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Minuscule alcoves for GL_n and GSp_{2n}

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Abstract. Let μ be a minuscule coweight for either GL_n or GSp_{2n}, and regard μ as an element t_μ in the extended affine Weyl group $\tilde{W}$. We say that an element $x \in \tilde{W}$ is μ-admissible if there exists μ' in the Weyl group orbit of μ such that $x \leq t_{\mu'}$ in the Bruhat order on $\tilde{W}$. Our main result is that $x \in \tilde{W}$ is μ-admissible if and only if it is μ-permissible, where μ-permissibility is defined using inequalities arising naturally in the study of bad reduction of Shimura varieties.

Introduction

The main result of this paper concerns the extended affine Weyl group $\tilde{W}$ for GL_n. The group $\tilde{W}$ is the semidirect product of the symmetric group S_n and the group $\mathbf{Z}^n$; this group acts on $\mathbf{R}^n$, with S_n acting by permutations of the coordinates and with $\mathbf{Z}^n$ acting by translations.

Let d be an integer in the range $0 \leq d \leq n$ and let ω_d denote the vector $(1, \dots, 1, 0, \dots, 0) \in \mathbf{Z}^n$ in which 1 is repeated d times and 0 is repeated $n - d$ times.

Let $x \in \tilde{W}$. We say that x is *d-admissible* if there exists $\tau \in S_n$ such that x is less than or equal to $\tau(\omega_d) \in \mathbf{Z}^n \subset \tilde{W}$ in the Bruhat order on $\tilde{W}$.

Again let $x \in \tilde{W}$. For integers i in the range $0 \leq i \leq n-1$ we let v_i denote the vector in $\mathbf{Z}^n$ obtained by applying the affine transformation $x : \mathbf{R}^n \to \mathbf{R}^n$ to the vector $\omega_i \in \mathbf{Z}^n \subset \mathbf{R}^n$. We say that x is *minuscule* if

$$0 \leq v_i(m) - \omega_i(m) \leq 1 \qquad \forall i \in \{0, 1, \dots, n-1\} \quad \forall m \in \{1, 2, \dots, n\},$$

where $v_i(m)$ and $\omega_i(m)$ denote the m-th entries of the vectors v_i, ω_i respectively. We define the *size* of $x \in \tilde{W}$ to be the sum of the entries of the vector v_0. We say that x is *d-permissible* if it is minuscule of size d.

The main result of this paper, Theorem 3.5(3), states that $x \in \tilde{W}$ is d-admissible if and only if it is d-permissible. It is easy to see that d-admissibility implies

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d-permissibility; the converse seems to be harder. Theorem 4.5(3) is a completely analogous result for the group $G = GSp_{2n}$ of symplectic similitudes.

Let us now rephrase our results in a way that makes sense for any extended affine Weyl group $\tilde{W}$. Recall that $\tilde{W}$ has the form $W \ltimes X$, where W is the finite Weyl group and X is the lattice of cocharacters. Inside X is the coroot lattice X_0, and there is a canonical surjective homomorphism

$$c : \tilde{W} \to X/X_0$$

that is trivial on W and induces the canonical surjection from X to X/X_0.

Let $\mu \in X$. When we regard μ as an element in the translation subgroup of $\tilde{W}$, we sometimes denote it by t_μ. We say that $x \in \tilde{W}$ is *μ-admissible* if there exists $\tau \in W$ such that $x \le t_{\tau(\mu)}$ in the Bruhat order on $\tilde{W}$.

Let P_μ denote the convex hull in $X_{\mathbf{R}} := X \otimes_{\mathbf{Z}} \mathbf{R}$ of the points $\tau(\mu)$ as τ ranges through the finite Weyl group W. Let $\bar{\mathbf{a}}$ denote the closure of the base alcove $\mathbf{a}$ in $X_{\mathbf{R}}$; thus

$$\bar{\mathbf{a}} = \{v \in X_{\mathbf{R}} : \tilde{\alpha}(v) \ge 0 \text{ for all simple affine roots } \tilde{\alpha}\}.$$

We say that $x \in \tilde{W}$ is *μ-permissible* if $c(x) = c(t_\mu)$ and $xv - v \in P_\mu$ for all $v \in \bar{\mathbf{a}}$ (equivalently, for all "vertices" v of $\bar{\mathbf{a}}$, where by a "vertex" we mean an element lying in a face of minimal dimension).

When G is GL_n or GSp_{2n} and μ is a minuscule coweight, Theorems 3.5(3) and 4.5(3) say that x is μ-admissible if and only if it is μ-permissible. In the case of GL_n the point is that for $\mu = \omega_d$, the convex hull P_μ is equal to the set of vectors $(x_1, x_2, \dots, x_n) \in \mathbf{R}^n$ such that $x_1 + x_2 + \cdots + x_n = d$ and $0 \le x_i \le 1$ for all $i \in \{1, 2, \dots, n\}$; similar considerations apply to GSp_{2n} and ω_n (see 12.4). Also, it is easy to check that μ-admissibility always implies μ-permissibility (see 11.2 for a proof). T. Haines pointed out to us that in light of Deodhar's results in [D] it seems unlikely that μ-permissibility always implies μ-admissibility, as we had first hoped. It would be interesting to clarify the situation.

The results and the methods of this paper are purely combinatorial, but the origin of the problems considered here is the study of the bad reduction of certain Shimura varieties. Indeed, consider the *local model* associated to the triple (G, μ, K) consisting of the algebraic group $G = GL_n$ over $\mathbf{Q}_p$, the minuscule coweight ω_d, and an Iwahori subgroup K of G, ([RZ, G1]). Then the special fiber of this local model has a natural stratification (into affine Schubert cells) whose strata are parametrized by the μ-permissible subset of the extended affine Weyl group $\tilde{W}$ for GL_n. Similar remarks apply to GSp_{2n}.

The concepts of μ-admissibility and μ-permissibility also play a role in the problem of determining the function in the Iwahori-Hecke algebra which describes the trace of the Frobenius on the sheaf of nearby cycles of a local model. We refer the reader to [G2, H1, H2, HN] for more details.

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1. Review of the Bruhat order

1.1. Notation

In this section we consider the Weyl group W of a root system R, which can be either an ordinary root system or an affine root system. We choose an order on R, so that $R = R_+ \coprod R_-$, where R_+ denotes the set of positive roots, and we denote by B the set of simple positive roots. For $\alpha \in R$ we denote by w_α the reflection in the root α. The map $\alpha \mapsto w_\alpha$ sets up a bijection from R_+ to the set T of reflections in W, and it also sets up a bijection from B to the set S of simple reflections in W. For $w \in W$ we denote by $l(w)$ the length of w with respect to the base B. We will recall from [1,2,4] the definition and properties of the Bruhat order. Note that [1, 2] concern ordinary Weyl groups, but we will cite only results that apply equally well to affine Weyl groups.

1.2. Definition of Bruhat order

Let $x, y \in W$. We write $x \to y$ if $yx^{-1} \in T$ and $l(y) = l(x) + 1$. We write $x \leq y$ if there exists a chain

$$x = x_1 \to x_2 \to \cdots \to x_{k-1} \to x_k = y$$

($k = 1$ is allowed, so that $x \leq x$). The partial order $\leq$ is called the Bruhat order on W. It is clear that $x \leq y$ if and only if $x^{-1} \leq y^{-1}$.

Lemma 1.3. *Let $x, y \in W$, $s \in S$ and assume that $x \leq y$. Then*

(1) *either $sx \leq y$ or $sx \leq sy$, and*
(2) *either $x \leq sy$ or $sx \leq sy$.*

Proof. This is Lemma 2.5 of [2]. □

Lemma 1.4. *Let $x, y \in W$. Suppose that $yx^{-1} \in T$ and $l(x) < l(y)$. Then $x \leq y$.*

Proof. This is Lemma 8.11 in [1]. The idea is to prove Lemma 1.4 by induction on $l(y)$, using Lemma 1.3 (if $l(y) > 0$ pick $s \in S$ such that $sy \to y$). Note however that Lemma 1.3 was not explicitly formulated in [1]. □

Corollary 1.5. *Let $x \in W$ and let α be a positive root. Then $x^{-1}\alpha$ is a positive root if and only if $x \leq w_\alpha x$. Equivalently, $x^{-1}\alpha$ is a negative root if and only if $w_\alpha x \leq x$.*

Proof. Use Corollary 2.3(ii) in [2] in addition to Lemma 1.4. □

Lemma 1.6. *Let $x, y \in W$ and let $x = s_1 \dots s_q$ be a reduced decomposition of x. Then $y \leq x$ if and only if there exists a strictly increasing sequence $i_1 < \cdots < i_k$ (possibly empty) of integers drawn from $\{1, \dots, q\}$ such that $y = s_{i_1} \dots s_{i_k}$.*

Proof. This is Proposition 2.8 of [2]. □

Lemma 1.7. *Let $\preceq$ be a partial order on W satisfying the following three properties.*

(1) *Let $x, y \in W$. If $x \leq y$, then $x \preceq y$.*
(2) *$x \preceq e$, then $x = e$. Here e denotes the identity element of W.*
(3) *Let $x, y \in W$ and $s \in S$. Suppose that $x \preceq y$. Then either $x \preceq sy$ or $sx \preceq sy$.*

Then the partial order $\preceq$ coincides with the Bruhat order $\leq$.

Proof. Suppose that $x \preceq y$. We must show that $x \leq y$. We use induction on $l(y)$. If $l(y) = 0$, then $y = e$ and (2) yields the desired result. Now suppose that $l(y) > 0$. Then there exists $s \in S$ such that $l(sy) = l(y) - 1$ and hence $sy \leq y$. By (3) either $x \preceq sy$ or $sx \preceq sy$. By the induction hypothesis either $x \leq sy$ or $sx \leq sy$. By Lemma 1.3 either $x \leq y$ or $x \leq sy$. Since $sy \leq y$ we conclude that in any case $x \leq y$. $\square$

Note that this lemma is a variant of Proposition 2.7 in [2], but that Proposition 2.7 applies only to ordinary root systems (the proof in [2] uses the existence of a longest element in W).

1.8. Bruhat order on extended affine Weyl groups

Let G be a connected reductive group over an algebraically closed field, and let A be a maximal torus in G. The Weyl group of G is the quotient group $N(A)/A$, where $N(A)$ denotes the normalizer of A in G. The Weyl group acts on the cocharacter group $X_*(A)$, and the extended affine Weyl group $\tilde{W}$ of G is by definition the semidirect product of W and $X_*(A)$. In case G is semisimple and simply connected, the extended affine Weyl group coincides with the affine Weyl group W_a of G. In general we write G_{sc} for the simply connected covering group of the derived group of G, and we write A_{sc} for the inverse image of A in G_{sc}. Then the cocharacter group $X_*(A_{\text{sc}})$ can be identified with a subgroup of $X_*(A)$, and thus the affine Weyl group $W_a = W \ltimes X_*(A_{\text{sc}})$ can be identified with a subgroup of the extended affine Weyl group $\tilde{W} = W \ltimes X_*(A)$. In fact the subgroup W_a is normal in $\tilde{W}$, so that we get a canonical surjective homomorphism

$$\tilde{W} \to X_*(A)/X_*(A_{\text{sc}}) \tag{1.8.1}$$

whose kernel is W_a.

Pick a Borel subgroup of G containing A. It determines a set of simple roots and also a set of simple affine roots. Let C be the subgroup of $\tilde{W}$ consisting of all elements that preserve the set of simple affine roots. Then $\tilde{W}$ is the semidirect product of C and the normal subgroup W_a, and thus C maps isomorphically to $X_*(A)/X_*(A_{\text{sc}})$ under the homomorphism (1.8.1).

As usual the Bruhat order and length function on the extended affine Weyl group $\tilde{W}$ are defined as follows. Let $x, x' \in \tilde{W}$. Then x, x' can be decomposed uniquely as $x = wc$ and $x' = w'c'$ with $c, c' \in C$ and $w, w' \in W_a$. Then by definition $x \leq y$ means that $w \leq w'$ and that $c = c'$. Moreover the length $l(x)$ of x is defined to be $l(x) := l(w)$.

2. An inheritance property of the Bruhat order

2.1. Notation

We keep the notation of 1.1. In addition we now consider an automorphism θ of the root system R having the property that θ preserves the set B of simple roots. If R is an ordinary root system we require that θ come from an automorphism of the real vector space on which the roots are linear functions, and if R is an affine root system we require that θ come from an automorphism of the real affine space on which the roots are affine linear functions. Thus we are in the situation considered in §1 of [3].

2.2. Fixed point group W^θ

The automorphism θ induces an automorphism, still called θ, of the group W, and this automorphism preserves the set S of simple reflections. Recall from [3] that the fixed point group $W^\theta := \{w \in W \mid \theta(w) = w\}$ is again the Weyl group of a root system. To describe the simple reflections in the Weyl group W^θ we must first recall that for any subset $X \subset S$ the subgroup W_X of W generated by X is a Weyl group having X as its set of simple reflections. Let Δ be an orbit of θ on S and suppose that the Weyl group W_Δ is finite. Then we denote by s_Δ the unique longest element of W_Δ; note that s_Δ is fixed by θ. Then the elements s_Δ, one for each orbit Δ such that W_Δ is finite, are the simple reflections in W^θ (see [3]).

Let $x \in W^\theta$. We write $l(x)$ for the length of x as element in W, and we write $l^\theta(x)$ for the length of x as element of W^θ. For any reduced decomposition $x = s_{\Delta_1} \dots s_{\Delta_q}$ of x as element of W^θ we have

$$l(x) = \sum_{i=1}^{q} l(s_{\Delta_i}) \tag{2.2.1}$$

(this is in [3], at least implicitly).

Proposition 2.3. *The Bruhat order $\leq$ on W^θ is inherited from the Bruhat order $\preceq$ on W. In other words, for any $x, y \in W^\theta$, the conditions $x \leq y$ and $x \preceq y$ are equivalent.*

Proof. We prove this result by verifying that the partial order $\preceq$ on W^θ satisfies the three hypotheses of Lemma 1.7. First we show that if $x \leq y$, then $x \preceq y$. Let

$$y = s_{\Delta_1} \dots s_{\Delta_q} \tag{2.3.1}$$

be a reduced decomposition for y as element of W^θ. By Lemma 1.6 the element x can be written as

$$x = s_{\Delta_{i_1}} \dots s_{\Delta_{i_k}} \tag{2.3.2}$$

for some strictly increasing sequence $i_1 < \cdots < i_k$. For each i choose a reduced decomposition for s_{Δ_i} as element of W; it follows from (2.2.1) that one obtains a reduced decomposition for y as element of W by substituting these decompositions of the elements s_{Δ_i} in the decomposition (2.3.1). Substituting the same reduced decompositions for the elements s_{Δ_i} into the expression (2.3.2), we see from Lemma 1.6 that $x \preceq y$.

Thus the partial order $\preceq$ satisfies the first hypothesis of Lemma 1.7, and it is obvious that it satisfies the second hypothesis as well. It remains to verify the third hypothesis. Let $x, y \in W^\theta$, let σ be a simple reflection in W^θ and suppose that $x \preceq y$. We must show that either $x \preceq \sigma y$ or $\sigma x \preceq \sigma y$.

Of course $l^\theta(\sigma y) = l^\theta(y) \pm 1$. If $l^\theta(\sigma y) = l^\theta(y) + 1$, then $y \leq \sigma y$. Therefore $x \preceq y \preceq \sigma y$ and we are done. So we may as well assume that $l^\theta(\sigma y) = l^\theta(y) - 1$, in which case $\sigma y \leq y$ and $\sigma y \preceq y$. It follows from (2.2.1) that $l(\sigma y) = l(y) - l(\sigma)$. Choose a reduced decomposition $\sigma = s_1 \dots s_q$ for σ in W and note that

$$l(s_i \dots s_q y) = l(s_{i+1} \dots s_q y) - 1$$

for all i in the range $1 \leq i \leq q$. It follows that

$$\sigma y \preceq s_2 \dots s_q y \preceq \cdots \preceq s_q y \preceq y.$$

Of course $l^\theta(\sigma x) = l^\theta(x) \pm 1$. First we assume that $l^\theta(\sigma x) = l^\theta(x) - 1$. As before it follows that

$$\sigma x \preceq s_2 \dots s_q x \preceq \cdots \preceq s_q x \preceq x.$$

Applying Lemma 1.3 to x, y and s_q we see that $s_q x \preceq s_q y$. Again applying Lemma 1.3, this time to $s_q x$, $s_q y$ and s_{q-1}, we see that $s_{q-1} s_q x \preceq s_{q-1} s_q y$. Continuing in this way, we see that $\sigma x \preceq \sigma y$, and we are done with this case.

Next we assume that $l^\theta(\sigma x) = l^\theta(x) + 1$. It follows that

$$x \preceq s_q x \preceq s_{q-1} s_q x \preceq \cdots \preceq s_2 \dots s_q x \preceq \sigma x.$$

Applying Lemma 1.3 to x, y and s_q, we see that $x \preceq s_q y$ (we used that $x \preceq s_q x$). We now note that in fact $x \preceq s_i x$ for any i in the range $1 \leq i \leq q$. Indeed, since σ is the longest element in the Weyl group W_Δ, where Δ is the θ-orbit in S corresponding to σ, we can find a reduced decomposition of σ ending with s_i rather than s_q. Applying Lemma 1.3 again, this time to x, $s_q y$ and s_{q-1}, we see that $x \preceq s_{q-1} s_q y$. Continuing in this way, we see that $x \preceq \sigma y$, and this concludes the proof. □

3. Main result for GL_n

3.1. Extended affine Weyl group of GL_n

Consider the general linear group GL_n for $n \geq 1$. Its extended affine Weyl group $\tilde{W}$ is the semidirect product of the symmetric group S_n and the group $\mathbf{Z}^n$; this group acts on $\mathbf{R}^n$, with S_n acting by permutations of the coordinates and with $\mathbf{Z}^n$ acting by translations. The affine Weyl group $W_a \subset \tilde{W}$ of SL_n is the semidirect product of S_n and the subgroup of $\mathbf{Z}^n$ consisting of all n-tuples $(a_1, \dots, a_n)$ such that $a_1 + \cdots + a_n = 0$. As usual we order the affine roots in such a way that the simple affine roots are the functions $(x_1, \ldots, x_n) \mapsto x_i - x_{i+1}$ $(1 \leq i \leq n-1)$ together with the affine linear function

$$(x_1, \ldots, x_n) \mapsto x_n - x_1 + 1.$$

The simple reflections are the transpositions $(i, i+1)$ in S_n $(1 \leq i \leq n-1)$ together with the affine linear transformation

$$(x_1, \ldots, x_n) \mapsto (x_n + 1, x_2, x_3, \ldots, x_{n-2}, x_{n-1}, x_1 - 1).$$

3.2. Alcoves

For a vector $v \in \mathbf{Z}^n$ we denote by $v(m)$ the m-th entry of v; thus $v = (v(1), \ldots, v(n))$. Also we write $\Sigma(v)$ for the sum of the entries of v; thus $\Sigma(v) = v(1) + \cdots + v(n)$. Given two vectors $u, v \in \mathbf{Z}^n$ we say that $u \leq v$ if $u(m) \leq v(m)$ for all m such that $1 \leq m \leq n$. An alcove for GL_n is a sequence $v_0, \ldots, v_{n-1}$ of elements $v_i \in \mathbf{Z}^n$ satisfying the following two conditions. Put $v_n := v_0 + (1, 1, \ldots, 1)$. Then the first condition is that

$$v_0 \leq v_1 \leq \cdots \leq v_{n-1} \leq v_n. \tag{3.2.1}$$

The second condition is that

$$\Sigma(v_i) = \Sigma(v_{i-1}) + 1 \tag{3.2.2}$$

for all i such that $1 \leq i \leq n$.

For $1 \leq i \leq n$ let e_i be the i-th standard basis vector in $\mathbf{Z}^n$, thus $e_i(j)$ is 0 unless $i = j$, in which case it is 1. For $0 \leq i \leq n$ let ω_i be the vector of the form $(1, 1, \ldots, 1, 0, 0, \ldots, 0)$ in which 1 is repeated i times and 0 is repeated $n-i$ times. The *standard alcove* is the sequence $\omega_0, \ldots, \omega_{n-1}$. The extended affine Weyl group $\tilde{W}$ acts on $\mathbf{Z}^n$ by affine linear transformations, and this action takes alcoves to alcoves ($x \in \tilde{W}$ sends $v_0, \ldots, v_{n-1}$ to $xv_0, \ldots, xv_{n-1}$). For any alcove $v_0, \ldots, v_{n-1}$ the vectors $v_1 - v_0, v_2 - v_1, \ldots, v_n - v_{n-1}$ are a permutation of the standard basis vectors $e_1, \ldots, e_n$ (in the case of the standard alcove the permutation is trivial). Thus we see that the extended affine Weyl group acts simply transitively on the set of alcoves. Using the standard alcove as base-point, we identify the extended affine Weyl group with the set of alcoves.

For any alcove $v_0, \ldots, v_{n-1}$ the integer $\Sigma(v_i) - \Sigma(\omega_i)$ is independent of i; we denote this integer by r and say that the *size* of the alcove is r. Two alcoves have the same size if and only if the corresponding elements in the extended affine Weyl group have the same image in the group $X_*(A)/X_*(A_{\mathrm{sc}})$ discussed in 1.8.

3.3. Minuscule alcoves

We say that an alcove $v_0, \ldots, v_{n-1}$ is *minuscule* if

$$\omega_i \leq v_i \leq \omega_i + (1, 1, \ldots, 1)$$

for all i in the range $0 \leq i \leq n-1$.

3.4. Minuscule cocharacters

We say that a vector $v \in \mathbf{Z}^n$ is *minuscule* if each of its entries is 0 or 1. Thus every minuscule vector v is a permutation of one of the vectors ω_i. If v is minuscule and $\Sigma(v) = i$, then v is a permutation of ω_i.

Theorem 3.5. *Let $v_0, \dots, v_{n-1}$ be an alcove and let x be the corresponding element of $\tilde{W}$.*

(1) *Suppose that the alcove $v_0, \dots, v_{n-1}$ is minuscule. Then v_0 is a minuscule vector, and when we regard v_0 as an element in the translation subgroup of $\tilde{W}$, we have the inequality $x \le v_0$ in the Bruhat order.*
(2) *Let v be a minuscule vector in $\mathbf{Z}^n$ and regard v as an element in the translation subgroup of $\tilde{W}$. Suppose that $x \le v$ in the Bruhat order. Then the alcove $v_0, \dots, v_{n-1}$ is minuscule.*
(3) *Let $0 \le r \le n$. The alcove $v_0, \dots, v_{n-1}$ is minuscule of size r if and only if there exists a permutation v of ω_r such that $x \le v$ in the Bruhat order.*

We will prove this theorem in Sect. 5.

Corollary 3.6. *Let $v_0, \dots, v_{n-1}$ be a minuscule alcove and let x be the corresponding element of $\tilde{W}$. Let i be an integer in the range $0 \le i \le n-1$ and define a vector $\mu_i \in \mathbf{Z}^n$ by $\mu_i := v_i - \omega_i$. Then μ_i is minuscule, and when we regard μ_i as an element of the translation subgroup of $\tilde{W}$, we have the inequality $x \le \mu_i$ in the Bruhat order.*

Proof. Let c be the element of $\tilde{W}$ defined (as affine linear automorphism of $\mathbf{R}^n$) by

$$(x_1, \dots, x_n) \mapsto (x_2, x_3, \dots, x_n, x_1 - 1).$$

Consider the alcove $v'_0, \dots, v'_{n-1}$ corresponding to the element $c^i x c^{-i} \in \tilde{W}$. An easy calculation shows that this new alcove is minuscule and that v'_0 is equal to the vector obtained by applying the linear part of the affine transformation c^i to the vector μ_i. Thus, viewing v'_0 as an element in the translation subgroup of $\tilde{W}$, we have the equality $v'_0 = c^i \mu_i c^{-i}$. By Theorem 3.5 (1) we have the inequality $c^i x c^{-i} \le c^i \mu_i c^{-i}$. Since c permutes the simple affine roots, the inner automorphism $w \mapsto c^i w c^{-i}$ of $\tilde{W}$ respects the Bruhat order. Therefore $x \le \mu_i$. □

4. Main result for Sp_{2n}

4.1. Extended affine Weyl group of the group GSp_{2n}

Now we turn to the group GSp_{2n} of symplectic similitudes on a symplectic space of dimension $2n$ (with $n \ge 1$). Its derived group is the symplectic group Sp_{2n}, which is simply connected. The extended affine Weyl group $\tilde{W}$ of GSp_{2n} can be realized as a subgroup of the extended affine Weyl group of GL_{2n}, and the affine Weyl group W_a of Sp_{2n} then occurs as a subgroup of the affine Weyl group of SL_{2n}.

Indeed, consider the automorphism θ of $\mathbf{R}^{2n}$ defined by

$$\theta(x_1, x_2, \ldots, x_{2n-1}, x_{2n}) = (-x_{2n}, -x_{2n-1}, \ldots, -x_2, -x_1);$$

then θ preserves the set of affine roots of SL_{2n}, and it also preserves the set of simple affine roots. The subgroup of fixed points of θ on the affine Weyl group of SL_{2n} is the affine Weyl group W_a of Sp_{2n}. We conclude from Proposition 2.3 that the Bruhat order on the affine Weyl group of Sp_{2n} is inherited from the Bruhat order on the affine Weyl group of SL_{2n}.

It is also true that the affine Weyl group of Sp_{2n} is the group of fixed points of θ on the extended affine Weyl group $S_{2n} \ltimes \mathbf{Z}^{2n}$ of GL_{2n}. In fact the group of fixed points of θ on S_{2n} is the Weyl group W_n of Sp_{2n}, and the group of fixed points of θ on the lattice $\mathbf{Z}^{2n}$ of translations is the lattice

$$\{(x_1, \ldots, x_n, -x_n, \ldots, -x_1) \mid (x_1, \ldots, x_n) \in \mathbf{Z}^n\}$$

of translations for Sp_{2n}. The extended affine Weyl group $\tilde{W}$ of GSp_{2n} is the following slightly larger subgroup of $S_{2n} \ltimes \mathbf{Z}^{2n}$, namely the semidirect product of the Weyl group W_n of the symplectic group with the translation group Y of all vectors $(x_1, \ldots, x_{2n}) \in \mathbf{Z}^{2n}$ such that there exists $c \in \mathbf{Z}$ such that

$$c = x_1 + x_{2n} = x_2 + x_{2n-1} = \cdots = x_n + x_{n+1}.$$

We now see from 1.8 that the Bruhat order on the extended affine Weyl group of GSp_{2n} is inherited from the Bruhat order on the extended affine Weyl group of GL_{2n}.

4.2. Alcoves

We use notation from 3.2, though now we are considering $\mathbf{Z}^{2n}$ rather than $\mathbf{Z}^n$. We want to define the set of alcoves for the group $G = GSp_{2n}$. We will refer to these as G-alcoves to distinguish them from alcoves for the group GL_{2n}. In fact the set of G-alcoves is defined as a subset of the set of alcoves for GL_{2n}. An alcove $v_0, \ldots, v_{2n-1}$ is a G-alcove if and only if the following condition is satisfied for some integer d:

$$v_{2n-i} = \mathbf{d} + \theta(v_i)$$

for $1 \leq i \leq 2n$, where $\mathbf{d}$ denotes the vector $(d, d, \ldots, d)$ (recall that the vector v_{2n} was defined in 3.2). Note that the standard alcove $\omega_0, \ldots, \omega_{2n-1}$ defined in 3.2 is a G-alcove (with $d = 1$). For any G-alcove $v_0, \ldots, v_{2n-1}$ it is clear that the vector v_0 belongs to the translation subgroup Y for G discussed in 4.1. It is also clear that the vectors $v_1 - v_0, v_2 - v_1, \ldots, v_{2n} - v_{2n-1}$ are a permutation of the standard basis vectors $e_1, \ldots, e_{2n}$, and that this permutation lies in the subgroup W_n of S_{2n}. From this discussion it is clear that the natural action of the extended affine Weyl group $\tilde{W}$ of G on the set of G-alcoves is simply transitive. Using the standard alcove as base-point, we identify $\tilde{W}$ with the set of G-alcoves.

4.3. Minuscule alcoves

We say that a G-alcove is *minuscule* if it is minuscule in the sense of 3.3.

4.4. Minuscule cocharacters

We say that a vector v in the subgroup Y of $\mathbf{Z}^{2n}$ is *minuscule* if it is minuscule in the sense of 3.4. Note that every minuscule vector in Y is either ω_0, ω_{2n} or permutation of the vector ω_n, this permutation coming from the subgroup W_n of S_{2n}.

Theorem 4.5. *Let $v_0, \dots, v_{2n-1}$ be a G-alcove and let x be the corresponding element of the extended affine Weyl group $\tilde{W}$ of $G = GSp_{2n}$.*

(1) *Suppose that the G-alcove $v_0, \dots, v_{2n-1}$ is minuscule. Then v_0 is a minuscule vector, and when we regard v_0 as an element in the translation subgroup Y of $\tilde{W}$, we have the inequality $x \le v_0$ in the Bruhat order.*
(2) *Let v be a minuscule vector in Y, the translation subgroup of $\tilde{W}$. Suppose that $x \le v$ in the Bruhat order. Then the G-alcove $v_0, \dots, v_{2n-1}$ is minuscule.*
(3) *The G-alcove $v_0, \dots, v_{2n-1}$ is minuscule of size n (see* 3.2 *for the definition of size) if and only if there exists a permutation $\tau \in W_n$ such that $x \le \tau(\omega_n)$ in the Bruhat order.*

Proof. Since the Bruhat order on $\tilde{W}$ is inherited from the Bruhat order on the extended affine Weyl group of GL_{2n}, the first two parts of this theorem follow from the corresponding parts of Theorem 3.5. The third part follows from the first two. □

5. Proof of Theorem 3.5

5.1. Strategy

We use the notation of Sect. 3. Let $x \in \tilde{W}$, let α be an affine root for GL_n, and let $w_\alpha \in W_a \subset \tilde{W}$ be the corresponding reflection. Then by Corollary 1.5 the elements x, $w_\alpha x$ are related by the Bruhat order, and the direction of the inequality is determined by whether the root $x^{-1}\alpha$ is positive or negative. Moreover the Bruhat order is generated by such elementary inequalities (by its very definition). Thus, in order to prove Theorem 3.5 we need to answer the following question: given a minuscule alcove $\mathbf{v}$ (we write $\mathbf{v}$ for the n-tuple $v_0, \dots, v_{n-1}$) and an affine root α, when is the alcove $w_\alpha \mathbf{v}$ minuscule?

5.2. Answer

As above we consider a minuscule alcove $\mathbf{v}$. For $0 \le k \le n$ we define a vector μ_k by putting $\mu_k := v_k - \omega_k$; note that $\mu_n = \mu_0$. The condition that the alcove $\mathbf{v}$ be minuscule is simply the condition that for all k the vector μ_k be minuscule. Thus each entry of each vector μ_k is 0 or 1. For $1 \leq m \leq n$ we define a subset K_m of $\{0, 1, 2, \ldots, n-1\}$ by putting

$$K_m := \{k \mid 0 \leq k \leq n-1 \quad \text{and} \quad \mu_k(m) = 1\}.$$

Let $1 \leq i < j \leq n$ and let $d \in \mathbf{Z}$. Then the affine linear function $\alpha = \alpha_{i,j;d}$ on $\mathbf{R}^n$ defined by

$$(x_1, \ldots, x_n) \mapsto x_i - x_j - d$$

is an affine root for GL_n. These are not the positive affine roots; nevertheless, up to sign they give all affine roots. We consider the reflection $w = w_{i,j;d}$ in the affine root α; note that w maps $(x_1, \ldots, x_n) \in \mathbf{R}^n$ to the vector

$$(\ldots, x_j + d, \ldots, x_i - d, \ldots)$$

(we have indicated only the i-th and j-th entries as the others are the same as the corresponding entries in $(x_1, \ldots, x_n)$). We want to express the condition that $w\mathbf{v}$ be minuscule.

To do so it is convenient to introduce some notation. We write $[i, j)$ for the set of integers k such that $i \leq k < j$. Let X be any subset of the set $\{0, 1, 2, \ldots, n-1\}$. We write X' for the complement of X in $\{0, 1, 2, \ldots, n-1\}$, and we write ξ_X for the characteristic function of the subset X.

It follows from the definitions that the alcove $w\mathbf{v}$ is minuscule if and only if

$$d - \xi_{[i,j)}(k) + \xi_{K_j}(k) \in \{0, 1\} \tag{5.2.1}$$

and

$$d - \xi_{[i,j)}(k) - \xi_{K_i}(k) \in \{0, -1\} \tag{5.2.2}$$

for all k in the range $0 \leq k \leq n-1$.

If condition (5.2.1) holds for all k, then $d = 0$ or $d = 1$ since the characteristic function $\xi_{[i,j)}(k)$ necessarily attains both the value 0 and the value 1 (for suitable k). A glance at the conditions (5.2.1) and (5.2.2) shows that if $d = 0$, then $w\mathbf{v}$ is minuscule if and only if

$$[i, j) \subset K_i' \cap K_j, \tag{5.2.3}$$

and if $d = 1$, then $w\mathbf{v}$ is minuscule if and only if

$$[i, j)' \subset K_j' \cap K_i. \tag{5.2.4}$$

5.3. *Position of the alcove with respect to the wall*

We continue with our discussion of the alcoves $\mathbf{v}$ and $w\mathbf{v}$. Since we have identified the set of alcoves with the extended affine Weyl group, we may transport the Bruhat order on the extended affine Weyl group over to the set of alcoves. Then we know that $\mathbf{v}$ and $w\mathbf{v}$ are related by the Bruhat order (Lemma 1.4), and we would like to know in which direction the inequality goes. The answer is given by Corollary 1.5. The affine root α defines a wall in $\mathbf{R}^n$ (the zero set of this affine linear function). Corollary 1.5 tells us that if the alcove $\mathbf{v}$ and the standard alcove lie on the same side of this wall, then $\mathbf{v} \leq w\mathbf{v}$, but if they lie on opposite sides of this wall, then $w\mathbf{v} \leq \mathbf{v}$. So we need to determine which side of the wall $\mathbf{v}$ is on.

Recall that the standard alcove is $\omega_0, \dots, \omega_{n-1}$. The first thing to do is to look at the values of our affine root α on the vectors $\omega_0, \dots, \omega_{n-1}$. There are two cases. Suppose first that $d \geq 1$. Then all these values are ≤ 0 and at least one is < 0. Therefore $\mathbf{v}$ and the standard alcove lie on opposite sides of the wall if and only if there exists k in the range $0 \leq k \leq n-1$ such that

$$\alpha(v_k) = \xi_{[i,j)}(k) + \xi_{K_i}(k) - \xi_{K_j}(k) - d > 0,$$

and it is easy to see that this condition holds if and only if $d = 1$ and $K_j' \cap K_i \cap [i, j)$ is non-empty.

Now suppose that $d \leq 0$. Then all values of α on the vectors $\omega_0, \dots, \omega_{n-1}$ are ≥ 0 and at least one is > 0. Therefore $\mathbf{v}$ and the standard alcove lie on opposite sides of the wall if and only if there exists k in the range $0 \leq k \leq n-1$ such that

$$\alpha(v_k) = \xi_{[i,j)}(k) + \xi_{K_i}(k) - \xi_{K_j}(k) - d < 0,$$

and it is easy to see that this condition holds if and only if $d = 0$ and $K_i' \cap K_j \cap [i, j)'$ is non-empty.

5.4. *Intervals in* $\mathbf{Z}/n\mathbf{Z}$

In order to simplify the conditions we derived in 5.3 we need a better understanding of the sets K_m. These subsets satisfy a very strong condition, which is best formulated when we identify the set $\{0, 1, 2, \dots n-1\}$ with $\mathbf{Z}/n\mathbf{Z}$. First we need some definitions.

Let k, l be distinct elements of $\mathbf{Z}/n\mathbf{Z}$. We denote by $[k, l)$ the following subset of $\mathbf{Z}/n\mathbf{Z}$: choose any representative $k_1 \in \mathbf{Z}$ for k, let $l_1 \in \mathbf{Z}$ be the unique representative for l satisfying $k_1 < l_1 < k_1 + n$, and then define $[k, l)$ to be the image under canonical surjection $\mathbf{Z} \to \mathbf{Z}/n\mathbf{Z}$ of the subset $\{x \in \mathbf{Z} \mid k_1 \leq x < l_1\}$. Here are two examples in case $n = 5$: $[2, 4) = \{2, 3\}$ and $[4, 2) = \{4, 0, 1\}$. Note that $[l, k)$ is always the complement to $[k, l)$ in $\mathbf{Z}/n\mathbf{Z}$.

We refer to $[k, l)$ as the *interval* in $\mathbf{Z}/n\mathbf{Z}$ with *lower endpoint* k and *upper endpoint* l. Note that any interval in $\mathbf{Z}/n\mathbf{Z}$ is non-empty and not equal to $\mathbf{Z}/n\mathbf{Z}$.

5.5. A property of the set K_m

So far we have not taken into account the condition (3.2.1) satisfied by our alcove $\mathbf{v}$. Translating this condition into a condition on the vectors μ_k, we find that for all k in the range $0 \le k \le n-1$, if $\mu_k(m) = 1$, then either $\mu_{k+1}(m) = 1$ or $m = k+1$. Translating this into a condition on the sets K_m (now viewing each set K_m as a subset of $\mathbf{Z}/n\mathbf{Z}$), we find that for all m the subset K_m of $\mathbf{Z}/n\mathbf{Z}$ is either empty, or all of $\mathbf{Z}/n\mathbf{Z}$, or else an interval $[?, m)$ in $\mathbf{Z}/n\mathbf{Z}$ with upper endpoint m.

5.6. Continuation of 5.3

We return to the discussion in 5.3. First consider the case in which $d \ge 1$. Then $\mathbf{v}$ and the standard alcove lie on opposite sides of the wall defined by α if and only if $d = 1$ and $K'_j \cap K_i \cap [i, j)$ is non-empty. Suppose that this condition is satisfied. In particular both K'_j and K_i must be non-empty. From 5.5 we conclude that K_i is either all of $\mathbf{Z}/n\mathbf{Z}$ or else an interval $[?, i)$ with upper endpoint i, and we also conclude that K'_j is either all of $\mathbf{Z}/n\mathbf{Z}$ or else an interval $[j, ?)$ with lower endpoint j. Using that K_i meets $[i, j)$, we see that K_i contains $[j, i)$. Similarly, using that K'_j meets $[i, j)$, we see that K'_j also contains $[j, i)$. Thus the condition that $K'_j \cap K_i \cap [i, j)$ be non-empty implies the condition that $K'_j \cap K_i$ contain $[j, i)$ and is moreover (trivially) equivalent to the condition that $K'_j \cap K_i$ not be contained in the complement $[j, i)$ of $[i, j)$. We conclude that the condition that $K'_j \cap K_i \cap [i, j)$ be non-empty is equivalent to the condition $[j, i) \subsetneq K'_j \cap K_i$.

At this point we have shown the following. In case $d \ge 1$, $\mathbf{v}$ and the standard alcove lie on opposite sides of the wall defined by α if and only if $d = 1$ and $[j, i) \subsetneq K'_j \cap K_i$. In case $d \le 0$, completely parallel reasoning shows that $\mathbf{v}$ and the standard alcove lie on opposite sides of the wall defined by α if and only if $d = 0$ and $[i, j) \subsetneq K'_i \cap K_j$.

Lemma 5.7. *Let $\mathbf{v}$ be a minuscule alcove, and let $w = w_{i,j;d}$ be the reflection in the affine Weyl group obtained from the affine root $\alpha = \alpha_{i,j;d}$, as in* 5.2.

(1) *If $\mathbf{v}$ and the standard alcove lie on opposite sides of the wall defined by α (equivalently, if $w\mathbf{v} \le \mathbf{v}$), then $w\mathbf{v}$ is minuscule.*
(2) *Suppose either that $d = 0$ and $[i, j) = K'_i \cap K_j$ or else that $d = 1$ and $[j, i) = K'_j \cap K_i$. Then $\mathbf{v} \le w\mathbf{v}$ and $w\mathbf{v}$ is minuscule.*

Proof. This follows immediately from the results in 5.6 and 5.2 (see (5.2.3) and (5.2.4)). $\square$

5.8. Proof of Theorem 3.5

First note that part (3) of Theorem 3.5 follows easily from parts (1) and (2) of the theorem. Next we prove part (2) of the theorem. Let v be a minuscule vector in $\mathbf{Z}^n$. We regard v as an element of the translation subgroup of $\tilde{W}$ and consider the corresponding alcove. It is obvious that this alcove, call it $\mathbf{a}$, is minuscule. Using part (1) of Lemma 5.7 repeatedly, we see that any alcove that is less than or equal to $\mathbf{a}$ is also minuscule, and this proves (2).

It remains to prove part (1) of Theorem 3.5. So let $\mathbf{v}$ be a minuscule alcove. Of course $\mathbf{v}$ is actually a sequence of vectors $v_0, \dots, v_{n-1}$. It is obvious that v_0 is a minuscule vector. Let $\mathbf{a}$ be the alcove corresponding to the element v_0, viewed as an element of the translation subgroup of $\tilde{W}$; of course $\mathbf{a}$ is given by the sequence $v_0 + \omega_0, \dots, v_0 + \omega_{n-1}$. We must show that $\mathbf{v} \le \mathbf{a}$.

If $\mathbf{v} = \mathbf{a}$, then we are done. Otherwise we will show that there exists a reflection w such that the following three conditions are satisfied: (i) $\mathbf{v} \le w\mathbf{v}$, (ii) $w\mathbf{v}$ is minuscule, and (iii) $wv_0 = v_0$. If $w\mathbf{v} = \mathbf{a}$ we are done. Otherwise we repeat the process. Since there are only finitely many minuscule alcoves, this process must eventually stop, and when it does, we have produced a chain of inequalities showing that $\mathbf{v} \le \mathbf{a}$.

Now we prove the existence of w (assuming that $\mathbf{v} \ne \mathbf{a}$). We need another way to view the condition $\mathbf{v} \ne \mathbf{a}$, so we introduce the following terminology. Recall (see 5.5) that for any $j \in \{1, \dots, n\}$ the set K_j is either empty, all of $\mathbf{Z}/n\mathbf{Z}$ or else an interval $[?, j)$ with upper endpoint j (these three possibilities are mutually exclusive). We say that j is *proper* if K_j is an interval. It follows immediately from the definitions that $\mathbf{v} \ne \mathbf{a}$ if and only there exists $j \in \{1, \dots, n\}$ such that j is proper.

For each proper $j \in \{1, \dots, n\}$ we define a positive integer N_j, as follows. If $0 \in K_j$ we put $N_j = |K_j|$, and if $0 \notin K_j$ we put $N_j = |K'_j|$. We now choose a proper element $j \in \{1, \dots, n\}$ for which the integer N_j is minimal. Since K_j is an interval with upper endpoint j, there exists a unique $i \in \{1, \dots, n\}$ with $i \ne j$ such that $K_j = [i, j)$. Of course i may or may not be less than j. We will show that if $i < j$ (respectively, $j < i$) there exists $d \in \{0, 1\}$ such that the reflection $w_{i,j;d}$ (respectively, $w_{j,i;d}$) satisfies conditions (i), (ii), and (iii) above.

To prove this assertion we begin by noting that since K'_i is either empty, $\mathbf{Z}/n\mathbf{Z}$, or else an interval with lower endpoint i, one of the two sets K_j, K'_i is a subset of the other. Either $0 \in K_j$ or $0 \notin K_j$, and either $0 \in K_i$ or $0 \notin K_i$. Thus there are four cases, each of which must be examined separately.

Suppose first that $0 \in K_j = [i, j)$. Thus $j < i$ and $N_j = |K_j|$. Suppose further that $0 \in K_i$. Since 0 lies in K_j but not in K'_i, it cannot be the case that $K_j \subset K'_i$. Therefore $K'_i \subset K_j$, which implies that $K'_j \cap K_i = K'_j = [j, i)$. Take $w = w_{j,i;0}$. It follows from Lemma 5.7(2) that (i) and (ii) hold (since $j < i$, one must switch the roles of i and j when applying the lemma). Since $0 \in K_i$ and $0 \in K_j$, the i-th and j-th coordinates of v_0 are both equal to 1. Since w simply interchanges the i-th and j-th coordinates of v_0, we see that (iii) holds.

We continue to suppose that $0 \in K_j = [i, j)$, but now we suppose that $0 \notin K_i$. In this case $N_i = |K'_i|$. By minimality of N_j we have $N_j \le N_i$, and hence it cannot be the case that $K'_i \subset K_j$. Therefore $K_j \subset K'_i$. Therefore $K'_i \cap K_j = K_j = [i, j)$. Take $w = w_{j,i;1}$. It follows from Lemma 5.7(2) that (i) and (ii) hold. Since $0 \notin K_i$ and $0 \in K_j$, the i-th and j-th coordinates of v_0 are 0 and 1 respectively. Since $(wv_0)(i) = v_0(j) - 1 = 0$ and $(wv_0)(j) = v_0(i) + 1 = 1$, we see that (iii) holds.

Now suppose that $0 \notin K_j = [i, j)$. Thus $i < j$ and $N_j = |K'_j|$. Suppose further that $0 \notin K_i$. Then, just as in the first case, we see that $w = w_{i,j;0}$ satisfies (i), (ii) and (iii). On the other hand, if $0 \in K_i$, then, just as in the second case, we see that $w_{i,j;1}$ satisfies (i), (ii) and (iii). This completes the proof of Theorem 3.5.

6. Complement to Theorem 3.5

6.1. Notation

In this section we use the notation of Sect. 3. Also we fix an integer r in the range $0 \le r \le n$ and consider the dominant minuscule vector ω_r defined in 3.2. As usual we also regard ω_r as an element in the translation subgroup of the extended affine Weyl group $\tilde{W}$ of GL_n.

Lemma 6.2. *Let* $\mathbf{v}$ *be a minuscule alcove, given by a sequence* $v_0, v_1, \dots, v_{n-1}$ *such that* $v_0 = \omega_r$. *Suppose that* w *is a reflection in* W_a *such that* $w\mathbf{v} \le \mathbf{v}$. *Then* $wv_0 = v_0$.

Proof. Choose i, j, d as in 5.2 such that $w = w_{i,j;d}$. By 5.6 our hypothesis that $w\mathbf{v} \le \mathbf{v}$ implies that either $d = 1$ and $[j, i) \subsetneq K'_j \cap K_i$, or else that $d = 0$ and $[i, j) \subsetneq K'_i \cap K_j$.

In the first case $0 \in [j, i)$ and therefore $0 \notin K_j$ and $0 \in K_i$, which means that the i-th coordinate of v_0 is 1 and the j-th coordinate of v_0 is 0. It follows that $w = w_{i,j;1}$ fixes v_0, as desired.

In the second case one sees easily that the condition $[i, j) \subsetneq K'_i \cap K_j$ implies that $K'_i \cup K_j = \mathbf{Z}/n\mathbf{Z}$ (use 5.5). In particular $0 \in K'_i \cup K_j$. If $0 \in K'_i$, then $\omega_r(i) = v_0(i) = 0$, and this implies that $v_0(j) = \omega_r(j) = 0$. On the other hand, if $0 \in K_j$, then $\omega_r(j) = v_0(j) = 1$, and this implies that $v_0(i) = \omega_r(i) = 1$. In any case we see that the i-th and j-th coordinates of v_0 are equal, and therefore $w = w_{i,j;0}$ fixes v_0, as desired. □

Theorem 6.3. *Let* $\mathbf{v} = v, \dots, v_{n-1}$ *be an alcove and let* x *be the corresponding element of* $\tilde{W}$. *Then* x *is less than or equal to* ω_r *in the Bruhat order on* $\tilde{W}$ *if and only if* $\mathbf{v}$ *is minuscule and* $v_0 = \omega_r$.

Proof. The implication $\Longleftarrow$ follows from Theorem 3.5(1). Now we prove the implication $\Longrightarrow$. It follows from Theorem 3.5(2) that $\mathbf{v}$ is minuscule. It follows from Lemma 6.2, applied repeatedly, that $v_0 = \omega_r$. □

Remark 6.4. It is essential for the truth of Theorem 6.3 that we only consider *dominant* minuscule vectors (all of which are of the form ω_r for some r). Indeed, consider an anti-dominant minuscule vector v (which necessarily has the form $(0 \dots, 0, 1, \dots, 1)$). It is easy to see that there is exactly one minuscule alcove $\mathbf{v} = v_0, \dots v_{n-1}$ such that $v_0 = v$, namely the translate by v of the standard alcove.

7. Complement to Theorem 4.5

7.1. Notation

We use the notation of §4. Thus $\tilde{W}$ now denotes the extended affine Weyl group of the group $G = GSp_{2n}$, and Y denotes its translation subgroup. We consider the dominant minuscule vector $\omega_n = (1, \dots, 1, 0, \dots, 0) \in Y$ (both 1 and 0 are repeated n times).

Theorem 7.2. *Let $\mathbf{v} = v_0, \dots, v_{2n-1}$ be a G-alcove and let x be the corresponding element of $\tilde{W}$. Then $x \le \omega_n$ in the Bruhat order on $\tilde{W}$ if and only if $\mathbf{v}$ is minuscule and $v_0 = \omega_n$.*

Proof. This follows from Theorem 4.5, Theorem 6.3 and the fact that the Bruhat order on $\tilde{W}$ is inherited from the Bruhat order on the extended affine Weyl group of GL_{2n}. $\square$

8. Review of Bruhat order on sets of cosets and double cosets

8.1. Double cosets

We use the notation of §1. In addition we fix two subsets I and J of the set S of simple reflections in W and denote by W_I and W_J the subgroups of W generated by I and J respectively. We consider the set $W_I \backslash W / W_J$ of double cosets with respect to (W_I, W_J). Of course this reduces to the set W/W_J of single cosets in case I is empty.

Recall from Bourbaki (Groupes et Algèbres de Lie, Ch. IV, §1, Exercise 3) that every double coset $W_I y W_J$ in W contains a unique element x_0 of minimal length, and that any element $x \in W_I y W_J$ can be written in the form $x = x_I x_0 x_J$, with $x_I \in W_I$, $x_J \in W_J$ and $l(x) = l(x_I) + l(x_0) + l(x_J)$.

8.2. Bruhat order on $W_I \backslash W / W_J$

Let $W_I x W_J$ and $W_I y W_J$ be double cosets, and let x_0 and y_0 be the unique elements of minimal length in $W_I x W_J$ and $W_I y W_J$ respectively. Recall that $W_I x W_J \le W_I y W_J$ in the Bruhat order on $W_I \backslash W / W_J$ if and only if $x_0 \le y_0$ in the Bruhat order on W.

Recall that the Bruhat order on $W_I \backslash W / W_J$ has the following two properties. First, if $x \le y$ in the Bruhat order on W, then $W_I x W_J \le W_I y W_J$. Second, if $W_I x W_J \le W_I y W_J$ and x is the unique element of minimal length in $W_I x W_J$, then $x \le y$. (The second property follows from the fact that $y_0 \le y$, where y_0 denotes the element of minimal length in $W_I y W_J$, and the first follows easily from Lemma 1.3.)

8.3. *Bruhat order on $W_I\backslash\tilde{W}/W_J$*

Now we return to the extended affine Weyl group $\tilde{W}$ discussed in 1.8. As in 8.1 we consider subgroups W_I and W_J of W_a generated respectively by subsets I and J of the set of simple reflections in W_a. We now define the Bruhat order on the set $W_I\backslash\tilde{W}/W_J$ of double cosets. Let $W_I\tilde{x}W_J$ and $W_I\tilde{y}W_J$ be double cosets, and let $\tilde{x}_0$ and $\tilde{y}_0$ be the unique elements of minimal length in $W_I\tilde{x}W_J$ and $W_I\tilde{y}W_J$ respectively. Then, by definition, $W_I\tilde{x}W_J \le W_I\tilde{y}W_J$ in the Bruhat order on $W_I\backslash\tilde{W}/W_J$ if and only if $\tilde{x}_0 \le \tilde{y}_0$ in the Bruhat order on $\tilde{W}$.

The Bruhat order on $W_I\backslash\tilde{W}/W_J$ has the following two properties. First, if $\tilde{x} \le \tilde{y}$ in the Bruhat order on $\tilde{W}$, then $W_I\tilde{x}W_J \le W_I\tilde{y}W_J$. Second, if $W_I\tilde{x}W_J \le W_I\tilde{y}W_J$ and $\tilde{x}$ is the unique element of minimal length in $W_I\tilde{x}W_J$, then $\tilde{x} \le \tilde{y}$.

9. Replacing alcoves by general faces (for GL_n)

The purpose of this section is to extend Theorem 3.5 by considering faces more general than alcoves. We use the notation and terminology of Sect. 3.

9.1. *Faces of type I*

Fix a non-empty subset $\bar{I}$ of $\mathbf{Z}/n\mathbf{Z}$ and let $I \subset \mathbf{Z}$ denote the inverse image of $\bar{I}$ under the canonical surjection $\mathbf{Z} \to \mathbf{Z}/n\mathbf{Z}$. We consider families $\mathbf{v} = (v_i)_{i\in I}$, indexed by I, of vectors $v_i \in \mathbf{Z}^n$. Such a family is called a *face of type I* if it satisfies the following three conditions:

$$v_{i+n} = v_i + (1, 1, \dots, 1) \text{ for all } i \in I, \tag{9.1.1}$$

$$v_i \le v_j \text{ for all } i, j \in I \text{ such that } i \le j, \tag{9.1.2}$$

$$\Sigma(v_i) - \Sigma(v_j) = i - j \text{ for all } i, j \in I. \tag{9.1.3}$$

Of course, in case $\bar{I} = \mathbf{Z}/n\mathbf{Z}$ a face of type I is simply an alcove (associate to the family $\mathbf{v} = (v_i)_{i\in I}$ the sequence $v_0, \dots, v_{n-1}$). We denote by $\boldsymbol{\omega}$ our usual standard alcove. Thus $\boldsymbol{\omega} = (\omega_i)_{i\in\mathbf{Z}}$, and for i in the range $0 \le i \le n$ the vector ω_i is the one defined in 3.2.

The group $\tilde{W}$ acts transitively on the set $\mathcal{F}_I$ of faces of type I. As base-point we take the unique face of type I such that $v_i = \omega_i$ for all $i \in I$. We use this base-point to identify the set $\mathcal{F}_I$ with the coset space $\tilde{W}/W_I$, where W_I is defined as the stabilizer in $\tilde{W}$ of our base-point. Note that W_I is contained in W_a and is a parabolic subgroup of that Coxeter group.

Now suppose that $\bar{J}$ is a non-empty subset of $\bar{I}$ and let J be the inverse image of $\bar{J}$ under $\mathbf{Z} \to \mathbf{Z}/n\mathbf{Z}$. Then there is a $\tilde{W}$-equivariant surjection

$$\mathcal{F}_I \to \mathcal{F}_J, \tag{9.1.4}$$

defined by $(v_i)_{i\in I} \mapsto (v_i)_{i\in J}$.

9.2. Minuscule faces

Let $\mathbf{v} = (v_i)_{i \in I}$ be a face of type I. We say that $\mathbf{v}$ is *minuscule* if

$$\omega_i \le v_i \le \omega_i + (1, 1, \ldots, 1) \text{ for all } i \in I. \tag{9.2.1}$$

We denote by $\mathcal{M}_I$ the set of minuscule faces of type I. Note that the subset $\mathcal{M}_I$ of $\mathcal{F}_I$ is preserved by the action of the subgroup W_I of $\tilde{W}$.

Proposition 9.3. *Let $\bar{J}$ be a non-empty subset of $\bar{I}$. Then the restriction of the map* (9.1.4) *to $\mathcal{M}_I$ is a surjection $\mathcal{M}_I \to \mathcal{M}_J$.*

Proof. It is obvious that $\mathcal{M}_I$ maps into $\mathcal{M}_J$. What we must show is that the map $\mathcal{M}_I \to \mathcal{M}_J$ is surjective. To prove this in general it suffices to prove it in the case in which $\bar{I} = \mathbf{Z}/n\mathbf{Z}$. Let $(v_j)_{j \in J}$ be an element in $\mathcal{M}_J$. We need to complete $(v_j)_{j \in J}$ to a minuscule alcove $(v_j)_{j \in \mathbf{Z}}$ by defining suitable additional vectors $v_{j'}$ for integers j' in the range $0 \le j' \le n-1$ with $j' \notin J$. We can do this step-by-step, starting with an integer j' such that $j' - 1 \in J$. These considerations show that the proposition is a consequence of the following lemma. □

Lemma 9.4. *Let k, l be integers such that $k < l \le k + n$ and let $v_k, v_l \in \mathbf{Z}^n$. Suppose that the vectors $v_k - \omega_k$, $v_l - \omega_l$ and $v_l - v_k$ are minuscule and suppose further that $\Sigma(v_l) - \Sigma(v) = l - k$. Then there exists a vector $v_{k+1} \in \mathbf{Z}^n$ such that $v_{k+1} - \omega_{k+1}$, $v_{k+1} - v_k$ and $v_l - v_{k+1}$ are minuscule and $\Sigma(v_{k+1}) - \Sigma(v_k) = 1$.*

Proof. Recall from 3.2 the standard basis vectors e_m in $\mathbf{Z}^n$. For any $m \in \mathbf{Z}$ we let r be the unique integer congruent to m modulo n and in the range $1 \le r \le n$, and we define e_m to be the basis vector e_r. Note that with our extended definitions of ω_i and e_i, the equality $\omega_{i+1} - \omega_i = e_{i+1}$ holds for all $i \in \mathbf{Z}$.

We are going to take v_{k+1} of the form $v_{k+1} = v_k + e_m$ for suitable m with $1 \le m \le n$. For any choice of m the vector v_{k+1} satisfies the conditions that $v_{k+1} - v_k$ be minuscule and that $\Sigma(v_{k+1}) - \Sigma(v_k) = 1$. It is clear that $v_l - v_{k+1}$ is minuscule if and only if m satisfies

$$v_l(m) - v_k(m) = 1. \tag{9.4.1}$$

Let k' be the unique integer in the range $1 \le k' \le n$ that is congruent to $k+1$ modulo n. Then $v_{k+1} - \omega_{k+1}$ is equal to $(v_k - \omega_k) + e_m - e_{k'}$, and this vector is minuscule (recall that $v_k - \omega_k$ is minuscule by hypothesis) if and only if one of the following two conditions holds:

$$m = k', \tag{9.4.2}$$

$$m \ne k' \text{ and } v_k(m) - \omega_k(m) = 0 \text{ and } v_k(k') - \omega_k(k') = 1. \tag{9.4.3}$$

Since $v_l - v_k$ is minuscule and $\Sigma(v_l) - \Sigma(v_k) = l - k$, exactly $l - k$ of the entries of $v_l - v_k$ are 1 and the remaining entries are all 0. There are two cases. In the first case $v_l(k') - v_k(k') = 1$. Then $m = k'$ satisfies both (9.4.1) and (9.4.2), and we are done.

In the second case $v_l(k') - v_k(k') = 0$. The vectors $v_l - v_k$ and $\omega_l - \omega_k$ are both minuscule of size $l - k$, and they are distinct (since their k'-th entries are different). Therefore we can choose an integer m in the range $1 \le m \le n$ such that $v_l(m) - v_k(m) = 1$ (in other words, m satisfies (9.4.1)) and such that

$$\omega_l(m) - \omega_k(m) = 0. \tag{9.4.4}$$

We claim that m satisfies (9.4.3) as well; this will conclude the proof. It is obvious that $m \neq k'$ (since m satisfies (9.4.1) and k' does not). Next we check that $v_k(m) = \omega_k(m)$. Since $v_l - \omega_l$ is minuscule, we have $v_l(m) \le \omega_l(m) + 1$. But $\omega_l(m) = \omega_k(m)$ (by (9.4.4)) and $v_l(m) = v_k(m) + 1$ (by (9.4.1)). Therefore $v_k(m) \le \omega_k(m)$. Since $v_k - \omega_k$ is minuscule, we also have $\omega_k(m) \le v_k(m)$. Therefore $v_k(m) = \omega_k(m)$, as desired.

Finally we check that $v_k(k') - \omega_k(k') = 1$. Since $v_l - \omega_l$ and $v_k - \omega_k$ are minuscule, we have $\omega_l(k') \le v_l(k')$ and $v_k(k') \le \omega_k(k') + 1$. Moreover $v_l(k') = v_k(k')$ in the case under consideration, and $\omega_l(k') = \omega_k(k') + 1$ (obvious). Combining these inequalities yields $v_k(k') - \omega_k(k') = 1$, as desired. □

9.5. *Size of a face of type I*

Let $\mathbf{v} = (v_i)_{i \in I}$ be a face of type I. Then the integer $r = \Sigma(v_i) - \Sigma(\omega_i)$ remains constant as i varies through the set I, and we call this integer r the *size* of $\mathbf{v}$.

Theorem 9.6. *Let $\mathbf{v} = (v_i)_{i \in I}$ be a face of type I, and let xW_I be the corresponding element of $\tilde{W}/W_I$.*

(1) *Suppose that $\mathbf{v}$ is minuscule. Let $i \in I$ and put $\mu_i := v_i - \omega_i$. Then μ_i is a minuscule vector, and when we regard μ_i as an element in the translation subgroup of $\tilde{W}$, we have the inequalities $xW_I \le \mu_i W_I$ and $W_I x W_I \le W_I \mu_i W_I$.*
(2) *Let μ be a minuscule vector in $\mathbf{Z}^n$ and regard μ as an element in the translation subgroup of $\tilde{W}$. Suppose that $W_I x W_I \le W_I \mu W_I$ in the Bruhat order on $W_I \backslash \tilde{W}/W_I$. Then the face $\mathbf{v}$ is minuscule.*
(3) *Let $0 \le r \le n$. The face $\mathbf{v}$ is minuscule of size r if and only if there exists a permutation μ of ω_r such that $W_I x W_I \le W_I \mu W_I$ in the Bruhat order on $W_I \backslash \tilde{W}/W_I$.*

Proof. First we prove (1). It follows from Proposition 9.3 that by changing x within its coset xW_I, we may assume that the alcove determined by x is minuscule. Then by Corollary 3.6 the vector μ_i is minuscule and $x \le \mu_i$. From 8.3 it follows that $xW_I \le \mu_i W_I$ and $W_I x W_I \le W_I \mu_i W_I$.

Next we prove (2). Our assumption that $W_I x W_I \le W_I \mu W_I$ implies (use 8.3) that $x_0 \le \mu$, where x_0 denotes the element of minimal length in the double coset $W_I x W_I$. Since μ is minuscule by hypothesis, Theorem 3.5 implies that the alcove determined by x_0 is minuscule. Therefore the face of type I corresponding to $x_0 W_I$ is minuscule. But $\mathbf{v}$ lies in the W_I-orbit of $x_0 W_I$, and W_I preserves the subset $\mathcal{M}_I$ of $\mathcal{F}_I$. Therefore $\mathbf{v}$ is minuscule.

Finally we note that (3) follows from (1) and (2). □

10. Replacing alcoves by general faces (for GSp_{2n})

The purpose of this section is to extend Theorem 4.5 by considering faces more general than alcoves.

10.1. G-faces of type I

Let $\bar{I}$ be a non-empty subset of $\mathbf{Z}/2n\mathbf{Z}$, and let $\mathbf{v} = (v_i)_{i\in I}$ be a face of type I in the sense of 9.1. Let $-I$ denote the set of integers $-i$ such that $i \in I$. Then for any $d \in \mathbf{Z}$ we define a face $\Theta_d(\mathbf{v})$ of type $-I$ by $\Theta_d(\mathbf{v})_{-i} = \theta(v_i) + d \cdot \mathbf{1}$. Here we have denoted by $\mathbf{1}$ the vector $\omega_{2n} = (1, 1, \dots, 1, 1)$, and θ is the automorphism of $\mathbf{Z}^{2n}$ defined in 4.1. The operations Θ_d are of order 2.

We suppose from now on that I is *symmetric* in the sense that $I = -I$. Then a *G-face of type* (I, d) is a face $\mathbf{v}$ of type I in the sense of 9.1 such that

$$\mathbf{v} = \Theta_d(\mathbf{v}). \tag{10.1.1}$$

A *G-face of type* I is a face of type I that is a G-face of type (I, d) for some $d \in \mathbf{Z}$. (The value of d is uniquely determined by the face.) Of course, in case $\bar{I} = \mathbf{Z}/2n\mathbf{Z}$ a G-face of type I is simply a G-alcove (associate to $\mathbf{v} = (v_i)_{i\in\mathbf{Z}}$ the sequence $v_0, \dots, v_{2n-1}$). Let $\mathcal{F}_I^G$ be the set of G-faces of type I.

For any non-empty symmetric subset $\bar{J}$ of $\bar{I}$ there is an obvious map (comp. 9.1)

$$\mathcal{F}_I^G \longrightarrow \mathcal{F}_J^G \tag{10.1.2}$$

which is equivariant for the action of the extended affine Weyl group $\tilde{W}$ of GSp_{2n}. It is well-known that this map is surjective, but this will also follow from the next lemma.

As base point for the action of $\tilde{W}$ on $\mathcal{F}_I^G$ we take the unique G-face of type I such that $v_i = \omega_i$ for all $i \in I$. Using this base point we identify $\mathcal{F}_I^G$ with $\tilde{W}/W_I$ where W_I is the stabilizer in $\tilde{W}$ of the base point. Then W_I is a parabolic subgroup of the affine Weyl group W_a of Sp_{2n}.

10.2. Extending G-faces

Let $\bar{J}$ be a non-empty symmetric subset of $\mathbf{Z}/2n\mathbf{Z}$, and let J be its inverse image under the canonical surjection $\mathbf{Z} \to \mathbf{Z}/2n\mathbf{Z}$. Let $k \in J$ and suppose that $k+1 \notin J$. We let l be the smallest integer in J such that $l > k$; thus $k < l \le k + 2n$. Define $\bar{I}$ by $\bar{I} := \bar{J} \cup \{\bar{k} + \bar{1}, -(\bar{k} + \bar{1})\}$. (For $m \in \mathbf{Z}$ we write $\bar{m}$ for the class of m modulo $2n$.) Thus $\bar{I}$ is symmetric, contains $\bar{J}$, and has either 1 or 2 more elements than $\bar{J}$ does.

We consider the canonical map (10.1.2)

$$\pi : \mathcal{F}_I^G \to \mathcal{F}_J^G.$$

We are interested in the fiber $\pi^{-1}(\mathbf{v})$ of π over an element $\mathbf{v} = (v_j)_{j \in J}$ in $\mathcal{F}_J$. We associate to an element $\mathbf{w} = (w_i)_{i \in I}$ in $\pi^{-1}(\mathbf{v})$ the vector $w := w_{k+1}$. Clearly w satisfies

$$v_k \leq w \leq v_l \tag{10.2.1}$$

and

$$\Sigma(w) = 1 + \Sigma(v_k). \tag{10.2.2}$$

Lemma 10.3. *The map* $\mathbf{w} \mapsto w$ *is a bijection from* $\pi^{-1}(\mathbf{v})$ *to the set of vectors* $w \in \mathbf{Z}^{2n}$ *satisfying* (10.2.1) *and* (10.2.2).

Proof. The injectivity of our map is immediate from (9.1.1) and (10.1.1). To prove surjectivity we start with $w \in \mathbf{Z}^{2n}$ satisfying (10.2.1) and (10.2.2), and we must construct $\mathbf{w} \in \pi^{-1}(\mathbf{v})$ such that $\mathbf{w} \mapsto w$.

Let $\bar{P} = \bar{J} \cup \{\bar{k} + \bar{1}\}$ and let $\bar{Q} = -\bar{P}$, so that $\bar{I} = \bar{P} \cup \bar{Q}$ and $P = -Q$. Since P, Q are not symmetric unless $P = Q$, it does not make sense to consider G-faces of type P or Q, but it does make sense to consider faces of type P or Q, and it is clear that there exists a unique face $\mathbf{x}$ of type P such that $x_{k+1} = w$ and $x_j = v_j$ for all $j \in J$.

Let d be the unique integer such that $\Theta_d(\mathbf{v}) = \mathbf{v}$. Define a face $\mathbf{y}$ of type Q by $\mathbf{y} := \Theta_d(\mathbf{x})$. Clearly $y_j = v_j$ for all $j \in J$.

We claim that

$$p \in P,\ q \in Q,\ p \leq q \implies x_p \leq y_q. \tag{10.3.1}$$

This is clear if there exists $j \in J$ such that $p \leq j \leq q$, so we now assume the contrary. By (9.1.1) it is harmless to assume that $p = k+1$, and then it is necessarily the case that $q = l - 1$ and $\bar{k} = -\bar{l}$. By (10.2.1) and (10.2.2) there exists a basis vector e_m such that $x_p = w = v_k + e_m$ and $e_m \leq v_l - v_k$. Since $\bar{k} = -\bar{l}$, the vector $v_l - v_k$ is symmetric under the operation of reversing its entries, and therefore $e_m + e_{m'} \leq v_l - v_k$, where $e_{m'}$ is the unique basis vector such that $e_{m'} = -\theta(e_m)$. Since $y_q = v_{q+1} - e_{m'}$, we conclude that $x_p \leq y_q$, as desired.

We also claim that

$$p \in P,\ q \in Q,\ q < p \implies y_q \leq x_p. \tag{10.3.2}$$

This is clear since there always exists $j \in J$ such that $q \leq j \leq p$. (Either $j = p$ or $j = p - 1$ will do.)

Suppose that $p \in P, q \in Q$ and $p = q$. Then $x_p \leq y_q$ from (10.3.1). Moreover the size of $\mathbf{x}$ and the size of $\mathbf{y}$ are both equal to the size of $\mathbf{v}$. Therefore $x_p = y_p$. Thus, without ambiguity, we may define w_i for $i \in I$ by putting $w_i = x_i$ if $i \in P$ and $w_i = y_i$ if $i \in Q$. It is immediate from (10.3.1) and (10.3.2) that $\mathbf{w} = (w_i)_{i \in I}$ is a G-face. By construction $\mathbf{w} \in \pi^{-1}(\mathbf{v})$ and $\mathbf{w} \mapsto w$. $\square$

10.4. Size of a G-face of type (I, d)

Let $\mathbf{v} = (v_i)_{i \in I}$ be a G-face of type (I, d). Then the *size* of $\mathbf{v}$ is defined to be its size considered as a face for GL_{2n}, see 9.5. Using (10.1.1) one sees easily that the size of $\mathbf{v}$ is nd.

10.5. Minuscule G-faces

Let $\mathbf{v} = (v_i)_{i \in I}$ be a G-face of type (I, d). Then $\mathbf{v}$ is called *minuscule* if it is minuscule in the sense of 9.2, i.e. when considered as a face of type I for GL_{2n}. We denote by $\mathcal{M}_I^G$ the set of minuscule G-faces of type I. Suppose $\mathbf{v}$ is minuscule. Then the vector v_0 is minuscule, and using (10.1.1) we see that $d \in \{0, 1, 2\}$. If $d = 0$, then $\mathbf{v}$ has size 0; therefore for all i the vector $v_i - \omega_i$ is minuscule and the sum of its entries is 0, which implies that $\mathbf{v} = (\omega_i)_{i \in I}$. If $d = 2$, then $\mathbf{v}$ has size $2n$, which is the same as the size of $\boldsymbol{\omega} + \mathbf{1}$; therefore for all i the vector $\omega_i + \mathbf{1} - v_i$ is minuscule and the sum of its entries is 0, which implies that $\mathbf{v} = (\omega_i + \mathbf{1})_{i \in I}$. Therefore $d = 1$ for all minuscule G-faces other than the two we just described.

Proposition 10.6. *Let $\bar{J} \subset \bar{I}$ be a non-empty symmetric subset of $\bar{I}$. Then the natural map*

$$\mathcal{M}_I^G \longrightarrow \mathcal{M}_J^G$$

is surjective.

Proof. It is enough to prove this in case $\bar{I} = \mathbf{Z}/2n\mathbf{Z}$. Enlarging $\bar{J}$ one step at a time, we then reduce to the case in which $\bar{J} \subset \bar{I}$ are as in 10.2. Now let $\mathbf{v} = (v_j)_{j \in J}$ be the minuscule G-face of type J that we wish to extend to a minuscule G-face of type I. From 10.5 it follows that $\mathbf{v}$ is of type (I, d) for $d \in \{0, 1, 2\}$ and also that our extension problem is trivial if $d = 0$ or $d = 2$. So we now assume that $d = 1$.

From Lemmas 9.4 and 10.3 it follows that there is a G-face $\mathbf{w}$ of type I such that $\mathbf{w} \mapsto \mathbf{v}$ and such that $u_1 := w_{k+1} - \omega_{k+1}$ is minuscule. We claim that $\mathbf{w}$ is in fact minuscule. By (9.1.1) it is enough to show that $u_2 := w_{-(k+1)} - \omega_{-(k+1)}$ is minuscule. Since the d-value for $\mathbf{w}$ is 1 and that for $\boldsymbol{\omega}$ is 0, we see that $u_2 = \mathbf{1} - r(u_1)$, where $r(u_1)$ is the vector obtained from u_1 by reversing the order of its entries. Therefore u_2 is indeed minuscule (since u_1 is). $\square$

Theorem 10.7. *Let $\mathbf{v} = (v_i)_{i \in I}$ be a G-face of type I and of size n. Let $x \cdot W_I$ be the corresponding element of $\tilde{W}/W_I$. Then $\mathbf{v}$ is minuscule if and only if there exists an element τ of the Weyl group W_n of Sp_{2n} such that $W_I x W_I \leq W_I \tau(\omega_n) W_I$ in the Bruhat order on $W_I \backslash \tilde{W}/W_I$, where as usual ω_n is the vector* $(1, \dots, 1, 0, \dots, 0)$.

Proof. Suppose that $\mathbf{v}$ is minuscule. Then it follows from Proposition 10.6 that by changing x within its coset $x \cdot W_I$ we may assume that x is minuscule of size n. It follows from Theorem 4.5 that $x \leq \tau(\omega_n)$ for some $\tau \in W_n$. From 8.3 it follows that $x \cdot W_I \leq \tau(w_n) W_I$ and $W_I x W_I \leq W_I \tau(w_n) W_I$. Conversely assume this last condition. Then $x_0 \leq \tau(\omega_n)$ where x_0 denotes the element of minimal length in $W_I x W_I$. By Theorem 4.5 x_0 is minuscule of size n and therefore the G-face of type I corresponding to $x_0 W_I$ is also minuscule. But $\mathbf{v}$ lies in the W_I-orbit of $x_0 W_I$ and W_I preserves the subset $\mathcal{M}_I^G$ of $\mathcal{F}_I^G$. Hence $\mathbf{v}$ is minuscule, of size n. $\square$

11. Admissibility implies permissibility

11.1. Notation

As in 1.8 we consider a general extended affine Weyl group $\tilde{W}$. As in the introduction we abbreviate $X_*(A)$ to X and we denote by c the canonical homomorphism (1.8.1) from $\tilde{W}$ to $X_*(A)/X_*(A_{\text{sc}})$. When we regard $\nu \in X$ as an element of the translation subgroup of $\tilde{W}$ we denote it by t_ν; thus $t_{\nu_1+\nu_2} = t_{\nu_1}t_{\nu_2}$.

11.2. Goal

Let $\mu \in X$. Our goal is prove that if $x \in \tilde{W}$ is μ-admissible, then it is also μ-permissible. (These terms were defined in the introduction.)

Let $\nu \in X$. It is immediate from the definition that t_ν is μ-permissible if and only if ν lies in P_μ (= the convex hull of the W-orbit of μ in $X_{\mathbf{R}}$) and $\nu - \mu \in X_*(A_{\text{sc}})$. In particular $t_{\mu'}$ is μ-permissible for every μ' in the W-orbit of μ in X. Therefore, in order to check that μ-admissibility implies μ-permissibility, we just need to prove the following lemma.

Lemma 11.3. *If $x \in \tilde{W}$ is μ-permissible and $y \leq x$ in the Bruhat order on $\tilde{W}$, then y is μ-permissible.*

Proof. By an obvious induction argument it is enough to prove the lemma in the special case in which $y = s_{\tilde{\alpha}}x$ for an affine root $\tilde{\alpha}$ which separates $x\mathbf{a}$ from $\mathbf{a}$. (As in the introduction $\mathbf{a}$ is the base alcove in $X_{\mathbf{R}}$.) Clearly $c(y) = c(x) = c(t_\mu)$, so all we must show is that for any v in the closure $\bar{\mathbf{a}}$ of $\mathbf{a}$, the point $s_{\tilde{\alpha}}xv - v$ lies in P_μ. Since x is μ-permissible, the point $xv - v$ lies in P_μ. Moreover P_μ is W-stable, and hence $s_\alpha(xv - v)$ lies in P_μ, where α denotes the vector part of the affine root $\tilde{\alpha}$. Therefore (by convexity of P_μ) it is enough to check that $s_{\tilde{\alpha}}xv - v$ lies on the line segment joining $xv - v$ and $s_\alpha(xv - v)$. This follows from the next lemma. □

Lemma 11.4. *Let $x \in \tilde{W}$ and let $\tilde{\alpha}$ be an affine root, with vector part α. Then $\tilde{\alpha}$ separates $x\mathbf{a}$ and $\mathbf{a}$ if and only if for all $v \in \bar{\mathbf{a}}$ the point $s_{\tilde{\alpha}}xv - v$ lies on the line segment joining $xv - v$ and $s_\alpha(xv - v)$.*

Proof. Let $v \in \bar{\mathbf{a}}$. Note that

$$s_{\tilde{\alpha}}xv - v = (xv - v) - \tilde{\alpha}(xv)\alpha^\vee$$
$$s_\alpha(xv - v) = (xv - v) - [\tilde{\alpha}(xv) - \tilde{\alpha}(v)]\alpha^\vee.$$

Therefore both $s_{\tilde{\alpha}}xv - v$ and $s_\alpha(xv - v)$ lie on the line through $xv - v$ in the direction $\alpha^\vee$. Moreover $s_{\tilde{\alpha}}xv - v$ lies between the other two points $\iff$ 0 lies between $\tilde{\alpha}(xv)$ and $\tilde{\alpha}(v)$ $\iff$ $\tilde{\alpha}$ (weakly) separates xv and v. □

12. ω_n-permissibility for GSp_{2n}

The introduction stated that μ-admissibility is equivalent to μ-permissibility for any minuscule coweight μ for $G = GSp_{2n}$. This is obvious if $\mu = 0$, and is true for μ if and only if it is true for $\mu + \mathbf{1}$. Thus it suffices to consider the case $\mu = \omega_n$. In view of Theorem 4.5(3) we just need to check that an element x in the affine Weyl group $\tilde{W}$ for G is ω_n-permissible if and only if the associated G-alcove $\mathbf{v} = (v_i)_{i \in \mathbf{Z}}$ is minuscule of size n. (As usual $v_i = x \cdot \omega_i$.)

12.1. The convex hull P_{ω_n}

For any vector $u \in \mathbf{R}^{2n}$ we define $r(u)$ to be the vector obtained from u by reversing the order of its entries: $r(u_1, \dots, u_{2n}) = (u_{2n}, \dots, u_1)$. It is easy to see that the convex hull P_{ω_n} is given by

$$P_{\omega_n} = \{u \in \mathbf{R}^{2n} : u + r(u) = \mathbf{1} \text{ and } 0 \le u \le \mathbf{1}\}.$$

Note that $\Sigma(u) = n$ for any $u \in P_{\omega_n}$.

12.2. The base alcove

For G the closure $\bar{\mathbf{a}}$ of the base alcove $\mathbf{a}$ is the set of points $u \in \mathbf{R}^{2n}$ such that $u_1 + u_{2n} = u_2 + u_{2n-1} = \cdots = u_n + u_{n+1}$ and

$$1 + u_{2n} \ge u_1 \ge u_2 \ge \cdots \ge u_n \ge u_{n+1}.$$

Put $\eta_i := (\omega_i + \omega_{2n-i})/2$; note that $\Sigma(\eta_i) = n$. The points $\eta_0, \eta_1, \dots, \eta_n$ serve as “vertices” for $\bar{\mathbf{a}}$; in other words each face of $\bar{\mathbf{a}}$ of minimal dimension (namely 1) contains a unique point η_i.

12.3. ω_n-permissibility

Our element $x \in \tilde{W}$ (with associated G-alcove $\mathbf{v}$) is ω_n-permissible if and only if

$$y_i \in P_{\omega_n} \quad \forall i \in \mathbf{Z}, \tag{12.3.1}$$

where

$$y_i := x\eta_i - \eta_i = (v_i + v_{2n-i} - \omega_i - \omega_{2n-i})/2.$$

(Since $X_*(A)/X_*(A_{\mathrm{sc}})$ is torsion free for $G = GSp_{2n}$, the condition $c(x) = c(t_{\omega_n})$ is redundant.)

Lemma 12.4. *The element $x \in \tilde{W}$ is ω_n-permissible if and only if the associated G-alcove $\mathbf{v}$ is minuscule of size n.*

Proof. The alcove $\mathbf{v}$ is minuscule of size n if and only if

$$v_i + r(v_{2n-i}) = 2 \cdot \mathbf{1} \quad \forall i \in \mathbf{Z} \tag{12.4.1}$$

and

$$0 \leq v_i - \omega_i \leq \mathbf{1} \quad \forall i \in \mathbf{Z}. \tag{12.4.2}$$

On the other hand the condition (12.3.1) for ω_n-permissibility holds if and only if

$$y_i + r(y_i) = \mathbf{1} \quad \forall i \in \mathbf{Z} \tag{12.4.3}$$

and

$$0 \leq y_i \leq \mathbf{1} \quad \forall i \in \mathbf{Z}. \tag{12.4.4}$$

Note that (12.4.1) is equivalent to (12.4.3). Indeed, there exists $k \in \mathbf{Z}$ such that $v_i + r(v_{2n-i}) = k \cdot \mathbf{1}$, and it is clear that $y_i + r(y_i) = (k-1) \cdot \mathbf{1}$. We now assume that (12.4.1) and (12.4.3) hold, and we show that under this assumption (12.4.2) and (12.4.4) are equivalent. First note that (12.4.4) follows immediately from (12.4.2) (by using (12.4.2) for both i and $2n - i$).

It remains to show that (12.4.4) implies (12.4.2). Using (12.4.1) together with (12.4.4), we see that

$$-\mathbf{1} \leq (v_i - \omega_i) - r(v_i - \omega_i) \leq \mathbf{1}. \tag{12.4.5}$$

It is enough to prove (12.4.2) for all i such that $0 \leq i \leq n$. We now fix i in this range. Then we have $v_{-i} \leq v_i \leq v_{2n-i}$, which (using (12.4.1)) can be rewritten as

$$\mathbf{1} \leq v_i + r(v_i) \leq 2 \cdot \mathbf{1}. \tag{12.4.6}$$

Fix m with $1 \leq m \leq n$. Put $a := v_i(m) - \omega_i(m)$ and $b := r(v_i)(m) - r(\omega_i)(m)$. Then (12.4.5) implies

$$-1 \leq a - b \leq 1 \tag{12.4.7}$$

while (12.4.6) implies

$$1 \leq a + b + \omega_i(m) + r(\omega_i)(m) \leq 2. \tag{12.4.8}$$

Since $\omega_i(m)$ is 0 or 1 and $r(\omega_i)(m)$ is 0, the inequalities (12.4.7) and (12.4.8) imply that both a and b belong to $\{0, 1\}$; since this is true for all m between 1 and n, we see that $0 \leq v_i - \omega_i \leq \mathbf{1}$, as desired. □

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