \kappa_M(\mathcal{P}_\mu)$. By the definition of $M(\mu)$ there exists $n \in N(L)$ with $mn \in \tilde{K}\pi^\mu\tilde{K}$. Using the Cartan decomposition of M we may write $m = k_M \cdot \pi^\nu \cdot k'_M$ with $k_M, k'_M \in \tilde{K}_M = M(O_L)$. Then $\kappa_M(m) = \kappa_M(\pi^\nu)$. Now $mn = k_M\pi^\nu \cdot n' \cdot k'_M$ with $n' \in N(L)$. Hence $\pi^\nu n' \in \tilde{K}\pi^\mu\tilde{K}$. By Satake (comp. [R]) this implies $\nu \in \mathcal{P}_\mu$.

Conversely, let $\nu \in \mathcal{P}_\mu$. Then by [R], Thm. 1.1. there exists $u \in U(L)$ such that $\pi^\nu u \in \tilde{K}\pi^\mu\tilde{K}$. Writing $u = u_M.n$ with $u_M \in U(L) \cap M(L)$ and $n \in N(L)$ we have $\pi^\nu \cdot u_M \in M(\mu)$. But the image of ν in $\pi_1(M)$ is equal to $\kappa_M(\pi^\nu u_M)$ and hence lies in $\kappa_M(M(\mu))$. □

Proposition 4.6. *Let $b \in M(L)$ be basic, and let $\mu \in X_*(A)$. The σ-conjugacy class of b in $G(L)$ meets $\tilde{K}\pi^\mu\tilde{K}$ if and only if $\kappa_M(b) \in \kappa_M(\mathcal{P}_\mu)$.*

Proof. This is a consequence of the results established so far. Indeed, the σ-conjugacy class of b in $G(L)$ meets $\tilde{K}\pi^\mu\tilde{K}$ iff the σ-conjugacy class of b in $M(L)$ meets $M(\mu)$. Now $M(\mu)$ is a union of $\tilde{K}_M$-double cosets. Applying Proposition 4.1 to each $\tilde{K}_M$-double coset (with M instead of G), we see that this holds iff $\kappa_M(b) \in \kappa_M(M(\mu))$. But by the previous lemma we may identify $\kappa_M(M(\mu))$ and $\kappa_M(\mathcal{P}_\mu)$. □

Recall that $\mu \in X_*(A)$ is called minuscule (in the large sense) if $\langle \mu, \alpha \rangle \in \{0, \pm 1\}$ for all roots α. It is well-known that κ_G induces a bijection (Bourbaki: Groupes et Algèbres de Lie, ch. VI, §2, ex. 2)

$$\{\mu \in X_*(A)_{\mathrm{dom}};\ \mu \text{ minuscule}\} \longrightarrow \pi_1(G) \quad . \tag{4.5}$$

Recall our parabolic subgroup $P = MN$. We let A_M be the maximal split torus in the center of M and let $X_M = X_*(A_M) \subset X_*(A)$. Then κ_M induces an injective map $X_M \hookrightarrow \pi_1(M)$ with finite cokernel.

Lemma 4.7. *Let $G = \mathrm{GL}_n$. Let $\mu \in X_*(A)_{\mathrm{dom}}$ be minuscule and let $x \in \pi_1(M)$. The following conditions on x are equivalent:*
(i) $x \in \kappa_M(W\mu)$
(ii) Let $\nu \in X_M \otimes \mathbf{Q}$ be the unique element mapping under $\kappa_M \otimes \mathbf{Q}$ to x. Then $\nu \in \mathrm{Conv}(W\mu)$.
If x satisfies these conditions, let $\tilde{\nu} \in X_*(A)$ be the unique M-dominant M-minuscule element mapping to x, cf. (4.5)*. Then $\tilde{\nu} \in W\mu$.*

Proof. Let $G = \mathrm{GL}_n$ and $M = M_{(m_1,\dots,m_r)}$. Since μ is minuscule, we may write $\mu = k \cdot \mathbf{1} + \omega_s$, where $0 \leq s < n$, and $k \in \mathbf{Z}$ and where we used the notation $\mathbf{1} = \omega_n = (1, \dots, 1)$. Since adding to μ an element of X_G does not affect the assertion of the lemma, we may assume $\mu = \omega_s$. But then it is obvious that

$$\kappa_M(W\mu) = \{(s_1, \dots, s_r);\ 0 \leq s_i \leq m_i,\ \Sigma_{i=1}^r s_i = s\} \quad . \tag{4.6}$$

Here we have used the identification

$$\pi_1(M) = \pi_1(\mathrm{GL}_{m_1} \times \dots \times \mathrm{GL}_{m_r}) = \mathbf{Z}^r \quad . \tag{4.7}$$

The element $\nu \in X_M \otimes \mathbf{Q}$ in (ii) is of the form

$$\nu = (\nu(1)^{m_1}, \dots, \nu(r)^{m_r}) \quad , \text{ with } m_i \cdot \nu(i) \in \mathbf{Z},\ \forall i = 1, \dots, r \quad . \tag{4.8}$$

Then $\nu \in \mathrm{Conv}(W\mu)$ iff

$$0 \le \nu(i) \le 1,\ \forall i = 1, \dots, r, \text{ and } \Sigma_{i=1}^r m_i \cdot \nu(i) = s \quad . \tag{4.9}$$

It is therefore obvious that by letting ν vary over this set, its image in $\pi_1(M)$ is equal to $\kappa_M(W\mu)$. If $x = (s_1, \dots, s_r) \in \kappa_M(W\mu)$, then the element $\tilde{\nu}$ is equal to

$$\tilde{\nu} = (1^{s_1}, 0^{m_1 - s_1}, 1^{s_2}, 0^{m_2 - s_2}, \dots, 1^{s_r}, 0^{m_r - s_r}) \quad ,$$

which obviously lies in $W\omega_s$. □

Proposition 4.8. *Let $G = \mathrm{GL}_n$ or $G = \mathrm{GSp}_{2n}$. Let $\nu \in X_{M,\mathbf{Q}} \cap X_*(A)_{\mathbf{Q},\mathrm{dom}}$ such that its image under $\kappa_M \otimes \mathbf{Q}$ lies in $\pi_1(M)$. Let $\tilde{\nu} \in X_*(A)$ be the unique M-dominant M-minuscule element with $\kappa_M(\tilde{\nu}) = \kappa_M(\nu)$. Let $[\tilde{\nu}]$ be the unique G-dominant element in $W\tilde{\nu}$. Then we have, for every $\mu \in X_*(A)_{\mathrm{dom}}$ with $\nu \le \mu$,*

$$\nu \le [\tilde{\nu}] \le \mu \quad .$$

Proof. The first inequality is obvious since, ν being central in M, we have

$$\nu \in \mathrm{Conv}(W_M\tilde{\nu}) \subset \mathrm{Conv}(W\tilde{\nu}) \quad .$$

Now we prove the second inequality. First we consider the case in which $G = \mathrm{GL}_n$, $M = M_{(m_1, \dots, m_r)}$. We note that if μ is minuscule, then by the previous lemma we have $\tilde{\nu} \in W\mu$. Hence $[\tilde{\nu}] = \mu$, which proves the proposition in this case. We now proceed by induction on n. As in the previous proof we write $\nu = (\nu(1)^{m_1}, \dots, \nu(r)^{m_r})$.

Assume first that there exists a maximal proper parabolic $P' = M'N'$ containing P such that $\nu \le^{M'} \mu$. This last condition means equivalently

$$\nu \in \mathrm{Conv}(W_{M'}\mu) \Longleftrightarrow \mu - \nu \in \sum_{\alpha^\vee \in \Delta_{M'}^\vee} \mathbf{R}_+ \alpha^\vee \quad .$$

Here for any standard Levi subgroup M, $\Delta_M^\vee$ denotes the set of simple coroots appearing in $U \cap M$.

The Levi subgroup M' corresponds to the partition $(\sum_{j=1}^k m_j, \sum_{j=k+1}^r m_j)$ for some k with $1 \le k \le r-1$. Let us subdivide the interval $[0, n]$ into the r subintervals $I(i)$, $i = 1, \dots, r$, with

$$I(1) = [0, m_1], I(1) = [m_1, m_1 + m_2], \dots, I(r) = [m_1 + \ldots + m_{r-1}, n] \quad . \tag{4.10}$$

Since $\nu \le^{M'} \mu$ we have $\kappa_{M'}(\nu) = \kappa_{M'}(\mu)$ which means

$$\sum_{j=1}^k m_j \nu(j) = \sum_{i \in I(1) \cup \ldots \cup I(k)} \mu_i \quad , \tag{4.11}$$

where we adopt the convention that $\mu_0 = 0$ in order to make sense of the right side of this equation. The converse is also true by the following lemma applied to $M' = M$.

Lemma 4.9. *Let $\nu \leq \mu$ and $\kappa_M(\nu) = \kappa_M(\mu)$. Then $\nu \leq^M \mu$.*

Proof. We have by assumption

$$\mu - \nu = \sum_{\alpha^\vee \in \Delta_G^\vee} c_\alpha \alpha^\vee \quad , \quad c_\alpha \in \mathbf{R}_+ \quad .$$

We want to show that $c_\alpha = 0$, $\forall \alpha^\vee \in \Delta_G^\vee \setminus \Delta_M^\vee$. But $\Delta_G^\vee \setminus \Delta_M^\vee$ maps to a basis of $X_{M,\mathbf{Q}}/X_{G,\mathbf{Q}}$. Since $\kappa_M(\mu - \nu) = 0$, we deduce

$$0 = \kappa_M(\mu - \nu) \equiv \sum_{\alpha^\vee \in \Delta_G^\vee \setminus \Delta_M^\vee} c_\alpha \alpha^\vee \bmod X_{G,\mathbf{Q}} \quad . \qquad \square$$

We will also need the following lemma.

Lemma 4.10. *Let $\mu \in X_*(A)_{\mathrm{dom}}$. Let $\nu \in X_*(A)$ be M-dominant with $\nu \leq^M \mu$. Then ν is G-dominant and $\nu \leq \mu$.*

Proof. We have

$$\mu - \nu = \sum_{\alpha^\vee \in \Delta_M^\vee} c_\alpha \alpha^\vee \quad , \quad c_\alpha \in \mathbf{R}_+ \quad .$$

Let $\beta \in \Delta_G \setminus \Delta_M$. Then $\langle \alpha^\vee, \beta \rangle \leq 0$, $\forall \alpha^\vee \in \Delta_M^\vee$. Hence

$$\langle \nu, \beta \rangle = \langle \mu, \beta \rangle - \sum_{\alpha \in \Delta_M^\vee} c_\alpha \cdot \langle \alpha^\vee, \beta \rangle \geq 0 \quad .$$

Finally, it follows trivially from $\nu \leq^M \mu$ that $\nu \leq \mu$. $\square$

We apply this lemma to M' and the unique M'-dominant element $[\tilde{\nu}]_{M'}$ in $W_{M'} \cdot \tilde{\nu}$. It satisfies $[\tilde{\nu}]_{M'} \leq^{M'} \mu$ by induction hypothesis. By the previous lemma we therefore have $[\tilde{\nu}]_{M'} = [\tilde{\nu}] \leq \mu$ which proves the proposition in this case.

Let us now assume that there is no proper Levi subgroup M' containing M such that $\nu \leq^{M'} \mu$. By Lemma 4.9 this means that for $k = 1, \ldots, r-1$ we have

$$\sum_{j=1}^{k} m_j \cdot \nu(j) < \sum_{i \in I(1) \cup \ldots \cup I(k)} \mu_i \quad . \tag{4.12}$$

In this case we are going to prove the assertion by induction on the height of μ. If μ is minuscule, the assertion is already proved. Otherwise there exists a positive coroot $\alpha^\vee$ such that $\mu' = \mu - \alpha^\vee$ is dominant. It suffices to show that $\nu \leq \mu'$ since then by induction hypothesis we have $[\tilde{\nu}] \leq \mu'$ and hence a fortiori $[\tilde{\nu}] \leq \mu$.

To prove $\nu \leq \mu'$ we introduce the partial sum functions for $i = 0, \ldots, n$,

$$\begin{aligned} N_i &= \sum_{\ell=1}^{i} \nu_\ell = \langle \nu, \omega_i \rangle, \quad M_i = \sum_{\ell=1}^{i} \mu_\ell = \langle \mu, \omega_i \rangle \quad , \\ M_i' &= \sum_{\ell=1}^{i} \mu_\ell' = \langle \mu', \omega_i \rangle \quad . \end{aligned} \tag{4.13}$$

We have to show that $N_i \leq M_i'$, $\forall i = 1, \ldots, n$, knowing that $N_i \leq M_i$, $\forall i = 1, \ldots, n$. These functions of i may be interpolated into continuous functions on

$[0,n]$ which are affine-linear on consecutive intervals $[0,1],[1,2],\dots$ and which are convex, since ν,μ and μ' are all dominant. Furthermore, the function N is affine-linear on the intervals $I(1),I(2),\dots,I(r)$. Hence it suffices to check that

$$N(x) \leq M'(x) \tag{4.14}$$

for the endpoints x of the intervals $I(1),I(2),\dots,I(r)$. At the left endpoint of $I(1)$ and right endpoint of $I(r)$ we have equality in (4.14). Now consider the remaining endpoints. By (4.12) we have $N(x) < M(x)$. Since both arguments are integers we conclude that $N(x) \leq M(x)-1$. On the other hand, since the positive coroot $\alpha^\vee$ is of the form $\alpha^\vee = e_i - e_j$ for $i<j$ (where $e_1,\dots,e_n$ is the natural basis of $X_*(A) = \mathbf{Z}^n$), it is obvious that $M'(x) \geq M(x)-1$, which proves (4.14) in this case. This completes the proof for GL_n.

Now let us assume that $G = \mathrm{GSp}_{2n}$ and M is the Levi subgroup obtained as the intersection (inside GL_{2n}) of GSp_{2n} and $M_{(m_1,\dots,m_r,2j,m_r,\dots,m_1)}$. The second equality that we need to prove for G and M follows from that same inequality for GL_{2n} and $M_{(m_1,\dots,m_r,2j,m_r,\dots,m_1)}$. To see this one simply checks that GSp_{2n} inherits all relevant concepts (minuscule, dominant, $\leq$) from GL_{2n}, and that the same is true for the two Levi subgroups). □

Another way of formulating the previous proposition is that $[\tilde\nu]$ is the unique minimal element of the set

$$\{\mu \in X_*(A)_{\mathrm{dom}};\ \nu \leq \mu\} \quad . \tag{4.15}$$

We can now prove the main result of this section.

Theorem 4.11. *Let $G = \mathrm{GL}_n$ or $G = \mathrm{GSp}_{2n}$. Let $b \in G(L)$, with associated Newton point $\nu = \overline{\nu}(b) \in X_*(A)_{\mathbf{Q},\mathrm{dom}}$. Let $\mu \in X_*(A)_{\mathrm{dom}}$ with $\nu \leq \mu$. Then the σ-conjugacy class of b in $G(L)$ meets $\tilde K \pi^\mu \tilde K$. Equivalently, there exists $h \in G(L)/\tilde K$ with $\mathrm{inv}(h, b\sigma(h)) = \mu$.*

Proof. After replacing b by a σ-conjugate, we may assume that $b \in M(L)$ is basic, for a standard Levi subgroup M [K I], 6.2. By Proposition 4.6 we have to show that $\kappa_M(b)$ lies in the image of $\mathcal{P}_\mu$ in $\pi_1(M)$. By Proposition 4.8. we find $\tilde\nu \in \mathcal{P}_\mu$ with $\kappa_M(\tilde\nu) = \kappa_M(b)$. □

By Mazur's inequality we may summarize the previous theorem as an equality of two subsets of $X_*(A)_{\mathrm{dom}}$: Given $b \in G(L)$ we have

$$\begin{aligned} &\{\mu \in X_*(A)_{\mathrm{dom}};\ \exists h \in G(L)/\tilde K \text{ with } \mathrm{inv}(h, b\sigma(h)) = \mu\} \\ =&\{\mu \in X_*(A)_{\mathrm{dom}};\ \overline{\nu}(b) \leq \mu\} \quad . \end{aligned} \tag{4.16}$$

Furthermore, by (4.15) this subset has a unique minimal element.

Remark 4.12. Let $b \in M(L)$ be basic such that M is the centralizer of $\nu = \overline{\nu}(b)$ (i.e. b is G-regular, comp. [K I], 6.2.; recall that for any σ-conjugacy class $[b]$ there exists a standard Levi subgroup M and an element $b \in [b]$ which satisfies these conditions). Let $\mu \in X_*(A)_{\mathrm{dom}}$ such that $mb\sigma(m)^{-1} \in \tilde K_M \pi^\mu \tilde K_M$ for some $m \in M(L)$. It follows that $\nu \leq^M \mu$, or, equivalently by Lemma 4.9, that $\nu \leq \mu$ and $\kappa_M(\nu) = \kappa_M(\mu)$. Conversely let $\mu \in X_*(A)_{\mathrm{dom}}$ such that $\nu \leq^M \mu$. Assume furthermore that there exists $g \in G(L)$ with $g^{-1}b\sigma(g) \in \tilde{K}\pi^\mu\tilde{K}$ and fix such an element. *Then, if $G = \mathrm{GL}_n$ or $G = \mathrm{GSp}_{2n}$, it follows that $g \in M(L)\cdot\tilde{K}$.*

Indeed, assume that $G = \mathrm{GL}_n$ and $M = M_{(m_1,\dots,m_r)}$. The isocrystal $(N, F) = (L^n, b\sigma)$ has slope vector

$$\nu = (\nu(1)^{m_1}, \dots, \nu(r)^{m_r}) \quad ,$$

where $\nu(1) > \nu(2) > \ldots > \nu(r)$. This chain of inequalities follows from the assumption that M is the centralizer of ν (equivalently, the break points of the Newton polygon of (N, F) occur at $m_1, m_1+m_2, \dots, m_1+\ldots+m_{r-1}$). On the other hand, $g\tilde{K}$ defines a lattice Λ in N such that $\mu(\Lambda) = \mathrm{inv}(\Lambda, F\Lambda) = \mu$. Now the assumption $\nu \leq^M \mu$ tells us that the Hodge polygon of Λ goes through all break points of the Newton polygon. Hence, by the Hodge-Newton decomposition [Ka], Thm. 1.6.1., we can write

$$\Lambda = \bigoplus_{i=1}^{r} \Lambda_i \quad , \tag{4.17}$$

where $\Lambda_i = \Lambda \cap N_i$ is the intersection with the isotypic component of slope $\nu(i)$ of N. If the lattice Λ_i corresponds to $g_i \cdot \mathrm{GL}_{m_i}(O_L)$, then $g\tilde{K} = m \cdot \tilde{K}$ where $m = (g_1, \dots, g_r) \in \prod_{i=1}^r \mathrm{GL}_{m_i}(L) = M(L)$ which proves the claim in this case. The case where $G = \mathrm{GSp}_{2n}$ is similar.

It seems likely that the above conclusion holds for more general groups. But we point out that the assumption that μ be G-dominant is essential; it is not enough to merely assume that μ is M-dominant with $\nu \leq^M \mu$, as the following example shows.

Let $G = \mathrm{GL}_3$, $M = M_{(2,1)}$ and

$$b = \begin{pmatrix} 0 & \pi^a & 0 \\ \pi^{a+1} & 0 & 0 \\ 0 & 0 & \pi^a \end{pmatrix} \quad , \tag{4.18}$$

where $a > 0$ is a fixed integer. Then $\nu = \overline{\nu}(b) = (a + \frac{1}{2}, a + \frac{1}{2}, a)$. Let $\mu = (2a+1, 0, a)$. Then μ is M-dominant but not G-dominant and $\nu \leq^M \mu$. We claim that there exists an element $g \in G(L) \setminus M(L).\tilde{K}$ with $g^{-1}b\sigma(g) \in \tilde{K}\pi^\mu\tilde{K}$. Let

$$b' = \begin{pmatrix} 0 & \pi^a & 0 \\ \pi^{a+1} & 0 & 0 \\ 0 & 1 & \pi^a \end{pmatrix} \quad . \tag{4.19}$$

Then $\overline{\nu}(b') = \nu$, hence b' is σ-conjugate to b and $\mathrm{inv}(O_L^3, b'\sigma(O_L^3)) = [\mu]$, hence $b' \in \tilde{K}\pi^\mu\tilde{K}$. But an element $g \in G(L)$ with $g^{-1}b\sigma(g) = b'$ lies in $M(L)\tilde{K}$ if and only if O_L^3 is decomposable with respect to the slope decomposition of L^3 for $b'\sigma$. It is easy to see that O_L^3 is not decomposable.

5. Restriction of scalars.

Let F' be an unramified field extension of degree f of F. Let V be a F'- vector space of dimension n. In the first part of this section we will be concerned with the group $G = R_{F'/F}(\mathrm{GL}(V))$. Let $\tilde{K} \subset G(L)$ be a special maximal parahoric subgroup defined over F. The coset space $G(L)/\tilde{K}$ can be described as follows.

We fix an embedding $F' \to L$. Then we can write

$$V \otimes_F L = \bigoplus_{j \in \mathbf{Z}/f\mathbf{Z}} N_j \quad , \tag{5.1}$$

with $N_j = \{v \in V \otimes_F L;\ (x \otimes 1) \cdot v = (1 \otimes \sigma^{-j}(x)) \cdot x,\ \forall x \in F'\}$.
Each summand is an L-vector space of dimension n. The coset space $G(L)/\tilde{K}$ parametrizes lattices for $O_{F'} \otimes_{O_F} O_L$ in $V \otimes_F L$, or equivalently $\mathbf{Z}/f\mathbf{Z}$-graded O_L-lattices,

$$\tilde{M} = \bigoplus_{j \in \mathbf{Z}/f\mathbf{Z}} \tilde{M}_j \quad , \tag{5.2}$$

where each $\tilde{M}_j$ is an O_L-lattice in N_j.

Next we fix a conjugacy class of one-parameter subgroups of G. Under the decomposition

$$G \otimes_F L = \prod_{j \in \mathbf{Z}/f\mathbf{Z}} \mathrm{GL}(N_j) \tag{5.3}$$

this corresponds to an f-tuple of dominant cocharacters of GL_n,

$$\boldsymbol{\mu} = (\mu_j)_{j \in \mathbf{Z}/f\mathbf{Z}} \quad , \quad \mu_j \in (\mathbf{Z}^n)_+ \quad , \quad \forall j = 1, \ldots, f \quad . \tag{5.4}$$

Finally, let $b \in G(L)$ and consider the σ-linear operator

$$F = b \cdot (\mathrm{id}_V \otimes \sigma) \tag{5.5}$$

on $V \otimes_F L$. Then deg $F = 1$ with respect to the grading (5.1). We introduce the set

$$\overset{\circ}{X}(\boldsymbol{\mu}, b)_K = \{\tilde{M} = \bigoplus_{j \in \mathbf{Z}/f\mathbf{Z}} \tilde{M}_j;\ \ \mathrm{inv}\ (\tilde{M}, F\tilde{M}) = \boldsymbol{\mu}\} \quad . \tag{5.6}$$

The last condition is equivalent to $\mathrm{inv}(\tilde{M}_j, F\tilde{M}_{j-1}) = \mu_j$, $\forall j \in \mathbf{Z}/f\mathbf{Z}$, where each invariant is considered as an element of $(\mathbf{Z}^n)_+$. Note that if $\mu_1, \ldots, \mu_f$ are all minuscule, then by the minimality of minuscule elements with respect to the partial order $\overset{!}{\leq}$ on $(\mathbf{Z}^n)_+$, the set $\overset{\circ}{X}(\boldsymbol{\mu}, b)_K$ coincides with the set $X(\mu, b)_K$ of section 3.

Theorem 5.1. *We have*

$$\overset{\circ}{X}(\boldsymbol{\mu}, b)_K \neq \emptyset \Longleftrightarrow [b] \in B(G, \boldsymbol{\mu}) \quad .$$

We note that the direct implication is just the group theoretic version of Mazur's theorem which was proved in [RR]. To make the set $B(G, \boldsymbol{\mu})$ more explicit, we note the Shapiro bijection [K II], 6.5.3.

$$B(G, \boldsymbol{\mu}) = B(G', \mu') \quad . \tag{5.7}$$

Here $G' = \mathrm{GL}(V)$ is defined over F' and

$$\mu' = \sum_j \mu_j \quad . \tag{5.8}$$

The map is obtained by associating to the σ-linear operator (5.5) of $V \otimes_F L$ the σ^f- linear operator on $N_0 = V \otimes_{F'} L$,

$$F^f : N_0 \longrightarrow N_0 \quad . \tag{5.9}$$

The condition that $[b] \in B(G, \boldsymbol{\mu})$ is equivalent to the condition that the slope vector $\boldsymbol{\nu} = \boldsymbol{\nu}(F^f) \in (\mathbf{Q}^n)_+$ be smaller than μ'. Let us now fix $b \in G(L)$ satisfying this condition and let us construct an element in $\overset{\circ}{X}(\boldsymbol{\mu}, b)_K$. Let $\tilde{M} = \bigoplus_j \tilde{M}_j$ be any $\mathbf{Z}/f\mathbf{Z}$-graded lattice and put

$$M_j = F^j \tilde{M}_{-j} \quad , \quad j = 0, \ldots, f \quad . \tag{5.10}$$

Then M_j is a lattice in N_0 and we obtain the following description of $\overset{\circ}{X}(\boldsymbol{\mu}, b)_K$:

$$\begin{aligned} \overset{\circ}{X}(\boldsymbol{\mu}, b)_K = &\{(M_0, M_1, \ldots, M_f);\ M_f = F^f M_0, \\ &\mathrm{inv}(M_j, M_{j+1}) = \mu_j,\ \forall j = 0, \ldots, f-1\} \end{aligned} \tag{5.11}$$

We now apply Theorem 4.11 to the isocrystal (V_0, F^f). Since $\boldsymbol{\nu} \leq \mu'$ we obtain the existence of a lattice M_0 in N_0 such that

$$\mathrm{inv}(M_0, F^f M_0) = \mu' \quad . \tag{5.12}$$

We put $M_f = F^f M_0$. To complete the proof of Theorem 5.1, it therefore remains to fill in the remaining lattices $M_1, \ldots, M_{f-1}$. That this can be done follows from the following well-known lemma.

Lemma 5.2. *Let $\mu_0, \ldots, \mu_{f-1} \in (\mathbf{Z}^n)_+$ be dominant vectors and let $\mu = \sum \mu_j$. Let M_0, M_f be lattices with $\mathrm{inv}(M_0, M_f) = \mu$. Then there exists a collection of lattices $M_1, M_2, \ldots, M_{f-1}$ such that $\mathrm{inv}(M_j, M_{j+1}) = \mu_j$, $\forall j = 0, \ldots, f-1$.* □

Now let $(V, \langle\ ,\ \rangle)$ be a symplectic vector space of dimension $2n$ over F' and let $G = R_{F'/F}(\mathrm{GSp}(V, \langle\ ,\ \rangle))$. Let $\tilde{K} \subset G(L)$ be a special maximal parahoric subgroup defined over F. The coset space $G(L)/\tilde{K}$ parametrizes $\mathbf{Z}/f\mathbf{Z}$-lattices $\tilde{M} = \bigoplus_{j \in \mathbf{Z}/f\mathbf{Z}} \tilde{M}_j$ which are selfdual with respect to $\langle\ ,\ \rangle \otimes L$ up to a scalar in $F' \otimes_F L$. Since the summands in (5.1) are orthogonal to one another, we may write

$$\tilde{M} = \bigoplus_j \tilde{M}_j \ , \ \text{where } \tilde{M}_j^\perp = c_j \cdot \tilde{M}_j,\ c_j \in L,\ \forall j \in \mathbf{Z}/f\mathbf{Z} \quad . \tag{5.13}$$

A conjugacy class of one-parameter subgroups of G corresponds to an f-tuple of dominant cocharacters of GSp_{2n},

$$\boldsymbol{\mu} = (\mu_j)_{j \in \mathbf{Z}/f\mathbf{Z}} \quad , \quad \mu_j \in X_*(A)_{\mathrm{dom}} \quad . \tag{5.14}$$

Here $X_*(A)$ denotes the cocharacter module for GSp_{2n} and $X_*(A)_{\mathrm{dom}} = X_*(A) \cap (\mathbf{Z}^{2n})_+$.

Finally, let $b \in G(L)$, with associated σ-linear operator $F = b \cdot (\mathrm{id}_V \otimes \sigma)$ on $V \otimes_F L$. We introduce the set

$$(5.15) \qquad \overset{\circ}{X}(\boldsymbol{\mu}, b)_K = \{\tilde{M} = \bigoplus_{j \in \mathbf{Z}/f\mathbf{Z}} \tilde{M}_j;\ \tilde{M}_j^\perp = c_j M_j \text{ for some } c_j \in L,\ \forall j, \\ \mathrm{inv}(\tilde{M}_j, \tilde{M}_{j+1}) = \mu_j,\ \forall j\} \ .$$

We introduce as before the lattices $M_j = F^j \tilde{M}_{-j}$ for $j = 0, \ldots, f$. Then, since b is a symplectic similitude, it follows that each lattice M_j is self-dual up to a scalar. We therefore may identify $\overset{\circ}{X}(\boldsymbol{\mu}, b)_K$ with

$$(5.16)\ \overset{\circ}{X}(\boldsymbol{\mu}, b)_K = \{(M_0, \ldots, M_f);\ M_f = F^f M_0,\ M_j^\perp = c_j M_j,\ \forall j, \\ \mathrm{inv}(M_j, M_{j+1}) = \mu_j,\ \forall j = 0, \ldots, f-1\} \ .$$

Theorem 5.3. *We have*

$$\overset{\circ}{X}(\boldsymbol{\mu}, b)_K \neq \emptyset \Longleftrightarrow [b] \in B(G, \boldsymbol{\mu}) \ .$$

We only sketch the proof which is analogous to the proof of Theorem 5.1. Again the direct implication follows from [RR]. To see the reverse implication, let us assume that $[b] \in B(G, \boldsymbol{\mu})$, or equivalently, that the slope vector of $F^f : N_0 \to N_0$ is smaller than $\mu' = \sum_j \mu_j$. An application of Theorem 4.11 shows that there exists a lattice M_0 which is selfdual up to a scalar such that $\mathrm{inv}(M_0, F^f M_0) = \mu'$. We put $M_f = F^f M_0$. Applying Lemma 5.2 we find a chain of lattices $M_0, M_1, \ldots, M_{f-1}$ such that $\mathrm{inv}(M_j, M_{j+1}) = \mu_j$ for $j = 0, \ldots, f-1$. But M_0 is selfdual up to a scalar and $\mu_0, \ldots, \mu_{f-1} \in X_*(A)_{\mathrm{dom}}$; this implies successively that $M_1, M_2, \ldots, M_{f-1}$ are all selfdual up to a scalar. Hence we have indeed found an element of $\overset{\circ}{X}(\mu, b)_K$. $\square$

6. An incidence variety

Let k be an algebraically closed field of characteristic p. We fix a positive integer f. For each $i \in \mathbf{Z}/f\mathbf{Z}$ we fix a vector space W_i, all of the same dimension $m > 0$. Furthermore, for each $i \in \mathbf{Z}/f\mathbf{Z}$ we fix a semi-linear map $\varphi_i : W_{i-1} \to W_i$ with respect to some automorphism σ_i of k and a semi-linear map $\psi_i : W_i \to W_{i-1}$ with respect to some automorphism τ_i of k. We assume that σ_i and τ_i are all powers (positive, negative, or zero) of the Frobenius automorphism of k. We impose the conditions

$$(6.1) \qquad \psi_i \circ \varphi_i = 0 \quad , \quad \varphi_i \circ \psi_i = 0 \quad , \quad \forall i \in \mathbf{Z}/f\mathbf{Z} \quad .$$

We might picture these data in a circular diagram. Whenever you turn back while traveling through this diagram you are killed (Orpheus condition).

$$
\begin{tikzcd}[column sep=small]
 & & W_1 \arrow[dll, shift left, "\psi_1"] \arrow[drr, shift left, "\varphi_2"] & & \\
W_0 \arrow[urr, shift left, "\varphi_1"] \arrow[d, shift left, "\psi_0"] & & & & W_2 \arrow[ull, shift left, "\psi_2"] \arrow[d, shift left] \\
\vdots \arrow[u, shift left, "\varphi_0"] & & & & \vdots \arrow[u, shift left]
\end{tikzcd}
\tag{6.2}
$$

The aim of the present section is to prove the following theorem.

Theorem 6.1. *There exists a collection of lines $\ell_i \subset W_i$ $(i \in \mathbf{Z}/f\mathbf{Z})$ such that*

$$\varphi_i(\ell_{i-1}) \subset \ell_i \quad , \quad \psi_i(\ell_i) \subset \ell_{i-1} \quad , \quad \forall i \in \mathbf{Z}/f\mathbf{Z} \quad .$$

Before starting the proof we make some comments. In the case $f = 1$ we are given a vector space $W \neq (0)$ and two semi-linear endomorphisms φ and ψ of W such that $\varphi\psi = \psi\varphi = 0$. In this case we are looking for a line ℓ in W which is carried into itself under φ and ψ. This is essentially the situation considered in the proof of Lemma 1.3 where the existence of such a line is established. In the case $f = 2$ we are looking at a diagram

$$
\begin{tikzcd}[column sep=large]
W_0 \arrow[r, shift left=3, "\varphi_1"] \arrow[r, shift right=1, "\psi_0"'] & W_1 \arrow[l, shift right=1, "\psi_1"'] \arrow[l, shift left=3, "\varphi_0"]
\end{tikzcd}
\tag{6.3}
$$

We are searching for a pair of lines (ℓ_0, ℓ_1) which are incident under $\varphi_0, \psi_0, \varphi_1, \psi_1$. In this case it is again possible to establish the existence of such a pair of lines by pure linear algebra. But already in this case a large number of case distinctions has to be made and this approach quickly gets out of hand for a larger number of vector spaces. Instead we use a density argument together with induction on f to reduce the problem to a special case that can be treated directly.

Proof of Theorem 6.1. The special case goes as follows. Put

$$\Phi := \varphi_f \varphi_{f-1} \ldots \varphi_2 \varphi_1, \tag{6.4}$$

a semilinear endomorphism of W_0, and assume that there exists a line ℓ_0 in W_0 such that $\Phi\ell_0 = \ell_0$. For $i = 1, \ldots, f-1$ define a line ℓ_i in W_i by $\ell_i := \varphi_i\varphi_{i-1} \ldots \varphi_2\varphi_1\ell_0$. Then $\varphi_i\ell_{i-1} = \ell_i$ and $\psi_i\ell_i = 0$ for all $i \in \mathbf{Z}/f\mathbf{Z}$, so this collection of lines solves our problem.

The following reduction technique will be needed in the induction on f. Suppose $f > 1$ and that there exists j such that ψ_j is bijective, in which case φ_j is automatically 0. Given any family of lines ℓ_i solving our problem, we must have

$\ell_{j-1} = \psi_j \ell_j$. Using ψ_j to identify W_{j-1} with W_j, and discarding the two maps ψ_j, $\varphi_j = 0$, we are left with $f-1$ vector spaces $\ldots, W_{j-2}, W_{j-1} \simeq W_j, W_{j+1}, \ldots$ and maps $\varphi_i, \psi_i \quad (i \neq j)$ between them. In other words we have a new problem of the same kind as our old one, but with f decreased by 1. There is an obvious bijection between solutions of the old and new problems.

The idea of the density argument is to fix f, W_i, σ_i, τ_i and then to consider the space M of all possible families of maps φ_i, ψ_i satisfying condition (6.1). More precisely, for any finite dimensional k-vector spaces W, W' and any integral power τ of Frobenius, we denote by $\mathrm{Hom}_\tau(W, W')$ the k-vector space of τ-linear maps $\varphi : W \to W'$, with scalar multiplication by $\alpha \in k$ defined by $(\alpha\varphi)(w) = \alpha(\varphi(w))$ (for all $w \in W$). Returning to our fixed data f, W_i, σ_i, τ_i, we now put

$$H_i := \mathrm{Hom}_{\sigma_i}(W_{i-1}, W_i) \times \mathrm{Hom}_{\tau_i}(W_i, W_{i-1}), \tag{6.5}$$

a finite dimensional k-vector space which we regard as a k-variety. Inside H_i we have the closed subvariety

$$M_i := \{(\varphi_i, \psi_i) \in H_i : \psi_i\varphi_i = 0 \text{ and } \varphi_i\psi_i = 0\}. \tag{6.6}$$

Finally we put $M := \prod_{i \in \mathbf{Z}/f\mathbf{Z}} M_i$, the space of all families of maps φ_i,ψ_i satisfying (6.1), which we are now regarding as a (reducible) algebraic variety over k.

Writing $\mathbf{P}_i$ for the projective space of lines in W_i, and writing $\mathbf{P}$ for the product $\mathbf{P} := \prod_{i \in \mathbf{Z}/f\mathbf{Z}} \mathbf{P}_i$, we consider the total incidence variety $I \subset M \times \mathbf{P}$ consisting of $(\varphi_i, \psi_i)_{i \in \mathbf{Z}/f\mathbf{Z}} \in M$ and $(\ell_i)_{i \in \mathbf{Z}/f\mathbf{Z}} \in \mathbf{P}$ such that

$$\varphi_i \ell_{i-1} \subset \ell_i \text{ and } \psi_i \ell_i \subset \ell_{i-1} \tag{6.7}$$

for all $i \in \mathbf{Z}/f\mathbf{Z}$. It is easy to see that I is closed in $M \times \mathbf{P}$, hence that the projection map $\pi : I \to M$ is proper. Thus $M' := \pi(M)$ is closed in M, and since Theorem 6.1 can be reformulated as the statement that $M' = M$, it is enough to show that M' is dense in M. For this we need a better understanding of the irreducible components of M.

Recall that all the vector spaces W_i have the same dimension $m > 0$. For any family $\mathbf{r} = (r_i)_{i \in \mathbf{Z}/f\mathbf{Z}}$ of integers r_i such that $0 \le r_i \le m$, we denote by $M_\mathbf{r}$ the subset of M consisting of families $(\varphi_i, \psi_i)_{i \in \mathbf{Z}/f\mathbf{Z}}$ such that $\mathrm{rank}(\varphi_i) = r_i$ for all $i \in \mathbf{Z}/f\mathbf{Z}$. (As usual $\mathrm{rank}(\varphi_i)$ is the dimension of the image of φ_i.) Thus M has been decomposed into finitely many locally closed subsets $M_\mathbf{r}$, and it is not hard to see that each subset $M_\mathbf{r}$ is irreducible. (In the linear case, *i.e.* when σ_i, τ_i are the identity, the projection map $(\varphi_i, \psi_i)_{i \in \mathbf{Z}/f\mathbf{Z}} \to (\varphi_i)_{i \in \mathbf{Z}/f\mathbf{Z}}$ makes $M_\mathbf{r}$ into a vector bundle over a homogeneous space for a product of general linear groups; in general $M_\mathbf{r}$ is homeomorphic to such a vector bundle, and is therefore irreducible.)

For each $\mathbf{r}$ as above we are going to define a non-empty open subset $U_\mathbf{r}$ of $M_\mathbf{r}$ such that $U_\mathbf{r} \subset M'$. This will show that M' is dense, as desired.

We define $U_\mathbf{r}$ to be the subset consisting of $(\varphi_i, \psi_i)_{i \in \mathbf{Z}/f\mathbf{Z}} \in M_\mathbf{r}$ satisfying the following two open conditions. The first is that $\mathrm{rank}(\psi_i) = m - r_i$ for all i (an open condition since $\mathrm{rank}(\psi_i) \le m - r_i$ follows from $\varphi_i\psi_i = 0$).

To formulate the second condition we again consider $\Phi = \varphi_f \varphi_{f-1} \ldots \varphi_2 \varphi_1$, the semilinear endomorphism of W_0 that appeared in our earlier discussion of the special case of the theorem. Note that $\mathrm{rank}(\Phi) \le r_{\min} := \min\{r_i : i \in \mathbf{Z}/f\mathbf{Z}\}$. Let

$\Phi' : \operatorname{im}\Phi \to \operatorname{im}\Phi$ denote the restriction of Φ to the image of Φ in W_0. The second condition is that $\operatorname{rank}(\Phi) = r_{\min}$ and that the map Φ' be invertible. This is again an open condition on $(\varphi_i, \psi_i)_{i \in \mathbf{Z}/f\mathbf{Z}} \in M_{\mathbf{r}}$.

We claim that $U_{\mathbf{r}}$ is non-empty. Choose a basis in each vector space W_i, so that we can represent the semilinear maps φ_i, ψ_i by $m \times m$ matrices. Write E_s for the $m \times m$ diagonal matrix $1^s 0^{m-s}$ and F_s for the $m \times m$ diagonal matrix $0^s 1^{m-s}$. Then put $\varphi_i = E_{r_i}\sigma_i$, $\psi_i = F_{r_i}\tau_i$. Clearly $(\varphi_i, \psi_i)_{i \in \mathbf{Z}/f\mathbf{Z}}$ lies in $U_{\mathbf{r}}$.

It remains to check that $U_{\mathbf{r}} \subset M'$. In other words for $(\varphi_i, \psi_i)_{i \in \mathbf{Z}/f\mathbf{Z}} \in U_{\mathbf{r}}$ we must show that there exists a solution to the problem of finding lines $\ell_i \subset W_i$ such that

$$\varphi_i \ell_{i-1} \subset \ell_i, \quad \psi_i \ell_i \subset \ell_{i-1}.$$

There are two cases.

Suppose first that $r_{\min} = 0$, so that there exists $j \in \mathbf{Z}/f\mathbf{Z}$ such that $r_j = 0$. Thus $\varphi_j = 0$ and it follows from the first open condition that ψ_j is bijective. In this case we are done by induction on f, as discussed earlier.

Now suppose that $r_{\min} > 0$. By the second open condition $\operatorname{im}\Phi \neq 0$ and the restriction of Φ to $\operatorname{im}\Phi$ is bijective. Therefore there exists a line ℓ_0 in $\operatorname{im}\Phi \subset W_0$ such that $\Phi\ell_0 = \ell_0$. From the special case treated directly at the beginning of this proof we know that suitable lines ℓ_i do exist, and thus the proof the theorem is complete. $\square$

Remark 6.2 (O. Gabber): The conclusion of Theorem 6.1 is not true without any hypotheses on the algebraically closed field k and the automorphisms σ_i and τ_i of k. Indeed, in the case when all ψ_i are zero, the theorem asserts the existence of an eigenvector of the semi-linear map $\Phi = \varphi_f \varphi_{f-1} \cdots \varphi_2 \varphi_1$. However, such an eigenvector need not exist in general.

7. General parahoric subgroups

In this section we will prove Conjecture 3.1 in the cases when $G = R_{F'/F}(\mathrm{GL}_n)$ or $G = R_{F'/F}(\mathrm{GSp}_{2n})$, where F' is an unramified extension of F.

We start with the first group. We recall some notation from section 5. Let F' be an unramified extension of degree f of F. Let V be a F'-vector space of dimension n. After fixing an embedding $F' \to L$, we have a decomposition (comp. (5.1)),

$$V \otimes_F L = \bigoplus_{j \in \mathbf{Z}/f\mathbf{Z}} N_j \quad . \tag{7.1}$$

Let $\bar{I} \subset \mathbf{Z}/n\mathbf{Z}$ be a non-empty subset. As in section 1 we denote by I the inverse image of $\bar{I}$ in $\mathbf{Z}$. A $\mathbf{Z}/f\mathbf{Z}$*-graded periodic lattice chain of type* $\bar{I}$ is a set of $\mathbf{Z}/f\mathbf{Z}$-graded O_L-lattices, one for each $i \in I$,

$$\tilde{M}^i = \bigoplus_{j \in \mathbf{Z}/f\mathbf{Z}} \tilde{M}^i_j \quad . \tag{7.2}$$

Here $\tilde{M}^i_j = \tilde{M}^i \cap N_j$. We require that, for fixed j the lattices M^i_j form a periodic lattice chain of type $\bar{I}$ in N_j, in the sense of (1.4). We denote by $X^G_{\bar{I}}$ the set of $\mathbf{Z}/f\mathbf{Z}$-graded periodic lattice chains of type $\bar{I}$. The conjugacy classes of parahoric subgroups of $G(L)$ defined over F are in one-to-one correspondence with the non-empty subsets $\bar{I}$ of $\mathbf{Z}/n\mathbf{Z}$. If $\tilde{K}$ is of type $\bar{I}$ we may identify $G(L)/\tilde{K}$ with $X^G_{\bar{I}}$.

We fix integers r_j with $0 \leq r_j \leq n$, $\forall j \in \mathbf{Z}/f\mathbf{Z}$. We denote by $\boldsymbol{\mu} = (\mu_j)_{j \in \mathbf{Z}/f\mathbf{Z}}$ the corresponding minuscule dominant coweight of G, with $\mu_j = \omega_{r_j}$.

Let $b \in G(L)$. Then b defines the σ-linear operator $F = b \cdot (\mathrm{id}_V \otimes \sigma)$ on $V \otimes_F L$, which is of degree 1 for the grading (7.1). Taking into account the identification of the μ-admissible set with the μ-permissible set for GL_n (compare the end of section 3) we may identify the set $X(\mu, b)_K$ of (3.7) in case K is of type $\bar{I}$ with the following set

$$X(\boldsymbol{\mu}, b)_{\bar{I}} = \{(\tilde{M}^i_j)_{j,i} \in X^G_{\bar{I}};\ \tilde{M}^i_j \supset F\tilde{M}^i_{j-1} \supset \pi\tilde{M}^i_j, \quad \dim_{\mathbf{F}} \tilde{M}^i_j / F\tilde{M}^i_{j-1} = r_j,\ \forall j \in \mathbf{Z}/f\mathbf{Z},\ \forall i \in I\} \ . \tag{7.3}$$

For each $i \in I$ and each j with $0 \leq j \leq f$ let us set $M^i_j = F^j \tilde{M}^i_{-j}$. Let $N = N_0$. Then for fixed j, the lattices $(M^i_j)_i$ form a periodic lattice chain of type $\bar{I}$ in N. In section 1 we denoted the set of periodic lattice chains of type $\bar{I}$ in N by $X_{\bar{I}}$. Let us continue to do so. We therefore obtain from an element of $X^G_{\bar{I}}$ an f-tuple of elements of $X_{\bar{I}}$. We see that in this way we may identify $X(\boldsymbol{\mu}, b)_K$ with the following set

$$X(\boldsymbol{\mu}, b)_{\bar{I}} = \{(M^i_j)_j \in X^f_{\bar{I}};\ M^i_f = F^f M^i_0, \quad M^i_j \supset M^i_{j+1} \supset \pi M^i_j,\ \dim_{\mathbf{F}} M^i_j / M^i_{j+1} = r_j \ , \quad \forall i \in I,\ j = 0, \ldots, f-1 \ \} \ . \tag{7.4}$$

Theorem 7.1. *Conjecture 3.1 holds for $G = R_{F'/F}(\mathrm{GL}(V))$.*

We proceed as in the proof of Proposition 1.1. If $\bar{I}$ consists of a single element, the statement (i) in Conjecture 3.1 follows from Theorem 5.1. Again in statement (ii) it suffices to deal with the case when K is an Iwahori subgroup, and this is then reduced to proving the surjectivity of the map

$$X(\boldsymbol{\mu}, b)_{\bar{I}} \longrightarrow X(\boldsymbol{\mu}, b)_{\bar{J}} \tag{7.5}$$

when $\bar{J} \subset \bar{I}$ differ by one element. It therefore suffices to show the following analogue of Lemma 1.3.

Lemma 7.2. *Consider a commutative diagram of inclusions of lattices in N,*

$$\begin{array}{ccccccccccc}
M_0 & \supset & M_1 & \supset & \ldots & \supset & M_{f-1} & \supset & M_f & = & F^f M_0 \\
\cup & & \cup & & & & \cup & & \cup & & \\
M'_0 & \supset & M'_1 & \supset & \ldots & \supset & M'_{f-1} & \supset & M'_f & = & F^f M'_0 \\
\cup & & \cup & & & & \cup & & \cup & & \\
\pi M_0 & \supset & \pi M_1 & \supset & \ldots & \supset & \pi M_{f-1} & \supset & \pi M_f & = & \pi F^f M_0
\end{array} \ ,$$

where $M_j \supset M_{j+1} \supset \pi M_j$ and $M'_j \supset M'_{j+1} \supset \pi M'_j$ with

$$\dim_{\mathbf{F}} M_j / M_{j+1} = \dim_{\mathbf{F}}\ M'_j / M'_{j+1} = r_j \quad \textit{for } j = 0, 1, \ldots, f-1 \ .$$

Assume that $M_j \neq M'_j$ for one j, or equivalently, for all j. Then there exists a chain of lattices

$$L_0 \supset L_1 \supset \ldots \supset L_{f-1} \supset L_f = F^f L_0$$

with the following properties
a) $L_j \supset L_{j+1} \supset \pi L_j$ *with* $\dim_{\mathbf{F}} L_j/L_{j+1} = r_j$, *for* $j = 0, \ldots, f-1$
b) $M'_j \subset L_j \subset M_j$ *with* $\dim_{\mathbf{F}} L_j/M'_j = 1$, *for* $j = 0, \ldots, f-1$.

Proof. Consider for $j = 0, \ldots, f-1$ the $\mathbf{F}$-vector space

$$W_j = M_j/M'_j \quad . \tag{7.6}$$

These vector spaces have all the same dimension ≥ 1. The inclusions $M_{j+1} \subset M_j$, resp. multiplication by π induce linear maps ψ resp. φ,

$$W_j \underset{\psi}{\overset{\varphi}{\rightleftarrows}} W_{j+1} \qquad j = 0, \ldots, f-2 \quad . \tag{7.7}$$

Similarly $F^f : M_0 \to M_{f-1}$ and $\pi \cdot (F^f)^{-1} : M_{f-1} \to M_0$ induce σ^f-linear resp. σ^{-f}-linear maps ψ resp. φ

$$W_0 \underset{\psi}{\overset{\varphi}{\leftrightarrows}} W_{f-1} \quad . \tag{7.8}$$

It is obvious that we obtain in this way a diagram of the form (6.2) which satisfies all hypotheses of Theorem 6.1. We infer the existence of lines $(\ell_j \subset W_j)_j$ which are incident under the system of maps φ and ψ. Let $L_j \subset M_j$ be the inverse image of ℓ_j, for $j = 0, \ldots, f-1$. Then we obtain a chain $L_0 \supset L_1 \supset \ldots \supset L_{f-1} \supset L_f = F^f L_0$ which has the required properties. □

Variant 7.3. Let F' be an unramified extension of degree f of F and let D be a division algebra with center F'. Let V be a D-vector space of dimension m. Let $G = R_{F'/F}(\mathrm{GL}_D(V))$. Then G is an inner form of $R_{F'/F}(\mathrm{GL}_n)$, where $n = md$ with $d^2 = \dim_{F'} D$. Since G is not quasisplit, Mazur's inequality and its converse do not apply directly. Still we will show by reduction to the case of $R_{F'/F}(\mathrm{GL}_n)$ that conjecture 3.1 holds also in this case. To simplify notations let us restrict ourselves to the case $f = 1$, i.e., $F' = F$.

Let O_D be the maximal order in D. Let $\tilde{F}$ be an unramified extension of F of degree d in D. Then we may write O_D as

$$O_D = O_{\tilde{F}}[\Pi]/(\Pi \cdot a = a^{\sigma^s} \cdot \Pi \quad \forall a \in O_{\tilde{F}} \quad , \quad \Pi^d = \pi) \quad . \tag{7.9}$$

Here Π is a uniformizer of O_D and s is inverse to the invariant of D in $\mathbf{Z}/d\mathbf{Z}$. Let us fix an embedding $\tilde{F} \to L$. Then we obtain an eigenspace decomposition analogous to (7.1),

$$V \otimes_F L = \bigoplus_{j \in \mathbf{Z}/d\mathbf{Z}} N_j \quad . \tag{7.10}$$

Each L-vector space N_j is of dimension n and deg $\Pi = s$ with respect to this grading. Put $N = N_0$. Let $\tilde{K} \subset G(L)$ be a parahoric subgroup maximal among those defined over F. Then $G(L)/\tilde{K}$ parametrizes the lattices M in $V \otimes_F L$ which are $O_D \otimes_{O_F} O_L$-invariant. Such a lattice is a free module of rank m over $O_D \otimes_{O_F} O_L$. We associate to M the periodic lattice chain in N,

$$\mathcal{L}(M) = \ldots \supset M_0 \supset \Pi M_{-s} \supset \Pi^2 M_{-2s} \supset \ldots \supset \Pi^d M_0 = \pi M_0 \supset \ldots \quad . \tag{7.11}$$

Here $M_j = M \cap N_j$, so that $M = \bigoplus_j M_j$. Then $\mathcal{L}(M)$ is a periodic lattice chain of type $\bar{I} = \{0, m, \ldots, (d-1)m\} \subset \mathbf{Z}/n\mathbf{Z}$. In this way we obtain a bijection

$$G(L)/\tilde{K} = X_{\bar{I}} \simeq \mathrm{GL}_n(L)/\tilde{K}_{\bar{I}} \quad . \tag{7.12}$$

Here $K_{\bar{I}}$ is a parahoric subgroup of GL_n defined over F and we have implicitly chosen a basis of the L-vector space N. More generally, we obtain a bijection between the sets of conjugacy classes of parahoric subgroups of $G(F)$ and the non-empty subsets $\bar{I} \subset \mathbf{Z}/n\mathbf{Z}$ which are invariant under the translation action $x \mapsto x+\overline{m}$ on $\mathbf{Z}/n\mathbf{Z}$. If K corresponds to $\bar{I}$, then again

$$G(L)/\tilde{K} = X_{\bar{I}} \simeq \mathrm{GL}_n(L)/\tilde{K}_{\bar{I}} \quad .$$

Let $b \in G(L)$. Then b defines the σ-linear operator $F = b \cdot (\mathrm{id}_V \otimes \sigma)$ on $V \otimes_F L$. The relation between the Newton point of b and the slope vector of F is given by

$$\boldsymbol{\nu}(F) = (\nu(b)^d) \quad . \tag{7.13}$$

Here $\nu(b)^d \in (\mathbf{Q}^{nd})_+$ is the vector which repeats d times each entry of $\nu(b) \in (\mathbf{Q}^n)_+$. Let $\tilde{K} \subset G(L)$ be a parahoric subgroup maximal among those defined over F. We identify $\tilde{W}^K \backslash \tilde{W} / \tilde{W}^K$ with $\tilde{W}^{\bar{I}} \backslash \tilde{W} / \tilde{W}^{\bar{I}}$, where $\bar{I} = m \cdot \mathbf{Z}/n\mathbf{Z}$ and where $\tilde{W}^{\bar{I}} = \tilde{W}^{\tilde{K}_{\bar{I}}}$. (Something analogous holds for any parahoric subgroup of $G(L)$ defined over F). Let $g, g' \in G(L)/\tilde{K}$. Let M and M' be the corresponding $O_D \otimes_{O_F} O_L$-stable lattices in $V \otimes_F L$, with corresponding decompositions $M = \bigoplus_{j \in \mathbf{Z}/d\mathbf{Z}} M_j$ and $M' = \bigoplus_{j \in \mathbf{Z}/d\mathbf{Z}} M'_j$ and corresponding periodic lattice chains $\mathcal{L}(M)$ and $\mathcal{L}(M')$. Let $\mu \in (\mathbf{Z}^n)_+$ be a dominant cocharacter of G. Then

$$\begin{aligned} \mathrm{inv}(g, g') \in \mathrm{Adm}_{\tilde{K}}(\mu) &\Longleftrightarrow \mathrm{inv}(\mathcal{L}(M), \mathcal{L}(M')) \in \mathrm{Adm}_{\tilde{K}_{\bar{I}}}(\mu) \\ &\Longleftrightarrow \mathrm{inv}(M_j, M'_j) \leq \mu \quad , \quad \forall j \in \mathbf{Z}/d\mathbf{Z} \quad . \end{aligned} \tag{7.14}$$

Assume now that $g' = b\sigma(g)$. Comparing the lattices M and M' in $V \otimes_F L$, we obtain from (7.14) that

$$\mathrm{inv}(M, FM) \leq (\mu^d) \quad . \tag{7.15}$$

(Here inv denotes the relative position of two lattices in $V \otimes_F L \simeq L^{nd}$.) By Mazur's inequality we infer that $\boldsymbol{\nu}(F) \leq (\mu^d)$. By (7.13) this implies that $\nu(b) \leq \mu$.

This proves one implication of statement (i) in Conjecture 3.1. The remaining assertions of the Conjecture follow from the case of GL_n and the preceding remarks connecting the case at hand to the case of GL_n. $\square$

We now turn to the case $G = R_{F'/F}(\mathrm{GSp}_{2n})$. We now let V be an F'-vector space of dimension $2n$ equipped with a non-degenerate symplectic form $\langle\ ,\ \rangle$. Let $G = R_{F'/F}(\mathrm{GSp}(V, \langle\ ,\ \rangle))$. The decomposition (7.1) is an orthogonal sum decomposition with respect to $\langle\ ,\ \rangle$. Hence each summand N_j is a symplectic vector space of dimension $2n$ over L.

The conjugacy classes of parahoric subgroups of $G(L)$ defined over F are in one-to-one correspondence with the non-empty symmetric subsets $\bar{I}$ of $\mathbf{Z}/2n\mathbf{Z}$. A $\mathbf{Z}/f\mathbf{Z}$-graded periodic lattice chain $(\tilde{M}_j^i)_{i,j}$ of type $\bar{I}$ is called selfdual, if for each $j \in \mathbf{Z}/f\mathbf{Z}$ the periodic lattice chain $\tilde{M}_j$ of type $\bar{I}$ is selfdual in the sense of (2.3). We denote by $X_{\bar{I}}^G$ the set of $\mathbf{Z}/f\mathbf{Z}$-graded selfdual periodic lattice chains of type $\bar{I}$. If $\tilde{K}$ is a parahoric subgroup of type $\bar{I}$ defined over F, we may identify the coset space $G(L)/\tilde{K}$ with $X_{\bar{I}}^G$.

We fix integers $r_j \in \{0, n, 2n\}$, $\forall j \in \mathbf{Z}/f\mathbf{Z}$. We denote by $\boldsymbol{\mu} = (\mu_j)_{j\in\mathbf{Z}/f\mathbf{Z}}$ the corresponding minuscule dominant coweight of G, with $\mu_j = \omega_{r_j}$.

Let $b \in G(L)$. Then b defines the σ-linear operator $F = b\cdot(\mathrm{id}_V \otimes \sigma)$ on $V \otimes_F L$. It is of degree 1 with respect to the grading (7.1) and there are scalars $c_j \in L$, $\forall j \in \mathbf{Z}/f\mathbf{Z}$ such that

$$\langle Fv, Fw\rangle = c_j \cdot \langle v, w\rangle \quad , \quad v, w \in N_j \quad . \tag{7.16}$$

We have the following description of the set $X(\mu, b)_K$ of (3.7) in case K is of type $\bar{I}$,

$$X^G(\boldsymbol{\mu}, b)_{\bar{I}} = \{(\tilde{M}_j^i)_{i,j} \in X_{\bar{I}}^G;\ (\tilde{M}_j^i) \in X(\boldsymbol{\mu}, b)_{\bar{I}}\} \quad . \tag{7.17}$$

Here $X(\boldsymbol{\mu}, b)_{\bar{I}}$ is the set (7.3) for the group $R_{F'/F}(\mathrm{GL}(V))$. In other words, the elements of $X^G(\boldsymbol{\mu}, b)_{\bar{I}}$ are the $\mathbf{Z}/f\mathbf{Z}$-graded selfdual periodic lattice chains $(\tilde{M}_j^i)$ of type $\bar{I}$ which satisfy

$$\tilde{M}_j^i \supset F\tilde{M}_{j-1}^i \supset \pi\tilde{M}_j^i \quad ; \quad \dim\, \tilde{M}_j^i/F\tilde{M}_{j-1}^i = r_j \ ,\ \forall i \in I \ ,\ \forall j \in \mathbf{Z}/f\mathbf{Z} \quad . \tag{7.18}$$

Theorem 7.4. *Conjecture 3.1 holds for $G = R_{F'/F}(\mathrm{GSp}(V, \langle\ ,\ \rangle))$.*

We proceed as in the proof of Proposition 2.1. If $\bar{I} = \{0\}$, the statement (i) of Conjecture 3.1 follows from Theorem 5.3. In statement (ii) it suffices to deal with the case when K is an Iwahori subgroup, and this is then reduced to proving the surjectivity of the map

$$X^G(\boldsymbol{\mu}, b)_{\bar{I}} \longrightarrow X^G(\boldsymbol{\mu}, b)_{\bar{J}} \quad , \tag{7.19}$$

in the situation considered in (2.5). In other words,

$$\bar{I} = \bar{J} \cup \{\overline{k} + \overline{1}, -(\overline{k} + \overline{1})\} \quad , \text{ where } \overline{k} \in \overline{J} \text{ with } \overline{k} + \overline{1} \notin \bar{J} \ .$$

We choose as in (2.5) a representative k of $\overline{k}$ in $\mathbf{Z}$ and denote by ℓ the next largest element in J. The assertion reduces to the corresponding statement for GL_{2n} (Lemma 7.2) with the following variant of Lemma 2.3.

Lemma 7.4. *Let $\bar{J} \subset \bar{I}$ as above. Let $(\tilde{M}_j^i)$ be a $\mathbf{Z}/f\mathbf{Z}$-graded selfdual periodic lattice chain of type $\bar{J}$. The set of refinements of $(\tilde{M}_j^i)$ into a $\mathbf{Z}/f\mathbf{Z}$-graded selfdual periodic lattice chain of type $\bar{I}$ is in one-to-one correspondence with the set of $\mathbf{Z}/f\mathbf{Z}$-graded lattices $(\tilde{M}_j)_{j\in\mathbf{Z}/f\mathbf{Z}}$ with the property that*

$$\tilde{M}_j^k \subset \tilde{M}_j \subset \tilde{M}_j^\ell \quad \textit{and} \quad \dim_{\mathbf{F}} \tilde{M}_j/\tilde{M}_j^k = 1, \quad \forall j \in \mathbf{Z}/f\mathbf{Z} \quad .$$

Proof. This follows from Lemma 2.3 applied to each direct summand N_j, $j \in \mathbf{Z}/f\mathbf{Z}$.

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