Arithmetic Hirzebruch Zagier cycles

by

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Introduction.

In this paper and its companion [**12**], we establish a relation between part of the height pairing of special cycles on a Shimura variety associated to an orthogonal group of signature $(n-1,2)$ and special values of derivatives of Fourier coefficients of certain Siegel Eisenstein series of genus n in two new cases. Our results provide some evidence in favor of the general program set forth in [**11**], where the case of Shimura curves was analyzed. In [**12**], we considered the case of (twisted) Siegel threefolds ($n=4$) while, in the present paper, we are concerned with the case of (twisted) Hilbert-Blumenthal surfaces ($n=3$).

For these surfaces, the special cycles include the modular and Shimura curves studied extensively by Hirzebruch and Zagier (comp. [**5**], [**2**]). Among other things, they showed that a generating function for the intersection numbers of such curves is an elliptic modular form of weight 2. The Hirzebruch-Zagier curves also account for all Tate classes on such surfaces rational over abelian extensions of $\mathbb{Q}$, [**4**]. Thus it is of interest to investigate the arithmetic analogues of these cycles on integral models of such surfaces, which are arithmetic threefolds.

The canonical model M over $\mathbb{Q}$ of a Hilbert-Blumenthal surface is defined as a moduli space of abelian varieties with real multiplications, and the Hirzebruch-Zagier curves on M can be defined as the loci where the abelian variety admits an extra endomorphism of a particular type, a special endomorphism. For a prime p of good reduction, a model $\mathcal{M}$ of M over $\mathbb{Z}_{(p)}$ can be defined by a moduli problem. We then consider a modular extension $\mathcal{Z}$ to $\mathcal{M}$ of a Hirzebruch-Zagier curve Z on M, again defined by imposing a special endomorphism. These are the arithmetic Hirzebruch-Zagier cycles of the title. More generally, one can consider the loci where the abelian variety carries several special endomorphims. If two

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independent endomorphisms are imposed one obtains points on the generic fiber M and curves on $\mathcal{M}$. If three independent endomorphisms are imposed, then the associated cycle $\mathcal{Z}$ is supported in the special fiber, but *need not have dimension* 0! In fact, it is sometimes possible to impose 4 independent special endomorphisms and still have a nonempty locus, which may have dimension 0 or 1.

It turns out that the case where p splits in the real quadratic field differs radically from the case where p is inert. The first case is similar to the case of modular curves, and in this case a special cycle $\mathcal{Z}$ cut out by three independent special endomorphisms consists of a finite number of points. The more interesting case is when p is inert. In this case, we determine the precise conditions which ensure that $\mathcal{Z}$ consists of a finite number of points.

For either type of p, when $\mathcal{Z}$ is an Artin scheme, we show that its length coincides with the derivative at $s=0$, the center of symmetry, of a Fourier coefficient of a Siegel Eisenstein series of genus 3 (Theorem 7.3 and Theorem 11.5). This result, which generalizes Theorem 3 of [**14**], is connected with local height pairings as follows. Suppose that $\mathcal{Z}_1$ (resp. $\mathcal{Z}_2$) is an arithmetic cycle defined by imposing 1 (resp. 2) special endomorphisms. The intersection of $\mathcal{Z}_1$ and $\mathcal{Z}_2$ is then the locus where a triple of endomorphisms is imposed, although the relative position of the first with respect to the second two is no longer specified. The locus where the triple is independent is then a finite union of special cycles $\mathcal{Z}_3$. The sum of the lengths of the corresponding local rings then gives the local contribution to the height pairing at p for the cycles Z_1 and Z_2 on the generic fiber, i.e., for a curve and a point on a Hilbert-Blumenthal surface. In fact, for this interpretation one must assume a conjecture explained in [**12**] concerning the vanishing of Tor terms in the intersection multiplicity. The case of a height pairing of a triple of curves can be handled in the same way. These results generalize Theorem 14.11 of [**11**].

We now give a more precise discussion of our results.

The twisted Hilbert-Blumenthal surfaces can be viewed as the Shimura varieties associated to certain rational quadratic forms. Let (V,Q) be a quadratic space of signature $(2,2)$ over $\mathbb{Q}$. The even part $C^+(V)$ of the Clifford algebra $C(V)$ is the base change of an indefinite quaternion algebra B_0 over $\mathbb{Q}$ to a real quadratic extension $\boldsymbol{k}$ of $\mathbb{Q}$. We will constantly use the exceptional isomorphism

$$\text{(I.1)} \qquad \mathrm{GSpin}(V) = G(\mathbb{Q}) = \{g \in C^+(V)^\times;\ \nu(g) \in \mathbb{Q}^\times\} \ .$$

Here (I.1) is the group of $\mathbb{Q}$-rational points of an algebraic group G over $\mathbb{Q}$ to which there is associated a Shimura variety with complex points given as follows:

$$M(\mathbb{C}) = Sh(G,\mathcal{D})_K = G(\mathbb{Q}) \setminus [\mathcal{D} \times G(\mathbb{A}_f)/K] \quad . \tag{I.2}$$

Here K is a compact open subgroup of the finite adele group $G(\mathbb{A}_f)$ and $\mathcal{D}$ is the space of oriented negative 2-planes in $V(\mathbb{R})$. This is a twisted version of a Hilbert-Blumenthal surface. In the degenerate case $\boldsymbol{k} = \mathbb{Q} \oplus \mathbb{Q}$, M is a product of two modular curves; when $\boldsymbol{k}$ is a field and $B_0 = M_2(\mathbb{Q})$, M is the usual Hilbert-Blumenthal surface associated to $\boldsymbol{k}$. The Shimura variety (I.2) is the moduli space of principally polarized abelian varieties of dimension 8 with level structure which are equipped with an action of $C(V) \otimes \boldsymbol{k}$ satisfying certain compatibilities. Let us fix a prime number p for which there exists a self-dual $\mathbb{Z}_{(p)}$-lattice Λ in $V(\mathbb{Q})$ (good reduction condition). We will always take K to be of the form $K = K^p.K_p$ where $K_p \subset G(\mathbb{Q}_p)$ is the stabilizer of Λ and where $K^p \subset G(\mathbb{A}_f^p)$ is sufficiently small. The modular interpretation of the Shimura variety then allows us to construct a smooth model $\mathcal{M}$ of (I.2) over $\operatorname{Spec} \mathbb{Z}_{(p)}$.

Hirzebruch and Zagier have defined special curves on Hilbert-Blumenthal surfaces. Expressed in adelic language, a prototype of such a curve is given by the inclusion of Shimura varieties

$$Sh(B_0^\times, \mathcal{D}_0)_{K_0} \hookrightarrow Sh(G,\mathcal{D})_K \quad . \tag{I.3}$$

Here B_0 is as before an indefinite quaternion algebra over $\mathbb{Q}$, K_0 is a compact open subgroup of the finite adeles $B_{0,\mathbb{A}_f}^\times$ and $\mathcal{D}_0$ is the space of oriented negative 2-planes in $V_0(\mathbb{R})$, where V_0 is the space of trace zero elements in B_0. In [**4**] the Hirzebruch-Zagier cycles are defined as the images of (I.3) under the Hecke correspondences defined by elements in $G(\mathbb{A}_f)$. Note, however, that Hirzebruch and Zagier do not use the adelic language and that their definition of these cycles is different.

In the present paper, we use yet another version of these cycles. As indicated above, we define the special cycles in a modular way by imposing *special endomorphisms* on the abelian varieties parametrized by $\mathcal{M}$. Here an endomorphism is special if it is self-adjoint for the Rosati involution and if it Galois commutes with the action of $C(V) \otimes \boldsymbol{k}$,

$$\iota(c \otimes a) \circ j = j \circ \iota(c \otimes a^\sigma) \quad , \quad c \in C(V),\ a \in \boldsymbol{k};\ <\sigma> = \ \mathrm{Gal}(\boldsymbol{k}/\mathbb{Q}) \quad . \tag{I.4}$$

The space of special endomorphisms is a finitely generated free $\mathbb{Z}_{(p)}$-module equipped with the quadratic form Q given by

$$j^2 = Q(j) \cdot 1.$$

For $T \in \mathrm{Sym}_n(\mathbb{Z}_{(p)})_{\geq 0}$ and a K-invariant open compact subset $\omega \subset V(\mathbb{A}_f)^n$ (with a certain integrality property at p), we define as in [**12**] a special cycle $\mathcal{Z}(T,\omega)$ by imposing an n-tuple $\mathbf{j}$ of special endomorphisms with $Q(\mathbf{j}) = T$ and satisfying a compatibility condition with ω. If $\det(T) \neq 0$, the generic fibre of $\mathcal{Z}(T,\omega)$ is a cycle of codimension n on the Hilbert-Blumenthal surface. For $n = 1$ we essentially obtain the classical Hirzebruch-Zagier curves. As in [**12**] our aim is to study the structure of these cycles and their intersection behaviour.

Let us fix positive integers $n_1, \ldots, n_r$ with $n_1 + \ldots + n_r = 3$. For each $i = 1, \ldots, r$ we choose $T_i \in \mathrm{Sym}_{n_i}(\mathbb{Z}_{(p)})_{>0}$ and $\omega_i \subset V(\mathbb{A}_f)^{n_i}$ as above. We form the fibre product of the corresponding special cycles,

$$\mathcal{Z} = \mathcal{Z}(T_1, \omega_1) \times_{\mathcal{M}} \ldots \times_{\mathcal{M}} \mathcal{Z}(T_r, \omega_r) \quad . \tag{I.5}$$

As in [**12**] we have a disjoint sum decomposition according to the value of the *fundamental matrix*

$$\mathcal{Z} = \coprod_{\substack{T \in \mathrm{Sym}_3(\mathbb{Z}_{(p)})_{\geq 0} \\ \mathrm{diag}(T) = (T_1, \ldots, T_r)}} \mathcal{Z}(T,\omega) \tag{I.6}$$

Here $\omega = \omega_1 \times \ldots \times \omega_r$. Concerning this decomposition we have the following results. Let $T \in \mathrm{Sym}_3(\mathbb{Z}_{(p)})_{\geq 0}$.

Theorem 1. *If $\det T \neq 0$, then $\mathcal{Z}(T,\omega)$ lies purely in characteristic p and consists only of supersingular points.*

Theorem 2. *Suppose that $\det T \neq 0$. If p is split in $\boldsymbol{k}$, then $\mathcal{Z}(T,\omega)$ is a union of isolated points. If p is inert in $\boldsymbol{k}$, then $\mathcal{Z}(T,\omega)$ is a union of isolated points if and only if $T \not\equiv 0 \bmod p$.*

Suppose that $\det T \neq 0$ and that $T \not\equiv 0 \bmod p$ when p is inert in $\boldsymbol{k}$. Define

$$< \mathcal{Z}(T,\omega) >_p = \sum_{\xi \in \mathcal{Z}(T,\omega)(\overline{\mathbb{F}}_p)} lg(\mathcal{O}_{\mathcal{Z}(T,\omega),\xi}) \quad .$$

In sections 7 and 11, we define a certain Siegel-Eisenstein series $E(h,s,\Phi)$ on $Sp_6(\mathbb{A})$ depending on ω which vanishes at $s=0$, the center of the critical strip. Our next result concerns the Fourier coefficients $E'_T(h,0,\Phi)$ of the derivative $E'(h,0,\Phi)$ at this point.

Theorem 3. *For T as above and for $h \in Sp_6(\mathbb{R})$,*

$$E'_T(h,0,\Phi) = -\frac{1}{2}\cdot C \cdot \log(p) \cdot < \mathcal{Z}(T,\omega) >_p \cdot W^2_T(h),$$

where C is a volume factor and $W^2_T(h)$ is a standard archimedean Whittaker function.

As explained above, via the decomposition (I.6), Theorems 1–3 have a direct bearing on the problem of intersection multiplicities of special cycles. The proofs here are quite similar to those in [**12**] and in fact easier since we are here in one dimension less. Ultimately, as in loc. cit., we rely for the proof of Theorem 3 on the results of Gross and Keating, [**3**], on deformations of homomorphisms of one-dimensional formal groups and of Kitaoka, [**8**], on local representation densities of quadratic forms.

These results for varying p can be combined. In fact we define a moduli problem represented by a scheme over Spec $\mathbb{Z}[\frac{1}{N}]$, for a suitable integer N, and special cycles $\mathcal{Z}(T,\omega_N)$ on it, whose base change to Spec $\mathbb{Z}_{(p)}$ are the cycles discussed above. Theorem 3 then shows that part of the Fourier expansion of $E'(h,0,\Phi)$ is a generating series for the degrees of certain of these cycles. In light of the result of [**14**], one might hope that the whole Fourier series has an interpretation of this kind.

The previous three theorems focus on the supersingular locus of the Hirzebruch-Zagier cycles. However, we also obtain a more global view of their geometry. Using the well known list of isogeny types in the special fiber, we are able to enumerate those that meet the image of a special cycle. Specifically, suppose that $T \in \mathrm{Sym}_n(\mathbb{Z}_{(p)})$ with $\det(T) \neq 0$. If $n=3$, the corresponding special cycle $\mathcal{Z}(T,\omega)$ lies entirely in the supersingular locus, while if $n=2$, the special cycle can meet at most one additional isogeny class, uniquely determined by T. Furthermore, a non-supersingular point of $\mathcal{M}$ in characteristic p is ordinary and we may use the Serre-Tate deformation theory to investigate the local structure of special cycles along the ordinary locus. Again it turns out that the divisibility of T by p is decisive.

Theorem 4. *Let $\overline{\mathcal{Z}}(T,\omega)^{\rm ord}$ be the ordinary part of the special fiber of the cycle $\mathcal{Z}(T,\omega)$.*
(i) Suppose that $n=1$, i.e., that $T\in\mathbb{Z}_{(p),>0}$. Then locally at each point, $\overline{\mathcal{Z}}(T,\omega)^{\rm ord}$ is the $p^{\mathrm{ord}_p(T)}$ -fold multiple of a smooth divisor.
(ii) Suppose that $n=2$, i.e., that $T\in\mathrm{Sym}_2(\mathbb{Z}_{(p)})_{>0}$. Then $\overline{\mathcal{Z}}(T,\omega)^{\rm ord}$ is an Artin scheme of length $p^{\mathrm{ord}_p(\det(T))}$ at each of its finitely many points.

Note that by Theorem 1, if $T\in\mathrm{Sym}_n(\mathbb{Z}_{(p)})_{>0}$ with $n\geq 3$, then $\overline{\mathcal{Z}}(T,\omega)^{\rm ord}$ is empty.

Now suppose that p is inert in $\boldsymbol{k}$ and that $p|T$. Then by Theorem 2, the special cycle $\mathcal{Z}(T,\omega)$ cut out by three independent special endomorphisms has dimension 1 (case of excess intersection). The supersingular locus of $\mathcal{M}$ is a union of projective lines, [**18**], and $\mathcal{Z}(T,\omega)$ is a union of certain of these. We give a complete enumeration of the lines in $\mathcal{Z}(T,\omega)$ in terms of the Bruhat-Tits building $\mathcal{B}$ of $G_{ad}(\mathbb{Q}_p)\simeq PGL_2(\mathbb{Q}_{p^2})$.

Theorem 5. *Suppose that $T=\mathrm{diag}(\varepsilon_1p^{a_1},\varepsilon_2p^{a_2},\varepsilon_3p^{a_3})$, where $\varepsilon_i\in\mathbb{Z}_p^\times$ and $0\leq a_1\leq a_2\leq a_3$. Associated to T is a triple $\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3)$ of anticommuting automorphisms of $\mathcal{B}$. Then, the dual graph to any connected component of $\mathcal{Z}(T,\omega)$ is the set of points in $\mathcal{B}$ which satisfy*

$$d(x,\mathcal{B}^{\beta_i})\leq\frac{1}{2}(a_i-1)\qquad\qquad i=1,\ 2,\ 3,$$

where $d(x,\mathcal{B}^{\beta_i})$ is the distance from x to the fixed point set $\mathcal{B}^{\beta_i}$ of β_i . In particular, the number of irreducible components of each connected component is the number of vertices $|\mathcal{T}(\boldsymbol{\beta})_0|$ of $\mathcal{B}$ lying in the intersection of these tubes, with specified radii, around the three fixed point sets. Furthermore, each connected component consists of a single projective line if and only if $a_1=a_2=1$ and a_3 is odd, and, in addition, $a_3=1$ if $-\varepsilon_2\notin\mathbb{Z}_p^{\times,2}$.

An explicit formula for the number $|\mathcal{T}(\boldsymbol{\beta})_0|$ can be obtained in general by a combinatorial effort, cf. for example, (8.19) and (8.25). On the other hand, $|\mathcal{T}(\boldsymbol{\beta})_0|$ can be computed as a twisted orbital integral, (8.30), and this integral can be expressed, in turn, as a representation density of quadratic forms! Combining these facts, we obtain the formula (Theorem 8.15)

$$|\mathcal{T}(\boldsymbol{\beta})_0|=(1-p^{-4})^{-1}\cdot\alpha_p(S,p^{-1}T), \tag{I.7}$$

which can be viewed as giving an explicit formula for the representation density on the right hand side. Such explicit formuli are of independent interest. Since the number of connected components can also be given as an orbital integral, we obtain the following result (Proposition 8.17):

$$\#\left\{\begin{matrix}\text{irreducible components}\\ \text{of } \mathcal{Z}(T,\omega)\end{matrix}\right\} = 2\cdot \operatorname{vol}(K)^{-1}\cdot TO_T(\varphi_p'')\cdot O_T(\varphi_f^p). \tag{I.8}$$

This formula is reminiscent of the expressions arising for the number of points of Shimura varieties over finite fields. It also suggests that there may be a modular generating function for the numbers of such components.

In the case where p is inert and $p|T$, it is an important open problem to find an analogue of Theorem 3 giving the contribution of $\mathcal{Z}(T,\omega)$ to an intersection multiplicity. A result of this type was obtained in [**13**] in the somewhat analogous case of bad reduction of Shimura curves. Note, however, that the p-adic uniformization which is available there is not available in our present case. The first degenerate cases that could be investigated are those where each connected component of $\mathcal{Z}(T,\omega)$ is a single projective line, as described in Theorem 5.

A similar analysis of the irreducible components of the degenerate cycles can be done in the Siegel case, to which we return in section 9. Here again, the projective lines in the supersingular locus lying in the image of a special cycle can be described in terms of the building of $G_{ad}(\mathbb{Q}_p)$, where G is now a rank 1 twisted form of the symplectic group of rank 2 over $\mathbb{Q}_p$. The description of the irreducible components again involves an analysis of the fixed point sets of certain anticommuting involutions, but this analysis is now considerably more complicated, and we draw heavily on the work of Kaiser on fixed point sets of tori in the building [**6**]. Nevertheless, we obtain a rather complete description. For example, Proposition 9.8 gives the precise conditions on T under which each connected component of a degenerate cycle (now associated to a 4-tuple of special endomorphisms) is irreducible – the analogue of the last statement of Theorem 5. The same method also works in the case of p-adic uniformization [**13**]. In all of these cases, the fact that $rk_{\mathbb{Q}_p} G_{ad} = 1$ seems absolutely crucial.

It seems to us that our results on degenerate special cycles constitute only the beginning of a circle of very interesting problems. This also explains the different and much more open-ended nature of this paper from its companion [**12**]. We hope that it can serve as a basis of further investigations.

We now give an overview of the structure of this paper. Section 0 contains preliminaries on linear algebra, in particular the exceptional isomorphism mentioned above. In section 1 we define the Shimura variety, the associated moduli problem and introduce the special cycles. In section 2 we extend these concepts to construct a model at a prime of good reduction. From this point through section 10, we assume that p is inert in $\boldsymbol{k}$. Section 3 contains an enumeration of the isogeny classes in the special fibre of $\mathcal{M}$ and their corresponding $\mathbb{Q}$-vector spaces of special endomorphisms. Section 4 is a presentation of the results of Stamm [**18**] on the structure of the supersingular locus of $\mathcal{M} \times_{\mathrm{Spec}\,\mathbb{Z}_{(p)}} \mathrm{Spec}\,\mathbb{F}_p$. In section 5 we determine the $\mathbb{Z}_p$-module of special endomorphisms of a supersingular Dieudonné module. This is then used in section 6 to prove our first main result (Theorem 6.1) which characterizes the special cycles with isolated supersingular points, cf. Theorem 2 above. In this section we also determine (by reduction to the theorem of Gross/Keating) the length of the local ring at an isolated point of a special cycle. Section 7 gives the relation to Eisenstein series and proves Theorem 3 above (Theorem 7.3 and Corollary 7.4). In section 8 we give the analysis of the set of irreducible components of $\mathcal{Z}(T,\omega) \cap \mathcal{M}^{ss}$ and in section 9 we treat the analogous question in the Siegel case. In section 10 we consider the ordinary locus of special cycles. We also investigate the compactness property of special cycles. In section 11, we consider the case of a prime p which splits in $\boldsymbol{k}$. In section 12, we consider special cycles on a moduli scheme over Spec $\mathbb{Z}[\frac{1}{N}]$ and the generating function for their degrees. The final section 13 contains some remarks on the interrelation of the special cycles in the various moduli problems considered in our series of papers [**11**], [**12**], [**13**], [**14**].

In conclusion we wish to thank Ch. Kaiser for considerable help with section 9. In particular, he corrected some errors in an earlier version and indicated to us how to obtain more complete results. We also thank A. Langer for interesting conversations on Hirzebruch-Zagier cycles. Stephen Kudla would like to thank the University of Cologne for its hospitality during June 1997. Michael Rapoport would like to thank the University of Maryland for its hospitality during March 1998. The support of the NSF and the DFG is gratefully acknowledged.

Table of contents

Notation: We use $\mathbb{Z}_{p^2}=W(\mathbb{F}_{p^2})$, the ring of Witt vectors of the finite field $\mathbb{F}_{p^2}$ and $\mathbb{Q}_{p^2}=\mathbb{Z}_{p^2}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. We fix a unit $\Delta\in\mathbb{Z}_p^\times-\mathbb{Z}_p^{\times,2}$ and write

$$\mathbb{Z}_{p^2}=\mathbb{Z}_p[\delta]/(\delta^2-\Delta)\quad .$$

By χ we denote the quadratic residue character on $\mathbb{Z}_p^\times$ resp. $\mathbb{F}_p^\times$. We also let $\mathbb{F}$ be a fixed algebraic closure of $\mathbb{F}_p$, $W=W(\mathbb{F})$, its ring of Witt vectors, $\mathcal{K}=W\otimes_{\mathbb{Z}}\mathbb{Q}$ the quotient field of W and σ their Frobenius endomorphism.

§0. Preliminaries on linear algebra.

In this section we begin by collecting some facts about the Clifford algebras of 4-dimensional quadratic spaces over a field F which will be used in the rest of the paper. In particular we will give a realization of the accidental isomorphism mentioned in the introduction, (I.1), comp. [**1**], pp. 31-33. We then specialize to the case $F=\mathbb{Q}$. When the signature of V is $(2,2)$, we construct the data needed to attach a Shimura variety to V.

Let (V,Q) be a quadratic space of dimension 4 over a field F of characteristic not 2 and let $C(V)=C^+(V)\oplus C^-(V)$ be its Clifford algebra. For a choice of an orthogonal basis $v_1,\ldots,v_4$ of V over F, with $Q(v_i)=a_i$, the element $\delta=v_1v_2v_3v_4\in C^+(V)$ satisfies $\delta^2=a_1a_2a_3a_4$. Up to multiplication by an element of $F^\times$, δ is independent of the choice of basis and $\det(V):=\delta^2\in F^\times/F^{\times,2}$ depends only on V. The algebra $\boldsymbol{k}:=F(\delta)$ is the center of $B:=C^+(V)$ and B is a quaternion algebra over $\boldsymbol{k}$. Let ι be the main involution of $C(V)$ which reduces to the identity on the elements of $V \subset C^-(V)$. Then ι fixes δ and hence induces the identity on $\boldsymbol{k}$. Furthermore it is obvious that the $+1$- resp. -1-eigenspaces of the action of ι on $C^-(V)$ are given by

$$\text{span}\{v_1, v_2, v_3, v_4\}\ ,\ \text{resp. span}\{v_2v_3v_4, v_1v_3v_4, v_1v_2v_4, v_1v_2v_3\}\ .$$

Hence we can recover V from $C(V)$ as follows.

Lemma 0.1.

$$V = \{x \in C^-(V);\ x^\iota = x\}\ .$$

Furthermore, for $x \in V$,

$$Q(x) = x^2 = x \cdot x^\iota = \nu(x)\ ,$$

where ν denotes the spinor norm. □

Lemma 0.2. *Let $v_0 \in V$ with $Q(v_0) = \alpha \neq 0$. Let $\sigma = \mathrm{Ad}(v_0)$ denote the adjoint automorphism of $C(V)$ induced by v_0.*

(i) $\delta^\sigma = -\delta.$

(ii) *The fixed algebra B_0 of σ in $B = C^+(V)$ is a quaternion algebra over F such that $B \simeq B_0 \otimes_F \boldsymbol{k}$.*

Proof. We may suppose that $v_0 = v_4$ (by rechoosing our basis if necessary). The relation $\delta v_0 = -v_0\delta$ is then obvious. To check (ii), consider the basis

$$1,\ i = v_1v_2,\ j = v_2v_3,\ k = ij = a_2v_1v_3,\ \delta, \delta i,\ \delta j,\ \delta k$$

for B. Since $v_0v_i = -v_iv_0$, for $i = 1,\ 2,\ 3$, it is clear that B_0 is the span of the first 4 basis vectors, and that $B = B_0 \otimes_F E$. Also, note that

$$B_0 = (-a_1a_2, -a_2a_3), \tag{0.1}$$

the cyclic algebra over F. □

We introduce the algebraic group G over F with

$$G(R) = \{g \in (B \otimes_F R)^\times;\ \nu(g) = g \cdot g^\iota \in R^\times\} \tag{0.2}$$

for any F-algebra R. Fix $v_0 \in V$ with $\alpha = Q(v_0) \neq 0$ and let again $\sigma = \mathrm{Ad}(v_0)$. Let

$$\tilde{V} = \{x \in B;\ x^\iota = x^\sigma\} \quad . \tag{0.3}$$

Then G acts on $\tilde{V}$ via

$$g : x \mapsto g \cdot x \cdot g^{-\sigma} \quad , \quad x \in \tilde{V} \quad .$$

Indeed,

$$\begin{aligned}
(gxg^{-\sigma})^\iota &= (g^{-\sigma})^\iota \cdot x^\iota \cdot g^\iota \\
&= \nu(g)^{-1} \cdot g^\sigma \cdot x^\sigma \cdot g^\iota \\
&= g^\sigma \cdot x^\sigma \cdot g^{-1} \\
&= (gxg^{-\sigma})^\sigma \quad .
\end{aligned}$$

On $\tilde{V}$ we have the quadratic form defined by

$$\tilde{Q}(x) = x \cdot x^\sigma = x \cdot x^\iota = \nu(x) \quad . \tag{0.4}$$

This quadratic form is preserved by G,

$$\tilde{Q}(gxg^{-\sigma}) = gxg^{-\sigma} \cdot (g^\sigma x^\sigma g^{-1}) = \tilde{Q}(x) \quad .$$

Lemma 0.3. *The map $x \mapsto x \cdot v_0$ induces an isometry*

$$(\tilde{V}, \alpha\, \tilde{Q}) \longrightarrow (V, Q) \quad .$$

Under this isometry the group G is identified with $\mathrm{GSpin}(V, Q)$.

Proof. In terms of the basis of B used in the proof of Lemma 0.2, $\tilde{V}$ is spanned by $1, \delta i, \delta j, \delta k$. Hence the map is a linear isomorphism from $\tilde{V}$ to V. For $x \in \tilde{V}$,

$$(x \cdot v_0)^2 = xv_0 \cdot xv_0 = xx^\sigma \cdot v_0^2 = \alpha\, \tilde{Q}(x) \quad ,$$

hence the map is an isometry. Under this map the action of G on $\tilde{V}$ gets carried into the usual conjugation action on V. In particular, V is automatically preserved by this action. This proves the last assertion. $\square$

Remark 0.4. Thus, we see that, starting with the quadratic space V and a non-isotropic vector $v_0 \in V$ with $\alpha = Q(v_0)$, we obtain a quaternion algebra B_0 with $C^+(V) = B_0 \otimes_F \boldsymbol{k}$. Conversely, let $\boldsymbol{k}$ be a semisimple F-algebra of dimension 2 with Galois automorphism σ. Let B_0 be a quaternion algebra over F and let $B = B_0 \otimes_F \boldsymbol{k}$, with automorphism $\sigma = \mathrm{id} \otimes \sigma$. Now choose $\alpha \in F^\times$ and define an algebra C of dimension 16 over F by

$$C = B\langle v_0 \rangle \quad ,$$

with the relations $v_0^2 = \alpha$ and $v_0 b = b^\sigma v_0$, for all $b \in B$. Extend the involution ι by $v_0^\iota = v_0$ and the automorphism σ by $v_0^\sigma = v_0$. Finally, let

$$V = \{x \in C^- = B \cdot v_0;\ x^\iota = x\} \quad ,$$

with quadratic form $Q(x) = x^2$. Then starting with (V, v_0) we recover B_0 and σ and these two constructions are inverse of one another in an obvious way.

From now on we take $F = \mathbb{Q}$ and assume that the signature of V is $(2, 2)$. There is a Shimura variety associated to such a V, as we now explain.

The choice of signature implies that $\delta^2 > 0$, so that $\boldsymbol{k}$ is a real quadratic field or $\mathbb{Q} \oplus \mathbb{Q}$, and that B is a totally indefinite quaternion algebra over $\boldsymbol{k}$.

Lemma 0.5. *Let $v_0 \in V$ with $Q(v_0) = \alpha > 0$ with associated fixed algebra B_0, as above. Let $\tau \in B_0^\times$ with $\tau^\iota = -\tau$ and $\tau^2 < 0$. Then the involution*

$$x \mapsto x^* = \tau x^\iota \tau^{-1}$$

is a positive involution of $C(V)$.

Proof. Since B is totally indefinite, if $\tau \in B^\times$ with $\tau^\iota = -\tau$ and $\tau^2 \ll 0$ (totally negative), then the involution $b \mapsto \tau b^\iota \tau^{-1}$ is a positive involution of B. If now $\alpha > 0$ and $\tau \in B_0^\times$ with $\tau^\iota = -\tau$ and $\tau^2 < 0$, then

$$\mathrm{tr}^0((b_1 + b_2 v_0)\,\tau(b_1 + b_2 v_0)^\iota \tau^{-1}) = \mathrm{tr}^0(b_1\,\tau b_1^\iota \tau^{-1}) + \alpha\,\mathrm{tr}^0(b_2\,\tau b_2^\iota \tau^{-1}) >> 0\,. \quad \square$$

Let $U_{\mathbb{Q}} = C(V)$, viewed as a $\mathbb{Q}$-vector space of dimension 16, and define

$$i : C(V) \otimes_{\mathbb{Q}} \boldsymbol{k} \to \mathrm{End}_{\mathbb{Q}}(U_{\mathbb{Q}}), \qquad i(c \otimes a)x = cxa.$$

Define a nondegenerate, $\mathbb{Q}$-valued, alternating form on $U_{\mathbb{Q}}$ by

$$<x,y> = \operatorname{tr}^0(y^\iota \tau x), \tag{0.5}$$

where tr^0 denotes the reduced trace on $C(V)$. Then

$$<i(c\otimes a)x,y> = <x, i(c^*\otimes a)y>. \tag{0.6}$$

Note that $c\otimes a\mapsto c^*\otimes a$ is a positive involution of $C\otimes \boldsymbol{k}$. Moreover, the action of $G = \operatorname{GSpin}(V)$ on $U_{\mathbb{Q}}$ by right multiplication commutes with the action of $C(V)\otimes\boldsymbol{k}$, and preserves the form $<\ ,\ >$ up to a scalar:

$$\begin{aligned} <xg,yg> &= \operatorname{tr}^0(g^\iota y^\iota \tau x g)\\ &= \operatorname{tr}^0(gg^\iota y^\iota\tau x)\\ &= \nu(g)\ <x,y>, \end{aligned}$$

where $\nu(g) = gg^\iota$.

Let $\operatorname{End}(U_{\mathbb{Q}}, i)$ denote the commutant of $i(C(V)\otimes\boldsymbol{k})$ in $\operatorname{End}(U_{\mathbb{Q}})$. Let $\mathcal{D}$ be the space of oriented negative 2-planes in $V(\mathbb{R})$. For $z\in\mathcal{D}$ with properly oriented orthogonal basis z_1, z_2, with $Q(z_i) = -1$, let $j_z = z_1z_2\in C^+(V)_{\mathbb{R}}$. Then $j_z^2=-1$, and there is an algebra homomorphism

$$h_z : \mathbb{C}\longrightarrow \operatorname{End}(U_{\mathbb{R}}, i)^{\mathrm{op}} \tag{0.7}$$

such that $h_z(\sqrt{-1}) = r(j_z)$ is right multiplication by j_z. Note that $G(\mathbb{R})$ is a subgroup of the invertible elements in $\operatorname{End}(U_{\mathbb{R}}, i)^{\mathrm{op}}$. Via (0.7), the space $\mathcal{D}$ can be identified with a $G(\mathbb{R})$-conjugacy class of homomorphisms $h_z:\mathbb{C}^\times\to G(\mathbb{R})$ satisfying the axioms required to define a Shimura variety $Sh(G,\mathcal{D})$. This variety has a canonical model over the reflex field $E(G,\mathcal{D})$.

Fix a point $z_0\in\mathcal{D}$ and let $U_{\mathbb{C}} = U_1\oplus U_2$ be the $\pm\sqrt{-1}$ eigenspaces of $h_{z_0}(\sqrt{-1})$. The reflex field $E(G,\mathcal{D})$ is then the field of definition of the isomorphism class of U_1 as a complex representation of $C(V)\otimes\boldsymbol{k}$.

Lemma 0.6. *(i)* $E(G,\mathcal{D}) = \mathbb{Q}$.
(ii) For $c\in C(V)$ *and* $a\in\boldsymbol{k}$,

$$\det(i(c\otimes a); U_1) = N^0(c)^2 N_{\boldsymbol{k}/\mathbb{Q}}(a)^4.$$

Proof. The space V has a $\mathbb{Q}$-basis $v_1, \dots, v_4$ such that the matrix for the quadratic form is $\operatorname{diag}(a_1, a_2, a_3, a_4)$, with a_1, $a_2 > 0$ and a_3, $a_4 < 0$. Let $z_j = \frac{1}{\sqrt{|a_j|}} v_j$ and let z be the oriented negative 2-plane spanned by z_3 and z_4. The element j_z lies in $C^+(V) \otimes \mathbb{Q}(\sqrt{a_3 a_4})$, and the $\pm\sqrt{-1}$ eigenspace of j_z on $U_{\mathbb{C}} = C(V) \otimes_{\mathbb{Q}} \mathbb{C}$ is preserved by any automorphism of $\mathbb{C}$ which is trivial on $\mathbb{Q}(\sqrt{a_3 a_4}, \sqrt{-1})$. In fact, for any automorphism σ of $\mathbb{C}$, we have either $U_1^\sigma = U_1$ or $U_1^\sigma = U_2$. But there is an element $h \in SO(V)(\mathbb{R})$ which acts by $h : z_1 \leftrightarrow z_2$, and $h : z_3 \leftrightarrow z_4$. There is an element $g \in G(\mathbb{R}) = \operatorname{GSpin}(V)(\mathbb{R})$ whose action on V by conjugation is h. Since $g j_z = -j_z g$, the action of g on $U_{\mathbb{C}}$ by right multiplication switches the eigenspaces U_1 and U_2, and commutes with the action of $C(V) \otimes \boldsymbol{k}$. Thus U_1 and U_2 are isomorphic as complex representations of $C(V) \otimes \boldsymbol{k}$, and the reflex field is indeed $\mathbb{Q}$. Also, since $C(V)_{\mathbb{C}} \simeq M_4(\mathbb{C})$,

$$\det(i(c \otimes a); U_{\mathbb{C}}) = N^0(c)^4 N_{\boldsymbol{k}/\mathbb{Q}}(a)^8 = \det(i(c \otimes a); U_1)^2. \quad \square$$

§1. Quaternionic Hilbert-Blumenthal surfaces and special cycles.

In this section we construct the quaternionic Hilbert-Blumenthal surfaces $Sh(G, \mathcal{D})$ as moduli spaces over $\mathbb{Q}$. We then define the special cycles, of codimension 1 or 2, on them by imposing additional special endomorphisms analogous to those in the companion paper [**12**]. For the standard Hilbert modular surfaces, we recover the Hirzebruch-Zagier curves.

We continue with the notation of the end of the last section. In particular (V, Q) is a quadratic space over $\mathbb{Q}$ of signature $(2,2)$ and $\tau \in B = C^+(V)$ is an element with $\tau^\iota = -\tau$ and $\tau^2 \in \mathbb{Q}$ with $\tau^2 < 0$. The map $x \mapsto x^* = \tau x^\iota \tau^{-1}$ is a positive involution of $C(V)$.

We fix a compact open subgroup K of $G(\mathbb{A}_f)$ and consider the following moduli problem over $\mathbb{Q}$. It associates to a scheme $S \in \mathrm{Sch}/\mathbb{Q}$ the set of isomorphism classes of 4-tuples $(A, \lambda, \iota, \bar{\eta})$, where

(i) A is an abelian scheme over S, up to isogeny,

(ii) $\lambda : A \xrightarrow{\sim} \hat{A}$, is a $\mathbb{Q}^\times$-class of polarizations on A,

(iii) $\iota : C(V) \otimes \boldsymbol{k} \longrightarrow \operatorname{End}^0(A)$ is a homomorphism such that

$$\iota(c \otimes a)^* = \iota(c^* \otimes a),$$

for the Rosati involution of $\operatorname{End}^0(A)$ and the involution $*$ of $C(V)$ introduced above, and

(iv) $\bar{\eta}$ is a K–equivalence class of $C(V) \otimes \boldsymbol{k}$–equivariant isomorphisms

$$\eta : \hat{V}(A) \xrightarrow{\sim} U(\mathbb{A}_f)$$

which preserve the symplectic forms up to a scalar in $\mathbb{A}_f^\times$.

Here $\hat{V}(A) = \prod_\ell T_\ell(A) \otimes \mathbb{Q}$. We refer to [**9**] for the precise explanation of these data. In addition, we impose the determinant condition:

$$\det(\iota(c \otimes a); \mathrm{Lie}(A)) = N^0(c)^2 N_{\boldsymbol{k}/\mathbb{Q}}(a)^4. \tag{1.1}$$

In particular, A has relative dimension 8 over S.

For $K \subset G(\mathbb{A}_f)$ sufficiently small, this moduli problem is represented by a smooth quasi-projective scheme $M = M_K$ over $\mathbb{Q}$. If $B = C^+(V) = \mathrm{End}(U_\mathbb{Q}, i)^{\mathrm{op}}$ is a division algebra, then M is in fact projective. Also, in general, M is a canonical model of the Shimura variety $Sh(G, h)_K$, and

$$M(\mathbb{C}) = Sh(G, h)_K(\mathbb{C}) \simeq G(\mathbb{Q})\backslash\mathcal{D} \times G(\mathbb{A}_f)/K, \tag{1.2}$$

where $\mathcal{D}$ is the space of oriented negative 2-planes in $V(\mathbb{R})$, as in section 0 above.

In the case where $\boldsymbol{k}$ is a (real) quadratic field and $B = M_2(\boldsymbol{k})$, $M(\mathbb{C})$ is a (union of) Hilbert-Blumenthal surfaces. If B is a division algebra, then $M(\mathbb{C})$ is a (union of) quaternionic Hilbert-Blumenthal surfaces. In the most degenerate case, when V is split over $\mathbb{Q}$, we have $\boldsymbol{k} = \mathbb{Q} \oplus \mathbb{Q}$ and $B = M_2(\mathbb{Q}) \times M_2(\mathbb{Q})$, and $M(\mathbb{C})$ is a product of modular curves.

Next, we construct certain algebraic cycles on these varieties.

Definition 1.1. *A special endomorphism of $U_\mathbb{Q}$ is an element $j \in \mathrm{End}(U_\mathbb{Q})$ such that, for $a \in \boldsymbol{k}$ and $c \in C(V)$,*

$$i(c \otimes a) \circ j = j \circ i(c \otimes a^\sigma), \tag{sp.1}$$

and

$$j^* = j, \tag{sp.2}$$

where $*$ denotes the adjoint with respect to $< \ , \ >$, cf. (0.6).

Since $\delta v = -v\delta$ for all $v \in V$, any endomorphism of $U_\mathbb{Q}$ satisfying the first condition in Definition 1.1 is given by right multiplication by an element of $C^-(V)$. Note that, for any $c \in C(V)$,

$$< xc, y > = < x, yc^\iota >, \tag{1.3}$$

so that, by Lemma 0.1, a special endomorphism is precisely one given by right multiplication by an element of V.

Definition 1.2. *A special endomorphism of* $(A, \lambda, \iota, \overline{\eta}) \in M(S)$ *is an element* $j \in \mathrm{End}^0(A)$ *such that*

$$\iota(c \otimes a) \circ j = j \circ \iota(c \otimes a^\sigma) \quad , \quad c \otimes a \in C(V) \otimes k$$

and

$$j^* = j \quad ,$$

for the Rosati involution of $\mathrm{End}^0(A)$*.*

For a special endomorphism j we have

$$j^2 = Q(j) \cdot \mathrm{id}, \quad \text{with } Q(j) \in \mathbb{Q} \quad ,$$

provided that the base scheme S is connected, comp. [**12**], Lemma 2.2. Therefore the space of special endomorphisms of $(A, \lambda, \iota, \overline{\eta})$ is a quadratic space in this case. The positivity of the Rosati involution implies that this quadratic space is positive-definite, comp. [**12**], Lemma 2.4.

Our cycles will be defined by imposing certain collections of special endomorphisms.

Fix n, with $1 \leq n \leq 4$. Let $\omega \subset V(\mathbb{A}_f)^n$ be a compact open subset, stable under the action of $K \subset G(\mathbb{A}_f)$, and let $T \in Sym_n(\mathbb{Q})$ be a symmetric matrix with $\det(T) \neq 0$. *We will soon assume that* T *is positive definite and* $n \leq 2$. The cases $n = 3$ or 4 will be relevant in positive characteristic.

For the additional data T, ω, we consider the functor which associates to each $S \in \mathrm{Sch}/\mathbb{Q}$ the set of isomorphism classes of 5-tuples $(A, \lambda, \iota, \bar{\eta}; \mathbf{j})$, where $(A, \lambda, \iota, \bar{\eta})$ is as before, and where $\mathbf{j}$ is an n-tuple of special endomorphisms $j_1, \ldots, j_n \in \mathrm{End}^0(A)$. The set of endomorphisms $\eta \circ \mathbf{j} \circ \eta^{-1}$ corresponding to $\mathbf{j}$ under an isomorphism $\eta : \hat{V}(A) \to U(\mathbb{A}_f)$ for $\eta \in \bar{\eta}$, are special endomorphisms of $U_{\mathbb{A}_f}$ and hence are given as right multiplication by elements of $V(\mathbb{A}_f)$. We require that

$$\eta \circ \mathbf{j} \circ \eta^{-1} \in \omega \subset V(\mathbb{A}_f)^n, \tag{1.4}$$

and that, in addition,

$$Q(\mathbf{j}) = T. \tag{1.5}$$

Here, as in the rest of the paper, for $\mathbf{x} \in V^n$, we put

$$Q(\mathbf{x}) = Q((x_1, \dots, x_n)) = \tfrac{1}{2}((x_i, x_j)) \in \mathrm{Sym}_n(\mathbb{Q}) \tag{1.6}$$

where $(\ ,\ )$ is the bilinear form on V with

$$(x, y) = Q(x+y) - Q(x) - Q(y), \tag{1.7}$$

so that $Q(x) = \frac{1}{2}(x, x)$. Since ω is a union of K-orbits and since the action of G on V preserves the quadratic form, conditions (1.4) and (1.5) are independent of the choice of η in the K-equivalence class $\bar{\eta}$.

Proposition 1.3. *The functor thus determined by T and ω has a coarse moduli scheme $Z(T, \omega)$. If K is sufficiently small, then $Z(T, \omega)$ is a fine moduli scheme and the natural forgetful morphism $Z(T, \omega) \to M_K$ is finite and unramified.* $\square$

Of course, if we impose too many special endomorphisms, the resulting space could be empty. The next result describes what happens in characteristic 0.

Proposition 1.4. *(i) $Z(T, \omega) \neq \emptyset$ implies that $n = 1$ or 2 and that $T \in Sym_n(\mathbb{Q})$ is positive definite and is represented by $V(\mathbb{Q})$.*
(ii) If T satisfies these conditions, fix $x \in V(\mathbb{Q})^n$ such that $Q(x) = T$. Let G_x be the stabilizer of x in G, and let

$$\mathcal{D}_x = \{ z \in \mathcal{D};\ z \perp x \ \}.$$

For $h \in G(\mathbb{A}_f)$, let $Z(x, h; K)$ be the image of the map

$$G_x(\mathbb{Q}) \backslash \mathcal{D}_x \times G_x(\mathbb{A}_f)/(G_x(\mathbb{A}_f) \cap hKh^{-1}) \longrightarrow G(\mathbb{Q}) \backslash \mathcal{D} \times G(\mathbb{A}_f)/K,$$

sending (z, g) to (z, gh). Also, let $\varphi_\omega \in S(V(\mathbb{A}_f)^n)$ be the characteristic function of the compact open set ω. Then

$$\mathrm{image}(Z(T, \omega)(\mathbb{C})) = \bigcup_{\{r; \varphi_\omega(h_r^{-1}x) = 1\}} Z(x, h_r; K),$$

where h_r runs over a set of representatives for the double cosets $G_x(\mathbb{A}_f) \backslash G(\mathbb{A}_f)/K$.

Under the map $h \mapsto h^{-1}x$, $G_x(\mathbb{A}_f) \backslash G(\mathbb{A}_f)$ is homeomorphic to a closed subset of $V(\mathbb{A}_f)^n$. The intersection of this set with the compact open subset ω is thus compact, and hence is a finite union of K-orbits. Thus the union on r is indeed finite.

Proof. Suppose that $\xi = (A, \lambda, \iota, \bar{\eta}; \mathbf{j}) \in Z(T, \omega)(\mathbb{C})$. Then, $H_1(A, \mathbb{Q})$ is a 16-dimensional vector space over $\mathbb{Q}$, which, by the determinant condition, is isomorphic to $U_{\mathbb{Q}}$ as a module over $C(V) \otimes \boldsymbol{k}$. Fix such an isomorphism $\mu : H_1(A, \mathbb{Q}) \xrightarrow{\sim} U_{\mathbb{Q}}$, and note that the complex structure on $H_1(A, \mathbb{R}) \simeq U_{\mathbb{R}}$ is given by right multiplication by an element j_z for some $z \in \mathcal{D}$. Then $\mu \circ \mathbf{j} \circ \mu^{-1}$ is an n–tuple of special endomorphisms of $U_{\mathbb{Q}}$, i.e.,

$$y = \mu \circ \mathbf{j} \circ \mu^{-1} \in V(\mathbb{Q})^n,$$

and, by condition (1.5),

$$Q(y) = Q(\mu \circ \mathbf{j} \circ \mu^{-1}) = T.$$

The components of y must commute with j_z, and it is not difficult to check that an element $y \in V \subset C(V)$ commutes with j_z if and only if $y \in z^{\perp}$, the orthogonal complement of z in $V_{\mathbb{R}}$, comp. [**12**], Lemma 2.3. Note that $z^{\perp}$ is a positive 2-plane in $V_{\mathbb{R}}$. Therefore

$$y = \mu \circ \mathbf{j} \circ \mu^{-1} \in (\, z^{\perp}\,)^n,$$

and, since $\det(T) \neq 0$, we must have $n = 1$ or 2, and T must be positive definite. This proves (i).

Replacing μ by $r(\gamma) \circ \mu$ for some $\gamma \in G(\mathbb{Q})$, changes y to $\gamma y \gamma^{-1}$. By Witt's Theorem, we may assume that $y = \mu \circ \mathbf{j} \circ \mu^{-1} = x$, our fixed n-tuple of endomorphisms, and that the complex structure is given by $r(j_z)$, where $z \in \mathcal{D}_x$. Note that μ can still be changed by an element of $G_x(\mathbb{Q})$. The composition $\eta \circ \mu^{-1} \in \mathrm{End}(U(\mathbb{A}_f), i)$ is given by right multiplication by an element $h \in G(\mathbb{A}_f)$, and condition (1.4) becomes

$$\begin{aligned} \eta \circ \mathbf{j} \circ \eta^{-1} &= \eta \circ \mu^{-1} \circ x \circ \mu \circ \eta^{-1} \\ &= r(h) \circ r(x) \circ r(h^{-1}) \\ &= r(h^{-1} x h) \in r(\omega), \end{aligned}$$

i.e., $h^{-1} \cdot x \in \omega$. Write $h = g h_r k$, with $g \in G_x(\mathbb{A}_f)$ and $k \in K$ for some double coset representative. This then gives the desired description of $\mathrm{image}(Z(T, \omega)(\mathbb{C}))$. $\square$

Remarks: (i) For $n=1$ and 2, the group theoretic cycles $Z(x,h;K)$ and their sums over double coset representatives were introduced in [**10**]. Proposition 1.4 gives them a modular interpretation in the case of signature $(2,2)$.
(ii) For $n=1$, the cycle $Z(T,\omega)$ is a divisor, rational over $\mathbb{Q}$, on the surface M. When $B=M_2(\boldsymbol{k})$ for a real quadratic field $\boldsymbol{k}$, for a suitable choice of $T\in\mathbb{Q}_{>0}$ and $\omega\subset V(\mathbb{A}_f)$, we recover the Hirzebruch-Zagier curves T_N on the Hilbert modular surface.
(iii) For $n=2$, the $Z(T,\omega)$'s are $\mathbb{Q}$–rational 0–cycles on M.

§2. Models over $\mathbb{Z}_{(p)}$ and arithmetic special cycles.

In this section, we define models of the quaternionic Hilbert-Blumenthal surfaces M_K and of the special cycles over $\mathbb{Z}_{(p)}$, in the case in which p is a prime of good reduction.

Fix a prime $p\neq 2$, and assume that there is a $\mathbb{Z}_{(p)}$-lattice $\Lambda\subset V(\mathbb{Q})$ which is self dual with respect to (1.7), i.e.,

$$\Lambda=\{x\in V(\mathbb{Q});(x,\Lambda)\subset\mathbb{Z}_{(p)}\ \}. \tag{2.1}$$

It follows that p is unramified in $\boldsymbol{k}$ and that B is split at each prime $\wp$ of $\boldsymbol{k}$ over p.

Let $\mathcal{O}_C=C(\Lambda)$ be the Clifford algebra of Λ. Then $\mathcal{O}_C$ is a $\mathbb{Z}_{(p)}$-order in $C(V)$ which is maximal at p and which is invariant under the main involution ι. Let $\mathcal{O}_{\boldsymbol{k}}$ be the ring of $\mathbb{Z}_{(p)}$-integers in $\boldsymbol{k}$, and note that $\mathcal{O}_{\boldsymbol{k}}\subset\mathcal{O}_C$. We also assume that $\Lambda=\tau\Lambda\tau^{-1}$, where $\tau\in B^\times$ is the element used to define the positive involution $*$. It follows that $\mathcal{O}_C$ is invariant under $*$. Let $U=\mathcal{O}_C\subset U_{\mathbb{Q}}$. This $\mathcal{O}_C\otimes\mathcal{O}_{\boldsymbol{k}}$-submodule of $U_{\mathbb{Q}}$ is self dual with respect to $<\ ,\ >$.

Having chosen this additional data with respect to p, and a compact open subgroup $K^p\subset G(\mathbb{A}_f^p)$, we can define a moduli problem over $\mathbb{Z}_{(p)}$ which associates to $S\in\mathrm{Sch}/\mathbb{Z}_{(p)}$ the set of isomorphism classes of 4-tuples $(A,\lambda,\iota,\overline{\eta^p})$ where

(i) A is an abelian scheme over S, up to prime to p isogeny,
(ii) $\lambda: A\xrightarrow{\sim}\hat{A}$, is a $\mathbb{Z}_{(p)}^\times$–class of principal polarizations on A,
(iii) $\iota:\mathcal{O}_C\otimes\mathcal{O}_{\boldsymbol{k}}\longrightarrow\mathrm{End}(A)\otimes\mathbb{Z}_{(p)}$ is a homomorphism such that

$$\iota(c\otimes a)^*=\iota(c^*\otimes a),$$

for the Rosati involution of $\mathrm{End}^0(A)$ determined by λ and the involution $*$ of $C(V)$, and

(iv) $\overline{\eta^p}$ is a K^p-equivalence class of $\mathcal{O}_C \otimes \mathcal{O}_{\boldsymbol{k}}$-equivariant isomorphisms

$$\eta^p : \hat{V}^p(A) \xrightarrow{\sim} U(\mathbb{A}_f^p)$$

which preserve the symplectic forms up to a scalar in $(\mathbb{A}_f^p)^\times$.

Again, the action of $\iota(c \otimes a)$ on $\mathrm{Lie}(A)$ is required to satisfy the determinant condition

$$\det(\iota(c \otimes a); \mathrm{Lie}(A)) = N^0(c)^2 N_{\boldsymbol{k}/\mathbb{Q}}(a)^4,$$

interpreted as an identity of polynomial functions with coefficients in $\mathcal{O}_S$, as in [**9**].

As explained in section 5 of [**9**], for $K^p \subset G(\mathbb{A}_f)$ sufficiently small, this moduli problem is represented by a smooth quasi-projective scheme $\mathcal{M}_{K^p}$ over $\mathbb{Z}_{(p)}$, which is, in fact, projective if $B = C^+(V) = \mathrm{End}(U_\mathbb{Q}, i)$ is a division algebra. Moreover, if K_p denotes the intersection of $(\mathcal{O}_C \otimes \mathbb{Z}_p)^\times$ with $G(\mathbb{Q}_p)$, and $K = K_p K^p$, then

$$M_K \simeq \mathcal{M}_{K^p} \times_{\mathrm{Spec}\,\mathbb{Z}_{(p)}} \mathrm{Spec}\,\mathbb{Q} \quad ,$$

so that $\mathcal{M}_{K^p}$ gives a model of M_K over $\mathbb{Z}_{(p)}$.

We next turn to the special cycles. Again, for a fixed n with $1 \le n \le 4$, we let $\omega^p \subset V(\mathbb{A}_f^p)^n$ be a compact open subset, stable under the action of K^p, and let $T \in Sym_n(\mathbb{Q})$, with $\det(T) \neq 0$. We point out that for $(A, \lambda, \iota, \overline{\eta^p}) \in \mathcal{M}_{K^p}(S)$ we may introduce the concept of a *special endomorphism* in analogy with Definition 1.2. If S is connected the special endomorphisms form a positive-definite quadratic space over $\mathbb{Z}_{(p)}$.

We consider the functor which associates to each scheme $S \in \mathrm{Sch}/\mathbb{Z}_{(p)}$, the set of isomorphism classes of 5-tuples $(A, \lambda, \iota, \overline{\eta^p}; \mathbf{j})$, with $(A, \lambda, \iota, \overline{\eta^p})$ as before, and with $\mathbf{j} \in \big(\mathrm{End}(A) \otimes \mathbb{Z}_{(p)}\big)^n$ an n-tuple of special endomorphisms satisfying (1.5), and the analogue of (1.4) with η^p in place of η and ω^p in place of ω.

Proposition 2.1. *(i) If K^p is sufficiently small, then the moduli problem just defined is representable by a scheme $\mathcal{Z}(T, \omega^p)$ which maps by a finite unramified morphism to $\mathcal{M}_{K^p}$.*
(ii) Let $\omega_p = (\Lambda \otimes \mathbb{Z}_p)^n \subset V(\mathbb{Q}_p)^n$. Then the generic fiber of $\mathcal{Z}(T, \omega^p)$ is

$$Z(T, \omega_p \times \omega^p) = \mathcal{Z}(T, \omega^p) \times_{\mathrm{Spec}\ \mathbb{Z}_{(p)}} \mathrm{Spec}\ \mathbb{Q}.$$

If $n \geq 3$, then the left side is to be interpreted as the empty scheme. $\square$

The reader should be warned, however, that, in contrast to the situation for $\mathcal{M}_{K^p}$, $\mathcal{Z}(T, \omega^p)$ is not flat over $\mathbb{Z}_{(p)}$ in general. Indeed, as already in [**12**], one of the most interesting cases arises when $n \geq 3$, so that the generic fiber of $\mathcal{Z}(T, \omega^p)$ is empty.

We add here a remark on Hecke correspondences prime to p. Let $g \in G(\mathbb{A}_f^p)$. The Hecke correspondence on $\mathcal{M}_{K^p}$ is defined by the diagram

$$\begin{array}{ccc} \mathcal{M}_{K^p \cap g^{-1} K^p g} & \xrightarrow{g} & \mathcal{M}_{g K^p g^{-1} \cap K^p} \\ \downarrow & & \downarrow \\ \mathcal{M}_{K^p} & \dashrightarrow & \mathcal{M}_{K^p} \end{array} \tag{2.2}$$

Here the solid horizontal morphism is defined by mapping $(A, \lambda, \iota, \overline{\eta^p})$ to $(A, \lambda, \iota, \overline{g \circ \eta^p})$ and the vertical arrows are defined by the natural inclusions of open compact subgroups. For T and ω^p as before, we obtain a morphism

$$\mathcal{Z}(T, \omega^p) \xrightarrow{g} \mathcal{Z}(T, g \cdot \omega^p) \tag{2.3}$$

which sends $(A, \lambda, \iota, \overline{\eta^p}, \mathbf{j})$ to $(A, \lambda, \iota, \overline{g \circ \eta^p}; \mathbf{j})$. It is obvious that this morphism may be combined with the natural forgetful morphisms from the special cycles to the moduli spaces to form a commutative diagram with three cartesian squares. We may consider this as a prolongation of the Hecke correspondence from $\mathcal{M}_{K^p}$ to itself to a correspondence from $\mathcal{Z}(T, \omega^p)$ to $\mathcal{Z}(T, g \cdot \omega^p)$.

From now until section 11 we will assume that p is inert in $\boldsymbol{k}$. This implies that $\boldsymbol{k}$ is a field (real quadratic since the signature of V is $(2, 2)$).

From now on we will denote by $\omega \subset V(\mathbb{A}_f^p)^n$ the set which is denoted by ω^p above. We also fix an algebraic closure $\mathbb{F}$ of the residue field of $\boldsymbol{k}$ at p.

§3. Isogeny classes and special endomorphisms.

In this section, we determine the $\mathbb{Q}$-vector space of special endomorphisms attached to each of the isogeny classes in the special fiber of $\mathcal{M}$. This information determines which isogeny classes can meet the various special cycles. We continue to assume that $\boldsymbol{k}$, the center of $C^+(V)$, is a quadratic field and that p is inert in $\boldsymbol{k}$.

For a point $\xi = (A, \lambda, \iota, \bar{\eta}^p) \in \mathcal{M}_{K^p}(\mathbb{F}) = \mathcal{M}_p$, we consider the isogeny class of the triple (A, λ, ι). Thus, A is an abelian variety of dimension 8 over $\mathbb{F}$ with an action $\iota : C(V) \otimes \boldsymbol{k} \hookrightarrow \mathrm{End}^0(A)$.

Recall that the Clifford algebra $C(V) = C^+(V) \oplus C^-(V)$ is a central simple algebra over $\mathbb{Q}$, with $C^+(V) = B$, a quaternion algebra over $\boldsymbol{k}$. Moreover, there is an isomorphism

$$
i : C(V) \otimes_{\mathbb{Q}} \boldsymbol{k} \simeq M_2(B), \tag{3.1}
$$

discussed in more detail below. Hence, up to isogeny, $A \simeq A_0^2$ where A_0 is a $4-$dimensional abelian variety with $\iota_0 : B \hookrightarrow \mathrm{End}^0(A_0)$. Thus, the possible isogeny classes of (A, ι)'s are the same as the possible isogeny classes of (A_0, ι_0)'s, and these are described by Theorem 5.4 of Milne, [**16**]:

- I_0. The supersingular class; $A_0 \simeq A_{00}^4$ for a supersingular elliptic curve A_{00}.
- I_1. The classes associated to imaginary quadratic extensions $E/\mathbb{Q}$; $A_0 \simeq A_{00}^4$ for an ordinary elliptic curve A_{00} with complex multiplication by E, i.e., $E \simeq \mathrm{End}^0(A_{00})$.
- I_2. The classes associated to totally imaginary quadratic extensions $\boldsymbol{k}'$ of $\boldsymbol{k}$ which are not of the form $\boldsymbol{k} \cdot E$ for any imaginary quadratic extension of $\mathbb{Q}$; $A_0 \simeq A_1^2$, where A_1 is a simple abelian surface with complex multiplication by $\boldsymbol{k}'$, i.e., $\boldsymbol{k}' \simeq \mathrm{End}^0(A_1)$.

In case I_1, let $\boldsymbol{k}' = \boldsymbol{k} \cdot E$. Then, in cases I_1 and I_2, the field $\boldsymbol{k}'$ splits B, and p splits in $\boldsymbol{k}'/\boldsymbol{k}$.

Note that type I_0 consists of supersingular abelian varieties while types I_1 and I_2 consist of ordinary abelian varieties.

For a point $\xi \in \mathcal{M}_p$, as above, the space of special endomorphisms of (A, λ, ι) is the $\mathbb{Q}-$vector space

$$
V' = V_\xi = \{ j \in \mathrm{End}^0(A) \ ; \ \iota(c \otimes a) \cdot j = j \cdot \iota(c \otimes a^\sigma), \text{ and } j^* = j \}. \tag{3.2}
$$

This space depends only on the isogeny class of (A, λ, ι), and carries a quadratic form defined by $j^2 = Q'(j) \cdot id_A$. In order to determine the quadratic space (V', Q') in each of the cases I_0, I_1 and I_2, we must first make the isomorphism (3.1) more explicit. In particular, we must keep track of the involution $*$ and the Galois action σ.

As in section 1, choose $v_0 \in V$, with $Q(v_0) = v_0^2 = \alpha \in \mathbb{Q}^\times$, and assume that $\alpha > 0$. Write

$$C(V) = C = B \oplus Bv_0. \tag{3.3}$$

As in section 1, let $\sigma = Ad(v_0)$, and recall that σ preserves B and extends the Galois automorphism σ on its center $\boldsymbol{k}$. The fixed algebra B_0 of σ in B is a quaternion algebra over $\mathbb{Q}$ with $B = B_0 \otimes_{\mathbb{Q}} \boldsymbol{k}$. As in Lemma 0.5, let $\tau \in B_0^\times$ be the element defining the positive involution $b^* = \tau b^\iota \tau^{-1}$ of $C(V)$. Since $v_0^\iota = v_0$, we have $v_0^* = v_0$.

Lemma 3.1. *For a choice of the decomposition (3.3), the isomorphism (3.1) is given explicitly by*

$$i : (b_1 + b_2 v_0) \otimes a \mapsto \begin{pmatrix} b_1 & \alpha b_2 \\ b_2^\sigma & b_1^\sigma \end{pmatrix} \cdot a,$$

for b_1 and $b_2 \in B$ and $a \in \boldsymbol{k}$. Under this isomorphism, the involution $c \otimes a \mapsto c^\iota \otimes a$ becomes

$$\beta \mapsto Ad(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix})({}^t\beta^\iota),$$

for $\beta \in M_2(B)$. The involution $c \otimes a \mapsto c^* \otimes a$ becomes

$$\beta \mapsto Ad(\begin{pmatrix} \tau\alpha & \\ & \tau \end{pmatrix})({}^t\beta^\iota),$$

and the automorphism $c \otimes a \mapsto c \otimes a^\sigma$ becomes

$$\beta \mapsto Ad(\begin{pmatrix} & \alpha \\ 1 & \end{pmatrix})(\beta^\sigma).$$

Proof. The space $U_{\mathbb{Q}} = C(V)$ is a two dimensional right vector space over B, and left-right multiplication gives an algebra homomorphism

$$C \otimes_{\mathbb{Q}} \boldsymbol{k} \longrightarrow \mathrm{End}_B(U_{\mathbb{Q}}), \qquad c \otimes a \mapsto \big(x \mapsto cxa\big). \tag{3.4}$$

This is an isomorphism of simple algebras over $\mathbb{Q}$. Taking 1 and v_0 as a B-basis for $U_{\mathbb{Q}}$, we obtain the isomorphism $C \otimes \boldsymbol{k} \to M_2(B)$ of the Lemma. For example,

$$b(x + v_0 y) = b(1, v_0)\begin{pmatrix} x \\ y \end{pmatrix} = (1, v_0)\begin{pmatrix} b & \\ & b^\sigma \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.5}$$

Now, for example,

$$
\begin{aligned}
i((b_1+b_2v_0)^\iota) &= i(b_1^\iota + v_0 b_2^\iota) = i(b_1^\iota + b_2^{\iota\sigma} v_0)\\
&= \begin{pmatrix} b_1^\iota & \alpha b_2^{\iota\sigma} \\ b_2^\iota & b_1^{\iota\sigma}\end{pmatrix} \qquad (3.6)\\
&= Ad(\begin{pmatrix} \alpha & \\ & 1\end{pmatrix})\begin{pmatrix} b_1^\iota & b_2^{\iota\sigma} \\ \alpha b_2^\iota & b_1^{\iota\sigma}\end{pmatrix}\\
&= Ad(\begin{pmatrix} \alpha & \\ & 1\end{pmatrix}){}^t i(b_1+b_2v_0)^\iota,
\end{aligned}
$$

as claimed. The other relations are similarly easily checked. $\square$

Note that the idempotents

$$e_1 = \frac{1}{2}(1\otimes 1 + \delta\otimes\delta^{-1}) \qquad \text{and} \qquad e_2 = \frac{1}{2}(1\otimes 1 - \delta\otimes\delta^{-1}) \tag{3.7}$$

in the subalgebra $\boldsymbol{k}\otimes\boldsymbol{k}\subset C\otimes_{\mathbb{Q}}\boldsymbol{k}$ have images

$$i(e_1) = \begin{pmatrix} 1 & \\ & 0\end{pmatrix} \qquad \text{and} \qquad i(e_2) = \begin{pmatrix} 0 & \\ & 1\end{pmatrix} \tag{3.8}$$

in $M_2(B)$, and that the decomposition $A\simeq A_0^2$ is then given by

$$A = i(e_1)A \times i(e_2)A. \tag{3.9}$$

Let λ_0 be the polarization of A_0 determined by

$$\lambda_0 = {}^t i(e_1)\circ\lambda\circ i(e_1), \tag{3.10}$$

and let $\beta\mapsto\beta'$ be the corresponding Roasti involution of $\mathrm{End}^0(A_0)$.

Proposition 3.2. *The isomorphism $A\simeq A_0^2$, determined by a decomposition (3.3), induces an isomorphism:*

$$V' = \left\{\begin{matrix} j\in\mathrm{End}^0(A)\\ \iota(c\otimes a)j = j\iota(c\otimes a^\sigma)\\ j^* = j\end{matrix}\right\} \simeq \left\{\begin{matrix} j_0\in\mathrm{End}^0(A_0)\\ \iota_0(b)j_0 = j_0\iota(b^\sigma)\\ j_0' = j_0\end{matrix}\right\}.$$

on the space of special endomorphisms. Here $j^2 = \alpha j_0^2{\cdot}1_2$, so that $Q(j) = \alpha\, Q_0(j_0)$ with $j_0^2 = Q_0(j_0)\, id_{A_0}$.

Proof. Having fixed the isomorphism (3.1), with $A = i(e_1)A\times i(e_2)A$, we have

$$\iota: C\otimes\boldsymbol{k} = M_2(B)\hookrightarrow M_2(\mathrm{End}^0(A_0))\simeq\mathrm{End}^0(A), \tag{3.11}$$

where the middle inclusion is induced by $\iota_0 : B \hookrightarrow \mathrm{End}^0(A_0)$. Recall that this map is required to satisfy $\iota(c \otimes a)^* = \iota(c^* \otimes a)$. If j is a special endomorphism, then, for $a \in \boldsymbol{k}$, $j \circ \iota(1 \otimes a) = \iota(1 \otimes a^\sigma) \circ j$, and $j \circ \iota(a \otimes 1) = \iota(a \otimes 1) \circ j$. In matrix form, these conditions imply that

$$j = \begin{pmatrix} & j_2 \\ j_3 & \end{pmatrix}, \tag{3.12}$$

with j_2 and $j_3 \in \mathrm{End}^0(A_0)$. The conditions $j \circ \iota(b \otimes 1) = \iota(b \otimes 1) \circ j$ and $j \circ \iota(v_0 \otimes 1) = \iota(v_0 \otimes 1) \circ j$ imply $j_2 = \alpha j_3$ and $j_3 \iota_0(b) = \iota_0(b^\sigma) j_3$.

Finally, we must impose the condition $j^* = j$. The polarization has the form:

$$\begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix} = \lambda : A_0^2 \simeq A \longrightarrow \hat{A} \simeq \hat{A_0}^2, \tag{3.13}$$

where, for example, $\lambda_2 = {}^t\iota(e_1)\lambda\iota(e_2)$, etc. But now, since the idempotents e_1 and e_2 which give the decomposition $A \simeq A_0^2$ lie in $\boldsymbol{k} \otimes \boldsymbol{k} \subset C \otimes \boldsymbol{k}$, and since the Rosati involution restricts to the identity on $\iota(\boldsymbol{k} \otimes \boldsymbol{k})$, we must have $\lambda_2 = \lambda_3 = 0$. Since $v_0^* = v_0$, we have ${}^t\iota(v_0)\lambda = \lambda\iota(v_0)$, and hence

$$\begin{pmatrix} & 1 \\ \alpha & \end{pmatrix} \begin{pmatrix} \lambda_1 & \\ & \lambda_4 \end{pmatrix} = \begin{pmatrix} \lambda_1 & \\ & \lambda_4 \end{pmatrix} \begin{pmatrix} & \alpha \\ 1 & \end{pmatrix}, \tag{3.14}$$

so that $\lambda_4 = \alpha\lambda_1$. Therefore, the Rosati involution on $M_2(\mathrm{End}^0(A_0))$ is given by

$$\beta \mapsto Ad(\begin{pmatrix} 1 & \\ & \alpha \end{pmatrix}^{-1})({}^t\beta'), \tag{3.15}$$

where $x \mapsto x'$ is the positive involution of $\mathrm{End}^0(A_0)$ determined by λ_1, applied componentwise to the matrix β. But then the condition

$$\begin{pmatrix} & \alpha j_3 \\ j_3 & \end{pmatrix} = j = j^* = \begin{pmatrix} & j_3' \\ \alpha j_3' & \end{pmatrix}, \tag{3.16}$$

is equivalent to $j_3' = j_3$. $\square$

We are now in a position to determine the space of special endomorphisms for each isogeny class in the special fiber. Proposition 3.2 shows that this space can be determined in terms of the triple $(A_0, \iota_0, ')$. Recall that, for $b \in B$,

$$\iota_0(b)' = \iota_0(\tau b^\iota \tau^{-1}), \tag{3.17}$$

and that $\boldsymbol{k} = \mathbb{Q}(\delta)$ with $\delta^2 = \Delta \in \mathbb{Q}^\times$.

We begin with the supersingular case I_0, so that $A_0 \simeq A_{00}^4$ for a supersingular elliptic curve A_{00}. Let $\mathbb{B}$ be the quaternion algebra over $\mathbb{Q}$ ramified precisely at infinity and p, and for $B = B_0 \otimes_{\mathbb{Q}} \boldsymbol{k}$, as above, let $B_0^\vee$ be the unique (up to isomorphism) quaternion algebra over $\mathbb{Q}$ such that $B_0 \otimes_{\mathbb{Q}} B_0^\vee \simeq M_2(\mathbb{B})$. Then we have

$$B_0 \otimes_{\mathbb{Q}} B_0^\vee \otimes_{\mathbb{Q}} M_2(\mathbb{Q}) \xrightarrow{\sim} M_4(\mathbb{B}) = \mathrm{End}^0(A_0). \tag{3.18}$$

We may assume that the embedding ι_0 of $B = B_0 \otimes_{\mathbb{Q}} \boldsymbol{k}$ is given by

$$\iota_0(b_0 \otimes a) = b_0 \otimes 1 \otimes i_0(a), \tag{3.19}$$

where $i_0 : \boldsymbol{k} \hookrightarrow M_2(\mathbb{Q})$ is determined by

$$i_0(\delta) = \begin{pmatrix} & 1 \\ \Delta & \end{pmatrix}. \tag{3.20}$$

By condition (3.17), we know the restriction of the Rosati involution $'$ of $\mathrm{End}^0(A_0)$ to $\iota_0(B)$. One positive involution of $B_0 \otimes B_0^\vee \otimes M_2(\mathbb{Q})$ which satisfies (3.17) is given by

$$\beta = b_0 \otimes b_1 \otimes c \mapsto \beta^\sharp := \tau b_0^\iota \tau^{-1} \otimes b_1^\iota \otimes Ad(\begin{pmatrix} 1 & \\ & \Delta \end{pmatrix})^t c. \tag{3.21}$$

Lemma 3.3. *Any positive involution $'$ of $\mathrm{End}^0(A_0)$ satisfying (3.17) has the form*

$$\beta \mapsto \beta' = \eta\, \beta^\sharp \eta^{-1},$$

where $\eta = 1 \otimes 1 \otimes i_0(\eta_0)$ for $\eta_0 \in \boldsymbol{k}^\times$ with $N_{\boldsymbol{k}/\mathbb{Q}}(\eta_0) > 0$.

Proof. Any involution of $B_0 \otimes B_0^\vee \otimes M_2(\mathbb{Q})$ has the form $\beta \mapsto \beta' = \eta \beta^\sharp \eta^{-1}$ for some invertible element η. The fact that $'$ is an involution is equivalent to the condition that $\eta^\sharp \eta^{-1} = \varepsilon \in \mathbb{Q}^\times$. Applying $\sharp$ to this condition, we find that $\varepsilon^2 = 1$. So there are two basic cases depending on $\varepsilon = \pm 1$. On the other hand, since $'$ also satisfies (3.17), η must lie in the centralizer of $B_0 \otimes 1 \otimes i_0(\boldsymbol{k})$, i.e., in $1 \otimes B_0^\vee \otimes i_0(\boldsymbol{k})$. We may identify $B_0^\vee \otimes M_2(\mathbb{Q})$ with $M_2(B_0^\vee)$ and write

$$\eta = \begin{pmatrix} \eta_1 & \eta_2 \\ \Delta\eta_2 & \eta_1 \end{pmatrix}, \tag{3.22}$$

with η_1 and $\eta_2 \in B_0^\vee$. Applying $\sharp$ yields

$$\eta^\sharp = \begin{pmatrix} \eta_1^\iota & \eta_2^\iota \\ \Delta\eta_2^\iota & \eta_1^\iota \end{pmatrix} = \varepsilon \begin{pmatrix} \eta_1 & \eta_2 \\ \Delta\eta_2 & \eta_1 \end{pmatrix}. \tag{3.23}$$

If $\varepsilon = -1$, we have $\eta_1^\iota = -\eta_1$ and $\eta_2^\iota = -\eta_2$.

Lemma 3.4. *If* $\eta^\sharp = -\eta$ *, then the involution* $\beta \mapsto \eta\beta^\sharp\eta^{-1}$ *is not positive.*

Proof. We may as well replace $B_0^\vee$ with $B_0^\vee \otimes \mathbb{R} = \mathbb{H}$. If $\kappa \in \mathbb{H}$ with $\kappa^\iota = -\kappa$, then the involution $z \mapsto \kappa z^\iota \kappa^{-1}$ is not positive. This follows from the fact that there is an element $z \in \mathbb{H}^\times$ such that $\kappa z = -z\kappa$, so that

$$\operatorname{tr}(z\kappa z^\iota\kappa^{-1}) = -\operatorname{tr}(\kappa z z^\iota \kappa^{-1}) = -2zz^\iota < 0. \tag{3.24}$$

For convenience, let

$$\tilde{\eta} = \eta\begin{pmatrix} 1 & \\ & \Delta \end{pmatrix} = \begin{pmatrix} \eta_1 & \Delta\eta_2 \\ \Delta\eta_2 & \Delta\eta_1 \end{pmatrix}, \tag{3.25}$$

so that, for $x \in M_2(B_0^\vee)$ or $M_2(\mathbb{H})$,

$$x' = \tilde{\eta}\left({}^t x^\iota\right)\tilde{\eta}^{-1}. \tag{3.26}$$

We want to consider the quantity $\operatorname{tr}(xx')$. If $g \in GL_2(\mathbb{H})$, then for $x \in M_2(\mathbb{H})$,

$$\operatorname{tr}\big(g^{-1}xg\tilde{\eta}\left({}^t(g^{-1}xg)^\iota\right)\tilde{\eta}^{-1}\big) = \operatorname{tr}(x(g\tilde{\eta}\,{}^t g^\iota)\left({}^t x^\iota\right)(g\tilde{\eta}\,{}^t g^\iota)^{-1}). \tag{3.27}$$

Thus the positivity of the involution associated to $\tilde{\eta}$ is equivalent to that of the involution associated to $g\tilde{\eta}\,{}^t g^\iota$. But now, if $\eta_1 \neq 0$,

$$\begin{pmatrix} 1 & \\ -\eta_2\Delta\eta_1^{-1} & 1 \end{pmatrix}\begin{pmatrix} \eta_1 & \Delta\eta_2 \\ \Delta\eta_2 & \Delta\eta_1 \end{pmatrix}\begin{pmatrix} 1 & -\eta_1^{-1}\Delta\eta_2 \\ & 1 \end{pmatrix} = \begin{pmatrix} \eta_1 & \\ & \Delta(\eta_1 - \Delta\eta_2\eta_1^{-1}\eta_2) \end{pmatrix}. \tag{3.28}$$

Thus, for $x = \begin{pmatrix} x_1 & 0 \\ 0 & 0 \end{pmatrix}$, we have

$$\operatorname{tr}(x(g\tilde{\eta}\,{}^t g^\iota){}^t x^\iota(g\tilde{\eta}\,{}^t g^\iota)^{-1}) = \operatorname{tr}(x_1\eta_1 x_1^\iota \eta_1^{-1}), \tag{3.29}$$

and this quantity can be negative, as we observed above. If $\eta_1 = 0$, an easier calculation yields the same conclusion, and Lemma 3.4 is proved. □

Continuing the proof of Lemma 3.3, we must have $\varepsilon = +1$, so that η_1 and $\eta_2 \in \mathbb{Q}$, and hence, $\eta = i_0(\eta_0)$ for $\eta_0 = \eta_1 + \delta\eta_2 \in \boldsymbol{k}^\times$. Applying (3.28) again, and taking $x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$, we find that

$$\operatorname{tr}(x(g\tilde{\eta}\,{}^t g^\iota){}^t x^\iota(g\tilde{\eta}\,{}^t g^\iota)^{-1}) = 2x_1x_1^\iota + 2\Delta\eta_1^{-2}N(\eta_0)x_2x_2^\iota + 2\Delta^{-1}\eta_1^2 N(\eta_0)^{-1}x_3x_3^\iota + 2x_4x_4^\iota.$$

This is positive provided $x \neq 0$ and $N(\eta_0) > 0$. □ □

Proposition 3.5. *Suppose that* $\xi = (A, \lambda, \iota, \overline{\eta^p}) \in \mathcal{M}_p^{\mathrm{ss}}$ *is a point in the supersingular isogeny class* I_0 *, and that the Rosati involution* λ_0 *of* A_0 *is given as in Lemma 3.3 for* $\eta_0 \in \boldsymbol{k}^\times$ *. Then the space of special endomorphisms (3.2) is a* 4 *–dimensional* $\mathbb{Q}$ *–vector space:*

$$V' \simeq \{\begin{pmatrix} a & b \\ -\Delta b & -a \end{pmatrix} \ ; \ a \in \mathbb{Q}, \ b \in B_0^\vee, \ b^\iota = -b \ \},$$

where $B_0^\vee$ *is the centralizer of* B_0 *in* $M_2(\mathbb{B})$ *, as above. The quadratic form on this space is*

$$Q_\xi(j) = \alpha N_{\boldsymbol{k}/\mathbb{Q}}(\eta_0)^{-1}(a^2 + \Delta N(b)).$$

Since α *and* $N(\eta_0)$ *are positive, this form is positive definite.*

Proof. For a special endomorphism j_0 of A_0 , we have $j_0 \iota_0(b) = \iota_0(b^\sigma) j_0$, and hence

$$j_0 \in 1 \otimes B_0^\vee \otimes \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \boldsymbol{k}. \tag{3.30}$$

As an element of $M_2(B_0^\vee)$, we can write

$$j_0 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ \Delta b_2 & b_1 \end{pmatrix}. \tag{3.31}$$

Then, recalling that $\eta^\sharp = \eta$, the condition

$$j_0' = \eta j_0^\sharp \eta^{-1} = j_0 \tag{3.32}$$

is equivalent to

$$(j_0 \eta)^\sharp = j_0 \eta. \tag{3.33}$$

Writing

$$j_0 \eta = \begin{pmatrix} c_0 & c_1 \\ -\Delta c_1 & -c_0 \end{pmatrix}, \tag{3.34}$$

we find that $c_0 = c_0^\iota \in \mathbb{Q}$ and $c_1^\iota = -c_1$. This gives the claimed description of V' . Also

$$(j_0 \eta)^2 = j_0^2 \eta^\sigma \eta, \tag{3.35}$$

so that

$$j_0^2 = N(\eta_0)^{-1}(c_0^2 + \Delta N(c_1)), \tag{3.36}$$

and, recalling Proposition 3.2, we obtain the claim expression for the quadratic form on V' . $\square$

In the case I_1 , we have $A_0 \simeq A_{00}^4$ and $\iota_0 : B \hookrightarrow \mathrm{End}^0(A_0) \simeq M_4(E)$, where $\mathrm{End}^0(A_{00}) = E$ and $\boldsymbol{k}' = \boldsymbol{k} \cdot E$ splits B .

Proposition 3.6. *(i) Let $E/\mathbb{Q}$ be an imaginary quadratic field, and let $E^\vee$ be the other imaginary quadratic subfield of the biquadratic field $\boldsymbol{k}\cdot E$. For the isogeny class I_1 associated to $E/\mathbb{Q}$, the space of special endomorphisms is isomorphic, as quadratic space over $\mathbb{Q}$, to $E^\vee$ with quadratic form $Q_\xi(j) = \alpha\alpha_1 N_{E^\vee/\mathbb{Q}}(j)$, where the additional scalar $\alpha_1 \in \mathbb{Q}^\times_{>0}$ is determined in the proof below.*
(ii) Let $\boldsymbol{k}'$ be a CM extension of $\boldsymbol{k}$ which is not a composite of $\boldsymbol{k}$ with an imaginary quadratic field. Then the space of special endomorphisms for the associated isogeny class I_2 is zero.

Proof. First assume that E splits the quaternion algebra B_0 over $\mathbb{Q}$. Then

$$B_0 \otimes_{\mathbb{Q}} E \otimes_{\mathbb{Q}} M_2(\mathbb{Q}) \simeq M_2(E) \otimes M_2(\mathbb{Q}) \simeq M_4(E), \tag{3.37}$$

so that we can take

$$\iota_0 : B = B_0 \otimes \boldsymbol{k} \longrightarrow B_0 \otimes E \otimes M_2(\mathbb{Q}), \qquad b_0 \otimes a \mapsto b_0 \otimes 1 \otimes i_0(a), \tag{3.38}$$

with $i_0 : \boldsymbol{k} \hookrightarrow M_2(\mathbb{Q})$ as before. The positive involution

$$\beta = b_0 \otimes e \otimes c \mapsto \beta^\sharp := \tau b_0^\iota \tau^{-1} \otimes \bar{e} \otimes Ad(\begin{pmatrix} 1 & \\ & \Delta \end{pmatrix})({}^t c) \tag{3.39}$$

satisfies $\iota_0(b)^\sharp = \iota_0(\tau b^\iota \tau^{-1})$. The most general such involution has the form

$$\beta \mapsto \beta' = \eta \beta^\sharp \eta^{-1}, \tag{3.40}$$

where $\eta \in 1 \otimes E \otimes i_0(\boldsymbol{k})$, the centralizer of $\iota_0(B)$. If we view $E \otimes M_2(\mathbb{Q}) \xrightarrow{\sim} M_2(E)$ and $E \otimes \boldsymbol{k} \xrightarrow{\sim} \boldsymbol{k}'$ with $i_0 : \boldsymbol{k}' \hookrightarrow M_2(E)$ corresponding to $1 \otimes i_0$, then $\eta = i_0(\eta_0)$ with $\eta_0 \in \boldsymbol{k}'^{,\times}$. Since $'$ is an involution, $\eta^\sharp \eta^{-1} = \lambda \in E^\times$, and $\lambda^\sharp \lambda = \bar{\lambda}\lambda = 1$. Writing $\lambda = \nu\bar{\nu}^{-1}$, we have $(\eta\nu)^\sharp = \eta\nu$. Since η and $\eta\nu$ give the same involution, we may assume that $\eta^\sharp = \eta$. Then, since $\eta^\sharp = i_0(\bar{\eta}_0)$, we have $\eta_0 \in \boldsymbol{k}^\times$. It is easy to check that the involution $'$ is positive if and only if $N_{\boldsymbol{k}/\mathbb{Q}}(\eta_0) > 0$.

If j_0 is a special endomorphism, then the condition $j_0 \iota_0(b) = \iota_0(b^\sigma) j_0$ implies that there is a $\mu_0 \in \boldsymbol{k}'^{,\times}$ so that, for $\mu = i_0(\mu_0)$,

$$j_0 = 1 \otimes \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \mu \in 1 \otimes \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} i_0(\boldsymbol{k}') \subset B_0 \otimes M_2(E). \tag{3.41}$$

Since

$$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}' = \eta \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}^\sharp \eta^{-1} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \eta^\sigma \eta^{-1}, \tag{3.42}$$

we have

$$(3.43)\qquad \begin{pmatrix}1&\\&-1\end{pmatrix} i_0(\mu_0) = j_0 = j_0' = \begin{pmatrix}1&\\&-1\end{pmatrix} i_0(\bar{\mu}_0^\sigma)\eta^\sigma\eta^{-1},$$

so that

$$(3.44)\qquad \rho := \mu_0\eta_0 = (\overline{\mu_0\eta_0})^\sigma \in E^\vee.$$

Hence

$$(3.45)\qquad j_0 = \begin{pmatrix}1&\\&-1\end{pmatrix}\eta^{-1} i_0(\rho),$$

and

$$(3.46)\qquad j_0^2 = N_{\boldsymbol{k}/\mathbb{Q}}(\eta)^{-1} N_{E^\vee/\boldsymbol{k}}(\rho)\, 1_2$$

as claimed, in this case.

In general, there is a quaternion algebra B_1 over $\mathbb{Q}$ such that $B = B_1 \otimes_{\mathbb{Q}} \boldsymbol{k}$ and such that E splits B_1. Let σ_1 be the automorphism of B which is given by $1\otimes\sigma$ on $B = B_1 \otimes \boldsymbol{k}$. There is then an element $h \in B^\times$ such that $b^\sigma = Ad(h)(b^{\sigma_1})$. The fact that σ and σ_1 have order 2 implies that $hh^{\sigma_1} \in \boldsymbol{k}^\times$. Also, $h^\sigma = hh^{\sigma_1}h^{-1} = h^{-1}hh^{\sigma_1} = h^{\sigma_1}$. Applying Theorem 90, we may adjust h so that, in fact $h^\sigma = h^{\sigma_1} = h^\iota$. In particular, $hh^\sigma = hh^{\sigma_1} = \nu(h) \in \mathbb{Q}^\times$.

We then have

$$(3.47)\qquad B_1 \otimes E \otimes \boldsymbol{k} \longrightarrow B_1 \otimes E \otimes M_2(\mathbb{Q}) \simeq M_2(E) \otimes M_2(\mathbb{Q}) \simeq M_4(E),$$

and an embedding

$$(3.48)\qquad i_1 : B = B_1 \otimes \boldsymbol{k} \hookrightarrow B_1 \otimes E \otimes M_2(\mathbb{Q}), \qquad b_1 \otimes a \mapsto b_1 \otimes 1 \otimes i_0(a).$$

Note that we have

$$(3.49)\qquad Ad(\begin{pmatrix}1&\\&-1\end{pmatrix}) i_0(a) = i_0(a^\sigma),$$

and

$$(3.50)\qquad Ad(\begin{pmatrix}1&\\&-1\end{pmatrix}) i_1(b) = i_1(b^{\sigma_1}).$$

Let

$$(3.51)\qquad \tau_+ = \begin{cases}\tau & \text{if } \nu(h) > 0,\\ \delta\tau & \text{if } \nu(h) < 0,\end{cases}$$

and define an involution on $B_1 \otimes E \otimes M_2(\mathbb{Q})$ by

$$(3.52)\qquad \beta = b_1 \otimes e \otimes c \mapsto \beta^\sharp := Ad(i_1(\tau_+))\big(b_1^\iota \otimes \bar{e} \otimes Ad(\begin{pmatrix}1&\\&\Delta\end{pmatrix})({}^tc)\big).$$

Lemma 3.7. *(i) The involution $\sharp$ is positive and satisfies*

$$i_1(b)^\sharp = i_1(\tau b^\iota \tau^{-1}).$$

(ii) Any positive involution with this restriction to B has the form

$$\beta \mapsto \beta' = \eta \beta^\sharp \eta^{-1},$$

where $\eta = i_0(\eta_0)$ for $\eta_0 \in \boldsymbol{k}^\times$ with $N_{\boldsymbol{k}/\mathbb{Q}}(\eta_0) > 0$.

Proof. The involution $\sharp$ preserves the subalgebra $B_1 \otimes M_2(\mathbb{Q})$, and it suffices to prove that its restriction to this subalgebra is positive. Since

$$M_2(\mathbb{Q}) = i_0(\boldsymbol{k}) \oplus \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} i_0(\boldsymbol{k}), \tag{3.53}$$

we have an isomorphism of right B-modules

$$B_1 \otimes M_2(\mathbb{Q}) \simeq B \oplus \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} B. \tag{3.54}$$

The left regular representation yields an algebra homomorphism:

$$\begin{aligned} B_1 \otimes M_2(\mathbb{Q}) \simeq B \oplus B \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} &\hookrightarrow M_2(B) \\ b_1 + b_2 \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} &\mapsto \begin{pmatrix} b_1 & b_2 \\ b_2^{s_1} & b_1^{s_1} \end{pmatrix}. \end{aligned} \tag{3.55}$$

We write $i_2 : B \to M_2(B)$ for the composition of this map with i_1. The restriction of the involution

$$\beta \mapsto Ad(i_2(\tau_+))\left({}^t\beta^\iota\right) \tag{3.56}$$

to $B_1 \otimes M_2(\mathbb{Q})$ coincides with $\sharp$, so it suffices to prove that (3.56) is positive on $M_2(B)$. But we have, setting $\varepsilon = \mathrm{sgn}(\nu(h))$,

(3.57)

$$\begin{aligned} &\mathrm{tr}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\iota \begin{pmatrix} \tau & \\ & \varepsilon\tau^{s_1} \end{pmatrix} \begin{pmatrix} a^\iota & c^\iota \\ b^\iota & d^\iota \end{pmatrix} \begin{pmatrix} \tau & \\ & \varepsilon\tau^{s_1} \end{pmatrix}^{-1} \right) \\ &\qquad = \mathrm{tr}(a\tau a^\iota \tau^{-1} + b\varepsilon\tau^{\sigma_1} b^\iota \tau^{-1} + c\tau c^\iota \varepsilon \tau^{-\sigma_1} + d\tau^{\sigma_1} d^\iota \tau^{-\sigma_1}). \end{aligned}$$

The quantities $\mathrm{tr}(a\tau a^\iota\tau^{-1})$ and $\mathrm{tr}(d\tau^{\sigma_1} d^\iota \tau^{-\sigma_1}) = \mathrm{tr}(d^{\sigma_1}\tau(d^\iota)^{\sigma_1}\tau^{-1})^\sigma$ in the last expression are positive for a and d nonzero. On the other hand,

$$\begin{aligned} \mathrm{tr}(b\varepsilon\tau^{\sigma_1} b^\iota \tau^{-1}) &= \varepsilon \mathrm{tr}(bh^{-1}\tau h b^\iota \tau^{-1}) \\ &= \varepsilon\nu(h)^{-1}\mathrm{tr}((bh^\iota)\tau(bh^\iota)^\iota\tau^{-1}) > 0, \end{aligned} \tag{3.58}$$

and similarly for $\mathrm{tr}(c\tau c^\iota \varepsilon \tau^{-\sigma_1})$. This proves (i).

The centralizer of $i_1(B)$ in $B_1 \otimes E \otimes M_2(\mathbb{Q})$ is $1 \otimes E \otimes i_0(\boldsymbol{k})$ and this can be identified with $1 \otimes i_0(\boldsymbol{k}')$ in $B_1 \otimes M_2(E)$. Note that this algebra centralizes $i_1(\tau_+)$. We can argue as above to prove (ii). $\square$

As in (3.41), we can write any special endomorphism in the form

$$j_0 = i_1(h)\left(1\otimes\begin{pmatrix}1 & \\ & -1\end{pmatrix}\mu\right) \in i_1(h)\left(1\otimes\begin{pmatrix}1 & \\ & -1\end{pmatrix} i_0(\boldsymbol{k}')\right) \subset B_1 \otimes M_2(E), \tag{3.59}$$

for $\mu = i_0(\mu_0)$ and $\mu_0 \in \boldsymbol{k}'$. To take into account the condition $j_0' = j_0$, we first note that, via (3.49) and (3.50),

$$\begin{aligned}\begin{pmatrix}1 & \\ & -1\end{pmatrix}' &= Ad(\eta)Ad(i_1(\tau_+))\left(\begin{pmatrix}1 & \\ & -1\end{pmatrix}\right)\\ &= \begin{pmatrix}1 & \\ & -1\end{pmatrix} i_0(\eta_0^\sigma)i_1(\tau_+^{\sigma_1})i_1(\tau_+^{-1})i_0(\eta_0^{-1})\\ &= \begin{pmatrix}1 & \\ & -1\end{pmatrix} i_0(\eta_0^\sigma\eta_0^{-1})i_1(\varepsilon\tau^{\sigma_1}\tau^{-1}),\end{aligned} \tag{3.60}$$

where $\varepsilon = \mathrm{sgn}(\nu(h))$, as above. Now

$$\begin{aligned} j_0' &= \eta j_0^\sharp \eta^{-1}\\ &= Ad(\eta)Ad(i_1(\tau_+))\left(i_0(\bar\mu_0)\begin{pmatrix}1 & \\ & -1\end{pmatrix} i_1(h^\iota)\right)\\ &= i_0(\bar\mu_0)\begin{pmatrix}1 & \\ & -1\end{pmatrix} i_0(\eta_0^\sigma\eta_0^{-1})i_1(\varepsilon\tau^{\sigma_1}\tau^{-1})i_1(\tau h^\iota\tau^{-1})\\ &= \begin{pmatrix}1 & \\ & -1\end{pmatrix} i_0(\bar\mu_0^\sigma\eta_0^\sigma\eta_0^{-1})i_1(\varepsilon\tau^{\sigma_1}h^\iota\tau^{-1}).\end{aligned} \tag{3.61}$$

Setting this equal to

$$j_0 = \begin{pmatrix}1 & \\ & -1\end{pmatrix} i_1(h^{\sigma_1})i_0(\mu_0), \tag{3.62}$$

recalling that $h^{\sigma_1} = h^\iota$, and dividing both sides by $\nu(h)$, we obtain

$$\begin{aligned} i_0(\mu_0) &= i_1(h)i_0(\bar\mu_0^\sigma\eta_0^\sigma\eta_0^{-1})i_1(\varepsilon\tau^{\sigma_1}h^{-1}\tau^{-1})\\ &= \varepsilon\, i_0(\bar\mu_0^\sigma\eta_0^\sigma\eta_0^{-1})i_1(h\tau^{\sigma_1}h^{-1}\tau^{-1})\\ &= \varepsilon\, i_0(\bar\mu_0^\sigma\eta_0^\sigma\eta_0^{-1}),\end{aligned} \tag{3.63}$$

since $h\tau^{\sigma_1}h^{-1} = \tau^\sigma = \tau$. Thus, if we set

$$\rho = \begin{cases} \mu_0\eta_0 & \text{if } \nu(h) > 0, \\ \delta\mu_0\eta_0 & \text{if } \nu(h) < 0, \end{cases} \tag{3.64}$$

then $\rho \in E^\vee$ and

$$j_0 = i_1(h)\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} i_0(\rho\eta_{0,+}^{-1}), \tag{3.65}$$

where $\eta_{0,+} = \eta_0$ if $\nu(h) > 0$ and $\eta_{0,+} = \delta\eta_0$ if $\nu(h) < 0$.

Finally, $j_0^2 = Q_0(j_0)\cdot id$ gives

$$Q_0(j_0) = \nu(h)N_{\boldsymbol{k}/\mathbb{Q}}(\eta_0)^{-1}N_{E^\vee/\mathbb{Q}}(\rho)\cdot\begin{cases} 1 & \text{if } \nu(h) > 0, \\ (-\Delta)^{-1} & \text{if } \nu(h) < 0, \end{cases} \tag{3.66}$$

as claimed. This proves the first part of Proposition 3.6.

Finally, consider case I_2 , where $A_0 \simeq A_1^2$ for a simple abelian surface A_1 with $\mathrm{End}^0(A_1) = \boldsymbol{k}'$ and with $\iota_0 : B \hookrightarrow \mathrm{End}^0(A_0) \simeq M_2(\boldsymbol{k}')$. But then the isomorphism

$$B \otimes_{\boldsymbol{k}} \boldsymbol{k}' \simeq M_2(\boldsymbol{k}'). \tag{3.67}$$

implies that $\iota_0(\boldsymbol{k})$ lies in the center of $\mathrm{End}^0(A_0)$, and so the space of special endomorphisms is zero. □ □

The results obtained so far impose restrictions on the isogeny classes which can be met by a special cycle $\mathcal{Z}(\omega, T)$. Recall that we assume that $\det(T) \neq 0$. If $\xi = (A, \lambda, \iota, \bar{\eta}^p; \mathbf{j}) \in \mathcal{Z}(\omega, T)(\mathbb{F})$, then the components of the collection $\mathbf{j}$ of special endomorphisms of (A, λ, ι) lie in V_ξ , and $Q_\xi(\mathbf{j}) = T$.

Corollary 3.8. *If $\xi \in \mathcal{Z}(\omega, T)(\mathbb{F})$, then T is represented by $V_\xi(\mathbb{Q})$. In particular:*
(i) The isogeny classes of type I_2 do not meet any special cycle.
(ii) If $n \geq 3$, then $\mathcal{Z}(\omega, T)$ lies in the supersingular locus and is empty unless $T > 0$.
(iii) When $n = 4$, $\mathcal{Z}(\omega, T)$ is empty unless T is equivalent over $\mathbb{Q}$ to the quadratic form of Proposition 3.5.
(iv) When $n = 3$, $\mathcal{Z}(\omega, T)$ is empty unless T is represented over $\mathbb{Q}$ by the quadratic form of Proposition 3.5.
(v) If $n = 2$, then the cycle $\mathcal{Z}(\omega, T)$ can meet at most one nonsupersingular isogeny class, namely, that for which T is equivalent over $\mathbb{Q}$ to the quadratic form Q_ξ of Proposition 3.6 (i).

§4. The structure of the supersingular locus.

In this section, we give a description of the supersingular locus $\mathcal{M}^{\rm ss}$ in the special fiber of $\mathcal{M} = \mathcal{M}_{K^p}$. The results are due Stamm [**18**], although his presentation is somewhat different. We continue to assume that p is inert in $\boldsymbol{k}$. Then $\mathcal{O}_C \otimes \mathbb{Z}_p \simeq M_4(\mathbb{Z}_p)$ and $\mathcal{O}_{\boldsymbol{k}} \otimes \mathbb{Z}_p \simeq \mathbb{Z}_{p^2}$. Also let $W = W(\mathbb{F})$ be the Witt ring of $\mathbb{F}$, $\mathcal{K} = W \otimes \mathbb{Q}$ its quotient field, and σ the Frobenius automorphism.

Fix a point $\xi = (A, \lambda, \iota, \bar{\eta}^p) \in \mathcal{M}^{\rm ss}(\mathbb{F})$ in the supersingular locus. Let $A(p)$ be the p-divisible group of A, and note that the inclusion

$$\iota : \mathcal{O}_C \otimes \mathbb{Z}_p \simeq M_4(\mathbb{Z}_p) \hookrightarrow \mathrm{End}(A(p)) \tag{4.1}$$

yields a decomposition $A(p) \simeq \mathcal{A}^4$ where $\mathcal{A}$ is a p-divisible group of dimension 2 and height 4 with an action $\iota_0 : \mathbb{Z}_{p^2} \hookrightarrow \mathrm{End}(\mathcal{A})$. Since A is supersingular, $A(p)$ and $\mathcal{A}$ are formal groups. We let $\mathrm{D}\mathcal{A}$ be the contravariant Dieudonné module of $\mathcal{A}$. This is a free W-module of rank 4 with operators F and V which are σ- and σ^{-1}-linear respectively, and with $FV = VF = p$. Let $\mathcal{L} = \mathrm{D}\mathcal{A} \otimes \mathbb{Q}$ be the associated isocrystal, with $\dim_{\mathcal{K}}(\mathcal{L}) = 4$. The polarization of A gives rise to a nondegenerate alternating form $<\,,\,>: \mathcal{L} \times \mathcal{L} \to \mathcal{K}$, with $< Fx, y > = < x, Vy >^\sigma$. Moreover, for $a \in \mathbb{Z}_{p^2}$,

$$< \iota_0(a)x, y > = < x, \iota_0(a)y > . \tag{4.2}$$

The fixed embedding $\mathbb{Z}_{p^2} \to W$ defines a $\mathbb{Z}/2$-grading

$$\mathcal{L}_0 = \{x \in \mathcal{L}; \iota_0(a)x = ax, \text{for } a \in \mathbb{Z}_{p^2}\} \tag{4.3}$$

and

$$\mathcal{L}_1 = \{x \in \mathcal{L}; \iota_0(a)x = a^\sigma x, \text{for } a \in \mathbb{Z}_{p^2}\}, \tag{4.4}$$

so that $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$, and F and V are endomorphisms of degree 1. Note that the two dimensional subspaces $\mathcal{L}_0$ and $\mathcal{L}_1$ are orthogonal to each other with respect to $<\,,\,>$. Let

$$G'_p = \{g \in GL(\mathcal{L});\ < gx, gy >= \nu(g)\cdot < x, y >,\ Fg = gF \text{ with } \nu(g) \in \mathbb{Q}_p^\times\}\ . \tag{4.5}$$

We are interested in W-lattices L in $\mathcal{L}$ which are stable under F and V and under the action of $\mathbb{Z}_{p^2}$, and such that $L^\perp = cL$ with respect to $<\,,\,>$. The lattice L is stable under $\mathbb{Z}_{p^2}$ if and only if $L = L_0 \oplus L_1$ with respect to the grading. For such a lattice L, we have

$$L_0 \supset FL_1 \supset pL_0, \tag{4.6}$$

and

$$L_1 \supset FL_0 \supset pL_1, \tag{4.7}$$

where all inclusions have index 1. This follows from the determinant condition, which implies that L_0/FL_1 and L_1/FL_0 have dimension 1 over $\mathbb{F}$.

Let X be the set of such lattices in $\mathcal{L}$.

Definition 4.1. *(i) For a lattice $L \in X$, the index i is critical for L if $F^2L_i = pL_i$.*
(ii) The lattice L is called superspecial if both indices 0 and 1 are critical for L.

Let $X_0 \subset X$ be the set of superspecial lattices.

Lemma 4.2. *For any $L \in X$, at least one index is critical.*

Proof. Note that $F^2L_i = pL_i = FVL_i$ if and only if $FL_i = VL_i$. Since we have inclusions

$$\begin{array}{ccc} FL_i & \subset & L_{i+1} \\ \cup & & \cup \\ pL_{i+1} & \subset & VL_i, \end{array}$$

we have that either i is critical or $L_{i+1} = FL_i + VL_i$. Suppose that there is no critical index, i.e., that $L_0 = FL_1 + VL_1$ and $L_1 = FL_0 + VL_0$. But then, using the fact that $FL_1 \supset pL_0$,

$$\begin{aligned} L_0 &= FL_1 + VL_1 \\ &= FL_1 + V(FL_0 + VL_0) = FL_1 + V^2L_0 \\ &= FL_1 + V^2(FL_1 + VL_1) = FL_1 + V^3L_1 \\ &= \dots \\ &= FL_1 + V^rL_i \quad \text{for all } r \geq 1 \\ &= FL_1, \end{aligned}$$

since the Dieudonné module is V-reduced. This contradicts the fact that L_0/FL_1 has dimension 1 over $\mathbb{F}$. $\square$

Note that the $\mathcal{K}$-vector space $\mathcal{L}_i$ has a σ^2-linear automorphism $p^{-1}F^2$. If i is a critical index for the lattice $L \in X$, then $p^{-1}F^2$ preserves L_i and defines an $\mathbb{F}_{p^2}$-rational structure on the $\mathbb{F}$-vector space L_i/pL_i and on the projective line $\mathbb{P}(L_i/pL_i)$.

Lemma 4.3. *(i) Suppose that* L_i *is a* W*-lattice in* $\mathcal{L}_i$ *such that* $F^2L_i = pL_i$. *For any line* ℓ *in the two dimensional* $\mathbb{F}$*-vector space* L_i/pL_i, *let* $L_{i+1} = F^{-1}(\ell + pL_i)$, *and let* $L = L_{i+1} \oplus L_i$. *Then* $L \in X$.
(ii) If $L = L_0 \oplus L_1 \in X$ *with* i *a critical index, let* $\ell \in \mathbb{P}(L_i/pL_i)$ *be the line* FL_{i+1}/pL_i. *Then* L *is recovered from* ℓ *by the construction of (i).*
(iii) The lattice $L = L_{i+1} \oplus L_i \in X$ *associated to* $\ell \in \mathbb{P}(L_i/pL_i)$ *is superspecial if and only if the line* ℓ *is rational over* $\mathbb{F}_{p^2}$.

Proof. Note that $FL_i = VL_i$. Since $L_{i+1} = F^{-1}(\ell + pL_i) \subset F^{-1}L_i$, we clearly have $FL_{i+1} \subset L_i$ and $VL_{i+1} \subset L_i$. Also, $L_{i+1} \supset F^{-1}pL_i = FL_i = VL_i$. Next, since the restriction of $<\ ,\ >$ to the two dimensional space $\mathcal{L}_i$ is a nondegenerate alternating form, we must have $< L_i, L_i >= cW$ for $c = p^r$, for some $r \in \mathbb{Z}$, and $L_i^{\perp} = c^{-1}L_i$. But then, writing $\ell + pL_i = Wy + pL_i$, we have

$$\begin{aligned} < \ell + pL_i, \ell + pL_i > &= < Wy + pL_i, Wy + pL_i > \\ &= < y, pL_i > \\ &= \ p < L_i, pL_i > \\ &= pcW. \end{aligned}$$

But then,

$$\begin{aligned} < L_{i+1}, L_{i+1} > &= < F^{-1}(\ell + pL_i), p^{-1}V(\ell + pL_i) > \\ &= p^{-1} < \ell + pL_i, \ell + pL_i >^{\sigma^{-1}} \\ &= cW, \end{aligned}$$

and so $(L_{i+1})^{\perp} = c^{-1}L_{i+1}$. This proves (i), and (ii) is obvious.

To prove (iii), observe that $FL_{i+1} = VL_{i+1}$ if and only if $\ell + pL_i = VF^{-1}(\ell) + pL_i$, and that $VF^{-1} = pF^{-2}$. $\square$

The set X can be described as the set of $\mathbb{F}$-points of a scheme over $\mathbb{F}_{p^2}$, cf. [**18**]. Let Y_i be the set of W-lattices in $\mathcal{L}_i$ which satisfy $F^2 L_i = pL_i$. Let

$$U_i = \mathcal{L}_i^{pF^{-2}} \quad . \tag{4.8}$$

Since pF^{-2} is a σ^{-2}-linear automorphism with all slopes equal to 0, U_i is a 2-dimensional $\mathbb{Q}_{p^2}$-vector space with $U_i \otimes_{\mathbb{Q}_{p^2}} \mathcal{K} = \mathcal{L}_i$ and each $L_i \in Y_i$ can be written as

$$L_i = \Lambda_i \otimes_{\mathbb{Z}_{p^2}} W \tag{4.9}$$

for the $\mathbb{Z}_{p^2}$-lattice $\Lambda_i = L_i^{pF^{-2}}$ of U_i. To each $L_i \in Y_i$ we associate the projective line

$$\mathbb{P}_{L_i} = \mathbb{P}(\Lambda_i / p\Lambda_i)$$

over $\mathbb{F}_{p^2}$. At each of its p^2+1 $\mathbb{F}_{p^2}$-rational points, $\mathbb{P}_{L_i}$ meets a single $\mathbb{P}_{L_{i+1}}$.

For the following result, cf. [**18**].

Proposition 4.4. *There is a natural G'_p-equivariant surjective map*

$$\coprod_{i=0,1} \coprod_{L_i \in Y_i} \mathbb{P}_{L_i}(\mathbb{F}) \longrightarrow X,$$

which induces a bijection

$$\coprod_{i=0,1} \coprod_{L_i \in Y_i} \Big(\mathbb{P}_{L_i}(\mathbb{F}) - \mathbb{P}_{L_i}(\mathbb{F}_{p^2}) \Big) \longrightarrow X - X_0.$$

For $L \in X_0$ the preimage consists of two points, one for each $i=0$ and $i=1$.

Let G' be the inner form of G (cf. (0.2)) given by the quaternion algebra B' over $\boldsymbol{k}$ which is ramified at the two archimedean primes and is isomorphic to B at all finite primes. Then there exists a morphism of schemes over $\operatorname{Spec} \mathbb{F}_{p^2}$ where $\mathcal{M}^{ss}_{K^p}$ denotes the supersingular locus of $\mathcal{M}_{K^p} \times_{\operatorname{Spec} \mathbb{Z}_{(p)}} \operatorname{Spec} \mathbb{F}_p$,

$$G'(\mathbb{Q}) \setminus \left[\left(\coprod_{i=0,1} \coprod_{L_i \in Y_i} \mathbb{P}_{L_i} \right) \times G'(\mathbb{A}_f^p)/K^p \right] \longrightarrow \mathcal{M}^{ss} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2} \quad . \tag{4.10}$$

This morphism can be identified with the normalization of $\mathcal{M}^{ss} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}$, cf. [**18**]. It induces a bijection

$$\coprod_{i=0,1} G'(\mathbb{Q}) \setminus G'(\mathbb{A}_f)/K_p.K^p \xrightarrow{\sim} \operatorname{Irred}(\mathcal{M}^{ss} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}) \quad . \tag{4.11}$$

Here on the right hand side is the set of geometric irreducible components of $\mathcal{M}^{ss}$ and K_p is a maximal open compact subgroup of $G'_p \cong G'(\mathbb{Q}_p)$ (the stabilizer of a $\mathbb{Z}_{p^2}$-lattice $\Lambda \subset U_i$).

Similarly we obtain for the set of superspecial points of $\mathcal{M}^{ss}$ a bijection

$$G'(\mathbb{Q}) \setminus G'(\mathbb{A}_f)/K'_p.K^p \xrightarrow{\sim} \mathcal{M}^{supersp}(\overline{\mathbb{F}}_p) \quad . \tag{4.12}$$

Here K'_p is an Iwahori subgroup of G'_p. Indeed, it follows from Lemma 5.1 below that all superspecial lattices are conjugate under G'_p (existence of a standard basis) and the stabilizer of a superspecial lattice is an Iwahori subgroup.

§5. Special endomorphisms of supersingular Dieudonné modules.

In this section we study the space V'_p (cf.(5.3)) of special endomorphisms of the isocrystal $\mathcal{L}$ associated to the supersingular isogeny class as in section 4. After giving an explicit description of this quadratic space over $\mathbb{Q}_p$, we provide a criterion for a given special endomorphism j of $\mathcal{L}$ to induce a special endomorphism of a lattice $L \in X$ (resp. every lattice in $\mathbb{P}_{L_i}$ for $L_i \in Y_i$). This information will be used to determine isolated supersingular point on special cycles (section 6) and supersingular components of special cycles (section 8).

We keep the notations of the previous section. We begin by considering the space of special endomorphisms of the $\mathcal{K}$-vector space $\mathcal{L}$. Recall that $\mathcal{K} = W \otimes \mathbb{Q}$. The grading $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$ gives a grading on the endomorphism ring:

$$\mathrm{End}(\mathcal{L})^{(0)} = \{j \in \mathrm{End}(\mathcal{L}) \ ; \ j\iota_0(a) = \iota_0(a)j\}, \tag{5.1}$$

and

$$\mathrm{End}(\mathcal{L})^{(1)} = \{j \in \mathrm{End}(\mathcal{L}) \ ; \ j\iota_0(a) = \iota_0(a^\sigma)j\}. \tag{5.2}$$

Let $\mathrm{End}(\mathcal{L}, F)$ be the algebra of endomorphisms which commute with F, and let

$$V'_p := \{j \in \mathrm{End}(\mathcal{L}, F)^{(1)} \ ; \ j^* = j\} \tag{5.3}$$

be the space of special endomorphisms of $\mathcal{L}$, where $*$ denotes the adjoint with respect to the alternating form $< \, , \, >$. Writing $j \in \mathrm{End}(\mathcal{L}, F)^{(1)}$ as

$$j = \begin{pmatrix} & j_1 \\ j_0 & \end{pmatrix}, \qquad j_0 \in \mathrm{Hom}(\mathcal{L}_0, \mathcal{L}_1) \text{, and } j_1 \in \mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_0) \text{,} \tag{5.4}$$

we have $j^* = j$ if and only if $j_1 = j_0^*$. Since j commutes with F, we also have $j_1 = Fj_0F^{-1}$ and $j_0 = Fj_1F^{-1}$, so that

$$V_p' \simeq \{j_0 \in \mathrm{Hom}(\mathcal{L}_0, \mathcal{L}_1) \ ; \ j_0 = F^2 j_0 F^{-2}, \ j_0^* = Fj_0F^{-1}\}. \tag{5.5}$$

If $x = j_0^* j_0 \in \mathrm{End}(\mathcal{L}_0, F^2)$, then x is self adjoint with respect to $< \, , \, >$ on the 2-dimensional space $\mathcal{L}_0$, and hence $x = a \cdot 1_{\mathcal{L}_0}$ for a scalar a. Thus, we have a quadratic form on V_p', defined by

$$j^2 = Q'(j) \cdot 1_{\mathcal{L}} \quad , \quad Q'(j) = j_0^* \cdot j_0 \quad . \tag{5.6}$$

Since the elements of G_p' commute with F, cf (4.5), we have an inclusion $G_p' \subset \mathrm{End}(\mathcal{L})^{(0),\times}$ and G_p' acts on V_p' by conjugation and preserves the quadratic form.

Choose a $\mathbb{Q}_{p^2}$-rational basis e_1, e_2 for $U_0 = \mathcal{L}_0^{pF^{-2}}$, with $< e_1, e_2 >= p^\ell$, and let $e_3 = Fe_1$ and $e_4 = p^{-1}Fe_2$, so that $< e_3, e_4 >= p^\ell$ as well. We call such a basis a *standard basis for* $\mathcal{L}$. Then, we may write

$$F = \begin{pmatrix} & & p & \\ & & & 1 \\ 1 & & & \\ & p & & \end{pmatrix} \sigma = \begin{pmatrix} & \pi \\ p\pi^{-1} & \end{pmatrix} \sigma, \tag{5.7}$$

where we set $\pi = \begin{pmatrix} p & \\ & 1 \end{pmatrix}$. If $j_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $j_0^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, and we have $jF = Fj$ if and only if

$$j_1 = p^{-1}\pi j_0^\sigma \pi, \qquad \text{and} \qquad j_1^\sigma = p^{-1}\pi j_0 \pi. \tag{5.8}$$

Applying σ to the first condition and comparing to the second, we have $j_0^{\sigma^2} = 1$. Recalling that $j_1 = j_0^*$, we obtain

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} pa^\sigma & b^\sigma \\ c^\sigma & p^{-1}d^\sigma \end{pmatrix}, \tag{5.9}$$

i.e., $d = pa^\sigma$, $c = -c^\sigma$, and $b = -b^\sigma$. Thus,

$$V_p' \simeq \{x = \begin{pmatrix} a & b \\ c & pa^\sigma \end{pmatrix} ; \ a, \, b, \, c \in \mathbb{Q}_{p^2}, \ b^\sigma = -b, \ c^\sigma = -c\} \tag{5.10}$$

is a 4-dimensional vector space over $\mathbb{Q}_p$ with quadratic form

$$Q'(x) = paa^\sigma - bc. \tag{5.11}$$

Also,

$$G'_p \simeq \{g \in GL_2(\mathbb{Q}_{p^2});\ \det(g) \in \mathbb{Q}_p^\times\}, \tag{5.12}$$

and $g \in G'_p$ acts on $x \in V'_p$ by

$$g:\ x \mapsto \pi^{-1} g^\sigma \pi x g^{-1}. \tag{5.13}$$

For any lattice $L \in X$, the ring $\mathrm{End}(L,F)$ is an order in the algebra $\mathrm{End}(\mathcal{L},F)$, and

$$N_L := \mathrm{End}(L,F) \cap V'_p \tag{5.14}$$

is a $\mathbb{Z}_p$-lattice in V'_p.

Lemma 5.1. *Suppose that $L \in X_0$ is a superspecial lattice with $< L, L >= p^\ell W$. Then, there exists a standard basis e_1, e_2, e_3, e_4 for $\mathcal{L}$ such that $L_0 = We_1 \oplus We_2$ and $L_1 = We_3 \oplus We_4$.* $\square$

We refer to the basis of Lemma 5.1 as a *standard basis* for $L \in X_0$.

Corollary 5.2. *If L is a superspecial lattice in $\mathcal{L}$, then with respect to a standard basis for L,*

$$\mathrm{End}(L,F)^{(0)} = \{\begin{pmatrix} A_0 & \\ & A_1 \end{pmatrix};\ A_0 = \begin{pmatrix} a & pb \\ c & d \end{pmatrix},\ \textit{and } A_1 = \pi^{-1} A_0^\sigma \pi,\ a,\ b,\ c,\ d \in \mathbb{Z}_{p^2}\},$$

and

$$\mathrm{End}(L,F)^{(1)} = \{\begin{pmatrix} & A_1 \\ A_0 & \end{pmatrix};\ A_0 = \begin{pmatrix} a & b \\ c & pd \end{pmatrix},\ \textit{and } A_1 = p^{-1} \pi A_0 \pi,\ a,\ b,\ c,\ d \in \mathbb{Z}_{p^2}\}.$$

In particular,

$$N_L = \mathrm{End}(L,F) \cap V'_p \simeq \{x = \begin{pmatrix} a & b \\ c & pa^\sigma \end{pmatrix};\ a,\ b,\ c \in \mathbb{Z}_{p^2},\ b^\sigma = -b,\ c^\sigma = -c\ \}.$$

If $L \in X_0$ is a superspecial lattice, then $p^{-1}F^2 L = L$, and we let Λ be the set of fixed points of $p^{-1}F^2$. This is a $\mathbb{Z}_{p^2}$-module of rank 4, and $\Lambda/F\Lambda$ is an $\mathbb{F}_{p^2}$-vector space of dimension 2. Note that the vectors e_i of a standard basis for L lie in Λ, and the images $\bar{e}_1$ and $\bar{e}_4$ give an $\mathbb{F}_{p^2}$-basis for $\Lambda/F\Lambda$. There is a natural reduction map:

$$\mathrm{red}_L : \mathrm{End}(L,F) \to \mathrm{End}(\Lambda/F\Lambda). \tag{5.15}$$

Lemma 5.3. *With respect to a fixed standard basis,*

$$\mathrm{red}_L : \mathrm{End}(L, F) \to \mathrm{End}(\Lambda/F\Lambda) \simeq M_2(\mathbb{F}_{p^2})$$

is given by

$$\mathrm{red}_L : \begin{pmatrix} A_0 & \\ & A_1 \end{pmatrix} \mapsto \begin{pmatrix} \bar{a} & \\ & \bar{d}^\sigma \end{pmatrix},$$

and

$$\mathrm{red}_L : \begin{pmatrix} & A_1 \\ A_0 & \end{pmatrix} \mapsto \begin{pmatrix} & \bar{b}^\sigma \\ \bar{c} & \end{pmatrix}.$$

Here $a \mapsto \bar{a}$ is the reduction map $\mathbb{Z}_{p^2} \to \mathbb{F}_{p^2}$, and the other notation is as in the Corollary 5.2.

Note that the grading on $M_2(\mathbb{F}_{p^2})$ is the checkerboard grading.

Let

$$\mathfrak{n}_L = \mathrm{red}_L(N_L) = \{x = \begin{pmatrix} & -b \\ c & \end{pmatrix};\ b,\ c \in \mathbb{F}_{p^2},\ \text{with } b^\sigma = -b,\ c^\sigma = -c\}, \tag{5.16}$$

with quadratic form $\bar{Q}'(x) = bc$. Then, we have the commutative diagram

$$\begin{array}{ccc} N_L & \xrightarrow{Q'} & \mathbb{Z}_p \\ \mathrm{red}_L \downarrow & & \downarrow \\ \mathfrak{n}_L & \xrightarrow{\bar{Q}'} & \mathbb{F}_{p^2} \end{array} \tag{5.17}$$

of quadratic spaces.

We next turn to the non-superspecial case.

Proposition 5.4. *(i) Suppose that $L \in X - X_0$ is a non-superspecial lattice with critical index i, and write $L = L_i \oplus L_{i+1}$. Then $j \in \mathrm{End}(\mathcal{L}, F)^{(1)}$, lies in $\mathrm{End}(L, F)^{(1)}$ if and only if*

$$j_i(L_i) \subset FL_i,$$

where

$$j = \begin{pmatrix} & j_1 \\ j_0 & \end{pmatrix},$$

as above.

(ii) If $L \in X - X_0$ is non-superspecial , and if $j \in N_L$ is a special endomorphism of L, so that $j \in \mathrm{End}(L, F)^{(1)}$ with $j^* = j$, then $\mathrm{ord}_p(Q'(j)) \geq 1$.

Proposition 5.5. *Fix an index* i *. Suppose that* $L_i \subset \mathcal{L}_i$ *is a* W *-lattice such that* $F^2 L_i = pL_i$ *, i.e.,* $\mathcal{L}_i \in Y_i$ *, in the notation of section 4. Then, a special endomorphism* $j \in V'_p$ *induces an endomorphism of every lattice in* $\mathbb{P}_{L_i}$ *if and only if*

$$j(L_i) \subset FL_i.$$

Corollary 5.6. *If* $j \in \mathrm{End}(L, F)^{(1)}$ *for one non-superspecial point* L *in* $\mathbb{P}_{L_i}$ *, then* $j \in \mathrm{End}(L', F)^{(1)}$ *for all points* L' *of* $\mathbb{P}_{L_i}$ *.*

Proof of Proposition 5.4. Since i is critical, $FL_i = VL_i$. Recall that $L_{i+1} = F^{-1}(\ell + L_i)$ where $\ell \in \mathbb{P}(L_i/pL_i)$ is a line, and that, since $i+1$ is not critical,

$$L_i = FL_{i+1} + VL_{i+1} \qquad \text{and} \qquad pL_i = FL_{i+1} \cap VL_{i+1}. \tag{5.18}$$

Recall that j_0 and j_1 commute with $VF^{-1} = (p^{-1}F^2)^{-1}$. Then, if $j(L) \subset L$, we have $j_i(L_i) \subset L_{i+1}$, and thus, $Fj_i(L_i) \subset FL_{i+1}$. Applying VF^{-1} to this, we get $Fj_i(L_i) \subset VL_{i+1}$. Hence

$$Fj_i(L_i) \subset FL_{i+1} \cap VL_{i+1} = pL_i, \tag{5.19}$$

i.e., cancelling an F ,

$$j_i(L_i) \subset VL_i = FL_i. \tag{5.20}$$

When i is critical and $i+1$ is not critical, condition (5.20) is in fact equivalent to the requirement that $j_i(L_i) \subset L_{i+1}$. Indeed, otherwise $L_{i+1} = j_i(L_i) + VL_i$, hence $VL_{i+1} = FL_{i+1}$, i.e. $i+1$ is critical. We must also check that $j_{i+1}(L_{i+1}) \subset L_i$. But, this is automatic, since

$$Fj_{i+1}(L_{i+1}) = j_i(FL_{i+1}) = j_i(\ell + pL_i) \subset j_i(L_i) \subset FL_i, \tag{5.21}$$

again by (5.20). Thus, condition (5.20) is all that is needed for $j(L) \subset L$, and (i) of the Proposition is proved.

To prove (ii), recall that for a special endomorphism j , $j_i^* j_i = Q'(j) \cdot 1_{L_i}$, by (5.6). Then, since, by (5.20),

$$< L_i, j_i^* j_i\,(L_i) > = < j_i(L_i), j_i(L_i) > \subset < FL_i, FL_i > = p < L_i, L_i >, \tag{5.22}$$

we have $j_i^* j_i\,(L_i) \subset pL_i$, and hence $\mathrm{ord}_p(Q'(j)) \geq 1$, as claimed. $\square$

Proof of Proposition 5.5. If j extends to every lattice in $\mathbb{P}_{L_i}$, then j preserves a non-superspecial lattice with critical index i, and hence, by Proposition 5.4 (i), $j_i(L_i) \subset FL_i$. Conversely, suppose that $j_i(L_i) \subset FL_i$. Let $L = L_i \oplus L_{i+1}$, with $L_{i+1} = F^{-1}(\ell + pL_i)$, be a lattice in $\mathbb{P}_{L_i}$. Then $j(L_i) \subset L_{i+1}$ if and only if $Fj(L_i) \subset FL_{i+1} = \ell + pL_i$. But $Fj(L_i) \subset F^2 L_i = pL_i$, so the required inclusion follows. Similarly, $j(L_{i+1}) \subset L_i$ if and only if $Fj(L_{i+1}) \subset FL_i$, but $Fj(L_{i+1}) = j(\ell + pL_i) \subset j(L_i) \subset FL_i$, as required. □

In section 8 below we will fix a special endomorphism $j \in V_p'$ of the isocrystal $\mathcal{L}$ and give a combinatorial description of the set of lattices $L_i \in Y_i$ for which j induces a special endomorphism of every lattice in $\mathbb{P}_{L_i}$.

§6. Isolated supersingular points of special cycles.

In this section we return to the special cycles introduced in section 2 and determine the isolated supersingular points on them. This allows us to characterize the isolated points of intersection of our special cycles and to obtain a formula for the 'intersection multiplicity' at such points (Corollary 6.3).

We fix n with $1 \leq n \leq 4$ and consider the special cycle $\mathcal{Z}(T, \omega)$, where $\omega \subset V(\mathbb{A}_f^p)^n$ and $T \in \mathrm{Sym}_n(\mathbb{Q})$. We fix a base point $\xi_0 = (A_0, \lambda_0, \iota_0, \overline{\eta}_0^p) \in \mathcal{M}_{K^p}^{ss}(\mathbb{F})$. As in section 4 we have a decomposition of the p-divisible group $A_0(p) = \mathcal{A}_0^4$ and we introduce the isocrystal $\mathcal{L} = D\mathcal{A}_0 \otimes \mathbb{Q}$. To every isogeny $\mu : \xi \xrightarrow{\sim} \xi_0$ with $\xi = (A, \lambda, \iota, \overline{\eta}^p) \in \mathcal{M}_{K^p}^{ss}(\mathbb{F})$ we then associate the Dieudonné lattice $L = \mu_*(D\mathcal{A})$ in $\mathcal{L}$ which lies in X. The point ξ will be called *superspecial* if $L \in X_0$ and *non-superspecial* if $L \in X - X_0$. This is independent of the choice of the isogeny μ. Suppose now that ξ is the image of $(A, \lambda, \iota, \overline{\eta}^p; \mathbf{j}) \in \mathcal{Z}(T, \omega)(\mathbb{F})$. Then $\mathbf{j}$ induces an n-tuple

$$\mathbf{j}_{\mathcal{L}} \in V_p'^{\,n} \tag{6.1}$$

of special endomorphisms of $\mathcal{L}$ with $\mathbf{j}_{\mathcal{L}} \in (N_L)^n$. Moreover the condition that $Q(\eta \circ \mathbf{j} \circ \eta^{-1}) = T$ implies that also

$$Q'(\mathbf{j}_{\mathcal{L}}) = T \quad , \tag{6.2}$$

with Q' as in (5.6).

Theorem 6.1. *Consider the set of supersingular points in the image of* $\mathcal{Z}(T,\omega)$ *, for* $T\in Sym_n(\mathbb{Q})$ *with* $\det(T)\neq 0$ *and for* $\omega\subset V(\mathbb{A}_f^p)^n$ *.*
(i) If $T\notin Sym_n(\mathbb{Z}_{(p)})$ *, i.e., if* T *is not* p*-integral, then* $\mathcal{Z}(T,\omega)\cap\mathcal{M}^{\rm ss}(\mathbb{F})$ *is empty.*
Assume that $T\in Sym_n(\mathbb{Z}_{(p)})$ *, and let* $\bar{T}\in Sym_n(\mathbb{F}_p)$ *be its reduction modulo* p *.*
(ii) If the rank of $\bar{T}$ *is greater than* 2 *, or if* $\bar{T}$ *has rank* 2 *and is anisotropic modulo its radical, then* $\mathcal{Z}(T,\omega)\cap\mathcal{M}^{\rm ss}(\mathbb{F})$ *is empty.*
(iii) If $\bar{T}\neq 0$ *, then* $\mathcal{Z}(T,\omega)\cap\mathcal{M}^{\rm ss}(\mathbb{F})$ *consists entirely of superspecial points.*
(iv) Suppose that $\bar{T}=0$ *and that* $\mathcal{Z}(T,\omega)\cap\mathcal{M}^{\rm ss}(\mathbb{F})\neq\phi$ *. If* L *is a non-superspecial point in the image of* $\mathcal{Z}(T,\omega)$ *, and if* $L\in\mathbb{P}_{L_i}$ *, then* $\mathbb{P}_{L_i}\subset\mathcal{Z}(T,\omega)$ *.*
(v) Suppose that $\bar{T}=0$ *and that* $L=L_0\oplus L_1\in X_0$ *is a superspecial lattice corresponding to a point of* $\xi\in\mathcal{M}^{\rm ss}(\mathbb{F})$ *in the image of* $\mathcal{Z}(T,\omega)$ *. As usual let* $\Lambda=L^{pF^{-2}}=\Lambda_0\oplus\Lambda_1$ *. Let* $M\subset N_L$ *be the* $\mathbb{Z}_p$*-submodule spanned by the components of* $\mathbf{j}_{\mathcal{L}}$ *, and let* $\mathfrak{m}=\mathrm{red}_L(M)$ *be the image of* M *in* $\mathfrak{n}_L\subset\mathrm{End}(\Lambda/F\Lambda)$ *.*

(a) *If* $\mathfrak{m}=0$ *, then both components* $\mathbb{P}_{L_0}$ *and* $\mathbb{P}_{L_1}$ *of* $\mathcal{M}^{\rm ss}$ *through* ξ *(cf. (4.10)) lie in* $\mathcal{Z}(T,\omega)\cap\mathcal{M}^{\rm ss}(\mathbb{F})$ *.*

(b) *If* $\mathfrak{m}\neq 0$ *, then* $\dim_{\mathbb{F}_p}\mathfrak{m}=1$ *. A basis vector* $\bar{j}\in\mathfrak{m}$ *is a nonzero endomorphism of*

$$\Lambda/F\Lambda=(\Lambda_0/F\Lambda_1)\oplus(\Lambda_1/F\Lambda_0)$$

of degree 1 with $\bar{j}^2=0$ *. Therefore the condition*

$$j(L_i)\subset F(L_i)$$

holds for precisely one index $i\in\mathbb{Z}/2$ *, in which case,* $\mathbb{P}_{L_i}$ *lies in* $\mathcal{Z}(T,\omega)\cap\mathcal{M}^{\rm ss}(\mathbb{F})$ *and* $\mathbb{P}_{L_{i+1}}$ *does not.*

Remark. In effect, if $p\nmid T$, then $\mathcal{Z}(T,\omega)^{ss}$ consists of (at most) an isolated (finite) set of superspecial points, while, if $p\mid T$, then the image of $\mathcal{Z}(T,\omega)$ in $\mathcal{M}^{ss}$ is either empty or consists of a union of components of $\mathcal{M}^{\rm ss}$. We will give a more complete description of the components which occur in section 8 below.

Proof. Suppose that $\tilde{\xi}=(A,\lambda,\iota,\bar{\eta}^p;\mathbf{j})\in\mathcal{Z}(T,\omega)$ lies above $\xi=(A,\lambda,\iota,\bar{\eta}^p)\in\mathcal{M}^{ss}(\mathbb{F})$. Since all of our assertions are 'local' at ξ , we fix an isogeny $\mu:\xi\xrightarrow{\sim}\xi_0$ and hence associated with ξ a Dieudonné lattice $L=L_0\oplus L_1\in X$ and n-tuple of special endomorphisms $\mathbf{j}_{\mathcal{L}}\in N_L^n\subset(\mathrm{End}(L,F)^{(1)})^n$. Let M be the $\mathbb{Z}_p$-submodule of N_L spanned by the components of $\mathbf{j}_{\mathcal{L}}$. Since Q' is $\mathbb{Z}_p$-valued on N_L , it follows that $T\in Sym_n(\mathbb{Z}_{(p)})$. This proves (i). (In fact, from the definitions in section 2 we know that the whole special fibre of $\mathcal{Z}(T,\omega)$ is empty if T is not p-integral.) Since we are assuming that $\det(T)\neq 0$, M must have rank n over $\mathbb{Z}_p$.

First suppose that ξ is a non-superspecial point. Then, by (ii) of Proposition 5.4, $\bar{T}=0$, since $Q'(j)\equiv 0 \mod p$ for every $j\in M$. This proves (iii).

If ξ is a superspecial point in the image of $\mathcal{Z}(T,\omega)$, then $\bar{T}$ is the matrix of inner products of the images of the components of $\mathbf{j}_{\mathcal{L}}$ under the reduction map $\mathrm{red}_L : N_L \to \mathfrak{n}_L$. Since $\mathfrak{n}_L$ is a hyperbolic plane over $\mathbb{F}_p$, it follows that $\bar{T}$ must have rank at most 2, and that, if the rank of $\bar{T}$ is 2, then $\mathfrak{m}=\mathfrak{n}_L$ and $\bar{T}$, modulo its radical, must be a hyperbolic plane. This proves (ii).

Finally, suppose that $\bar{T}=0$. If ξ is a non-superspecial point in $\mathcal{Z}(T,\omega)\cap\mathcal{M}^{\mathrm{ss}}(\mathbb{F})$, with critical index i, then, by Corollary 5.6, the whole component $\mathbb{P}_{L_i}$ lies in $\mathcal{Z}(T,\omega)\cap\mathcal{M}^{\mathrm{ss}}(\mathbb{F})$. This proves (iv). If ξ is a superspecial point in $\mathcal{Z}(T,\omega)\cap\mathcal{M}^{\mathrm{ss}}(\mathbb{F})$, then $\mathfrak{m}$ is an isotropic subspace of $\mathfrak{n}_L$ and thus has dimension 0 or 1. If $\mathfrak{m}=0$, then, for every $j\in M$, the condition $j(L_i)\subset FL_i$ is satisfied for both $i=0$ and 1. Thus, by Proposition 5.5, both components $\mathbb{P}_{L_0}$ and $\mathbb{P}_{L_1}$ of $\mathcal{M}^{\mathrm{ss}}$ through ξ lie in $\mathcal{Z}(T,\omega)\cap\mathcal{M}^{\mathrm{ss}}(\mathbb{F})$. If $\mathfrak{m}$ is an isotropic line in $\mathfrak{n}_L$, then there is a unique index i such that $j(L_i)\subset FL_i$ holds for all $j\in M$. Thus, in this case, $\mathbb{P}_{L_i}$ lies in $\mathcal{Z}(T,\omega)\cap\mathcal{M}^{\mathrm{ss}}(\mathbb{F})$, but $\mathbb{P}_{L_{i+1}}$ does not. This proves (v). $\square$

We now assume that $n\geq 3$ and that $T\in \mathrm{Sym}_n(\mathbb{Z}_{(p)})_{>0}$ and also that $\bar{T}\neq 0$. The first assumption implies that $\mathcal{Z}(T,\omega)$ lies over the supersingular locus $\mathcal{M}^{\mathrm{ss}}$ of the special fiber of $\mathcal{M}$. The second assumption implies that $\mathcal{Z}(T,\omega)=\mathcal{Z}(T,\omega)^{ss}$, if nonempty, consists of a finite set of isolated superspecial points.

We fix a supersingular point $\xi=(A,\lambda,\iota,\overline{\eta}^p;\mathbf{j})\in\mathcal{Z}(T,\omega)(\mathbb{F})$. Again we write $A(p)=\mathcal{A}^4$, where $\mathcal{A}$ is a 2-dimensional formal group of height 4, equipped with an action

$$\iota_0 : \mathbb{Z}_{p^2} \longrightarrow \mathrm{End}(\mathcal{A})$$

and with a principal quasi-polarization

$$\lambda_{\mathcal{A}} : \mathcal{A} \xrightarrow{\sim} \hat{\mathcal{A}}$$

such that $\iota_0(a)^*=\iota_0(a)$.

By the Serre-Tate Theorem, the formal completions at ξ of $\mathcal{M}$ and of $\mathcal{Z}(T,\omega)$

can be interpreted as versal deformation spaces:

$$\hat{\mathcal{M}}_\xi = \mathrm{Def}(\mathcal{A}, \lambda_{\mathcal{A}}, \iota_0), \tag{6.3}$$

and

$$\hat{\mathcal{Z}}(T,\omega)_\xi = \mathrm{Def}(\mathcal{A}, \lambda_{\mathcal{A}}, \iota_0; \mathbf{j}) = \mathrm{Def}(\mathcal{A}, \lambda_{\mathcal{A}}, \iota_0; M) \tag{6.4}$$

where M is the $\mathbb{Z}_p$-submodule of $\mathrm{End}(\mathcal{A})$ spanned by the components of $\mathbf{j}$. By our assumption $\overline{T} \neq 0$, the latter deformation space is the spectrum of a local Artin ring R_ξ. We let $e(\xi)$ be the length of R_ξ.

From now on, we assume that $n = 3$, and we reduce the computation of $e(\xi)$ to a result of Gross and Keating, [**3**].

Since, as always, $p \neq 2$, we may choose a $\mathbb{Z}_p$-basis $\psi_1, \ \psi_2, \ \psi_3$ for M such that the matrix for the restriction of the quadratic form Q_ξ to this basis is

$$T' = \mathrm{diag}(\varepsilon_1 p^{a_1}, \varepsilon_2 p^{a_2}, \varepsilon_3 p^{a_3}), \tag{6.5}$$

with $0 \le a_1 \le a_2 \le a_3$ and with units ε_i uniquely determined modulo squares. In particular, $\psi_i^2 = \varepsilon_i p^{a_i}$, and $\psi_i\psi_j + \psi_j\psi_i = 0$ for $i \neq j$.

Recall that p is inert in k so that $k_p = \mathbb{Q}_p(\delta)$ with $\delta^2 = \Delta \in \mathbb{Z}_p^\times$ not a square. Since $\overline{T} \neq 0$, we have $a_1 = 0$ and we may take $\varepsilon_1 = 1$ or $\varepsilon_1 = \Delta$, depending on whether or not ε_1 is a square. If $\varepsilon_1 = 1$ define idempotents

$$e_0 = \frac{1}{2}(1 + \psi_1) \quad , \quad e_1 = \frac{1}{2}(1 - \psi_1) \quad . \tag{6.6}$$

These yield a decomposition $\mathcal{A} \simeq \mathcal{A}_0 \times \mathcal{A}_1$ where $\mathcal{A}_i = e_i\mathcal{A}$ has dimension 1 and height 2. Since $e_i^* = e_i$, the polarization $\lambda_{\mathcal{A}}$ is of the form

$$\lambda_{\mathcal{A}} = \lambda_{\mathcal{A}_0} \times \lambda_{\mathcal{A}_1} : \mathcal{A}_0 \times \mathcal{A}_1 \xrightarrow{\sim} \hat{\mathcal{A}}_0 \times \hat{\mathcal{A}}_1 \quad . \tag{6.7}$$

Let $\underline{\delta} = \iota_0(\delta) \in \mathrm{End}(\mathcal{A})$. Then $\underline{\delta}$ is an isomorphism with $\underline{\delta} e_0 = e_1 \underline{\delta}$ and hence may be considered as an isomorphism $\mathcal{A}_0 \to \mathcal{A}_1$. Furthermore

$$\psi_i e_0 = e_1 \psi_i \ , \quad i = 2, 3 \quad . \tag{6.8}$$

We put

$$(\mu_1, \mu_2, \mu_3) = (\underline{\delta}, \psi_2, \psi_3) \quad , \tag{6.9}$$

so that (μ_1, μ_2, μ_3) is a triple of isogenies from $\mathcal{A}_0$ to $\mathcal{A}_1$. Since a principal quasi-polarization on a formal group of dimension 1 and height 2 deforms automatically and using the fact that ψ_2 and ψ_3 are special endomorphisms we see that

$$\text{(6.10)} \qquad \mathrm{Def}(\mathcal{A}, \lambda_{\mathcal{A}}, \iota_0; M) = \mathrm{Def}(\mathcal{A}_0, \mathcal{A}_1; \mu_1, \mu_2, \mu_3) \quad ,$$

where the right side is the locus inside the universal deformation space of the pair $(\mathcal{A}_0, \mathcal{A}_1)$ to which the triple of isogenies deforms. We note that the degree quadratic form on $\mathrm{Hom}(\mathcal{A}_0, \mathcal{A}_1)$ has matrix equal to

$$\text{(6.11)} \qquad T'' = \mathrm{diag}(\Delta, \varepsilon_2 p^{a_2}, \varepsilon_3 p^{a_3})$$

on the subspace of rank 3 spanned by the μ_i 's.

Next assume that $\varepsilon_1 = \Delta$ and write $\Delta^{-1} = a^2 + b^2$ with $a, b \in \mathbb{Z}_p^\times$. Let

$$\text{(6.12)} \qquad \psi_1' = a\underline{\delta} + b\psi_1 \in \mathrm{End}(\mathcal{A}) \quad .$$

Then ${\psi_1'}^2 = \mathrm{id}$. We define idempotents

$$\text{(6.13)} \qquad e_0 = \frac{1}{2}(1 + \psi_1') \ , \ e_1 = \frac{1}{2}(1 - \psi_1') \quad .$$

Again we obtain a decomposition $\mathcal{A} = \mathcal{A}_0 \times \mathcal{A}_1$ with $\mathcal{A}_i = e_i \mathcal{A}$. Since again $e_i^* = e_i$ the polarization $\lambda_{\mathcal{A}}$ again splits as a product. Put

$$\text{(6.14)} \qquad \psi_1'' = b\underline{\delta} - a\psi_1 \quad .$$

Then we have

$$\text{(6.15)} \qquad \psi_1'' e_0 = e_1 \psi_1'' \quad , \quad \text{and } \psi_i e_0 = e_1 \psi_i \quad , \quad i = 2, 3 \quad .$$

We put

$$\text{(6.16)} \qquad (\mu_1, \mu_2, \mu_3) = (\psi_1'', \psi_2, \psi_3)$$

and again obtain a triple of isogenies from $\mathcal{A}_0$ to $\mathcal{A}_1$. Since from $\mathcal{A}_0, \mathcal{A}_1$ and ψ_1'' we recover $\mathcal{A}$ with the action of $\underline{\delta}$ and ψ_1 we see that again

$$\text{(6.17)} \qquad \mathrm{Def}(\mathcal{A}, \lambda_{\mathcal{A}}, \iota_0; M) = \mathrm{Def}(\mathcal{A}_0, \mathcal{A}_1; \mu_1, \mu_2, \mu_3) \quad .$$

The degree quadratic form on $\mathrm{Hom}(\mathcal{A}_0, \mathcal{A}_1)$ has matrix

$$\text{(6.18)} \qquad T'' = \mathrm{diag}(1, \varepsilon_2 p^{a_2}, \varepsilon_3 p^{a_3})$$

on the subspace of rank 3 spanned by the μ_i 's.

We are now in a position to apply the results of Gross and Keating [**3**] about the length of the artinian W-scheme $\mathrm{Def}(\mathcal{A}_0, \mathcal{A}_1; \mu_1, \mu_2, \mu_3)$. By their results this length only depends on the $GL_3(\mathbb{Z}_p)$-equivalence class of the matrix T''. More precisely, Proposition 5.4 of [**3**] implies the following result.

Proposition 6.2. *The length of the local ring* $\mathcal{O}_{\mathcal{Z}(T,\omega),\xi}$ *is equal to* $e(\xi) = e_p(T)$ *with* $e_p(T)$ *given as follows. Let* T *be* $GL_3(\mathbb{Z}_p)$ *-equivalent to (6.5).*
(i) If a_2 *is even,*

$$e_p(T) = \sum_{i=0}^{a_2/2-1} (a_2 + a_3 - 4i)p^i + \frac{1}{2}(a_3 - a_2 + 1)p^{a_2/2}.$$

(ii) If a_2 *is odd,*

$$e_p(T) = \sum_{i=0}^{(a_2-1)/2} (a_2 + a_3 - 4i)p^i.$$

Note that the answer does not actually depend on the units ε_1 , ε_2 and ε_3 .

We conclude this section by indicating how the previous results can be applied to the intersection problem of special cycles. The calculus is the same as in the companion paper [**12**], esp. section 3.

We fix integers $n_1, \ldots, n_r$ with $1 \le n_i \le 3$ and with $n_1 + \ldots + n_r = 3$. For each i we choose $T_i \in \mathrm{Sym}_{n_i}(\mathbb{Q})_{>0}$ and a K^p -invariant open compact subset $\omega_i \subset V(\mathbb{A}_f^p)^{n_i}$. Let

$$\mathcal{Z} = \mathcal{Z}(T_1, \omega_1) \times_{\mathcal{M}} \ldots \times_{\mathcal{M}} \mathcal{Z}(T_r, \omega_r,) \tag{6.19}$$

be the fibre product of the corresponding special cycles. To each point $\xi \in \mathcal{Z}$ there is associated its *fundamental matrix*

$$T_\xi = Q((\mathbf{j}_1, \ldots, \mathbf{j}_n)) \in \mathrm{Sym}_3(\mathbb{Q}) \quad , \tag{6.20}$$

where $(\mathbf{j}_1, \ldots, \mathbf{j}_r)$ is the 3-tuple of special endomorphisms determined by the image of ξ via its images under the projections $\mathcal{Z} \to \mathcal{Z}(T_i, \omega_i)$. Since the function $\xi \mapsto T_\xi$ is locally constant, we obtain a disjoint sum decomposition

$$\mathcal{Z} = \coprod_T \mathcal{Z}_T \quad , \tag{6.21}$$

where $\mathcal{Z}_T$ is the union of those connected components of $\mathcal{Z}$ where the value of the fundamental matrix is equal to T . Note that T has the form

$$T = \begin{pmatrix} T_1 & * & \ldots & * \\ * & T_2 & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \ldots & T_r \end{pmatrix}. \tag{6.22}$$

In fact we may identify

$$\mathcal{Z}_T = \mathcal{Z}(T, \omega_1 \times \ldots \times \omega_r) \quad . \tag{6.23}$$

Hence we may apply the previous results to obtain the following:

Corollary 6.3. *Let* $\xi \in \mathcal{Z} = \mathcal{Z}(T_1, \omega_1) \times_{\mathcal{M}} \ldots \times_{\mathcal{M}} \mathcal{Z}(T_r, \omega_r)$ *with* $\det(T_\xi) \neq 0$ *. Then* $T_\xi \in \mathrm{Sym}_3(\mathbb{Z}_{(p)})_{>0}$ *and* ξ *lies over a supersingular point of* $\mathcal{M} \times_{\mathrm{Spec}\,\mathbb{Z}_{(p)}} \mathrm{Spec}\,\mathbb{F}_p$ *. Furthermore,* ξ *is an isolated point of* $\mathcal{Z}$ *if and only if* $T_\xi \not\equiv 0 \bmod p$ *. In this case the length of the local ring of* $\mathcal{Z}$ *at* ξ *is given by*

$$e(\xi) = lg(\mathcal{O}_{\mathcal{Z},\xi}) = e_p(T_\xi) \quad ,$$

with $e_p(T_\xi)$ *as in Proposition 6.2.*

Since the length dominates the local intersection multiplicity in the sense of Serre we deduce from the formulas for $e_p(T)$ as in [**12**], Cor. 6.2. the following result.

Corollary 6.4. *The cycles* $\mathcal{Z}(T_1, \omega_1), \ldots, \mathcal{Z}(T_r, \omega_r)$ *intersect transversally at the point* ξ *if and only if* $\mathrm{ord}(\det T_\xi) = 1$ *. In this case, the schemes* $\mathcal{Z}(T_i, \omega_i)$ *are regular at* ξ *and their tangent spaces give a direct sum decomposition of the tangent space of* $\mathcal{M}$ *at (the image of)* ξ *.*

§7. Representation densities and Eisenstein series.

In this section we will consider the total contribution of isolated intersection points to the intersection product of special cycles. Since the development is analogous to the corresponding part of the companion paper [**12**], sections 7-10, we will be brief.

For special cycles $\mathcal{Z}(T_1, \omega_1), \ldots, \mathcal{Z}(T_r, \omega_r)$ as at the end of the last section we define the contribution of the isolated intersection points to the total intersection number as

$$\langle \mathcal{Z}(T_1, \omega_1), \ldots, \mathcal{Z}(T_r, \omega_r) \rangle_p^{\mathrm{proper}} = \sum_{\xi} e(\xi) \quad . \tag{7.1}$$

Here ξ runs over the isolated points of $\mathcal{Z}(T_1, \omega_1) \times_{\mathcal{M}} \ldots \times_{\mathcal{M}} \mathcal{Z}(T_r, \omega_r)$. In the special case $r = 1$ we have the cycle $\mathcal{Z}(T, \omega)$ which lies over the supersingular locus of $\mathcal{M}$ and consists of isolated points if and only if $T \in \mathrm{Sym}_3(\mathbb{Z}_{(p)})$ is not divisible by p. In this case we use the notation

$$\langle \mathcal{Z}(T, \omega) \rangle_p = \sum_{\xi \in \mathcal{Z}(T,\omega)} e(\xi) \quad . \tag{7.2}$$

By (6.21) and (6.23) we have

$$\langle \mathcal{Z}(T_1, \omega_1), \ldots, \mathcal{Z}(T_r, \omega_r) \rangle_p^{\mathrm{proper}} = \sum_{T} \langle \mathcal{Z}(T, \omega) \rangle_p \quad , \tag{7.3}$$

where the summation is over $T \in \mathrm{Sym}_3(\mathbb{Z}_{(p)})_{>0}$ which have diagonal blocks $T_1, \ldots, T_r$ as in (6.22) and for which $p \nmid T$. Hence it suffices to determine the quantity (7.2).

By Proposition 6.2, the length of the local ring $\mathcal{O}_{\mathcal{Z}(T,\omega),\xi}$ depends only on T so that we may write

$$\langle \mathcal{Z}(T,\omega)\rangle_p = e_p(T) \cdot |\mathcal{Z}(T,\omega)(\mathbb{F})| \quad . \tag{7.4}$$

Here $e_p(T)$ is given by Proposition 6.2. Thus it only remains to determine $|\mathcal{Z}(T,\omega)(\mathbb{F})|$.

As at the end of section 4, let B' be the definite quaternion algebra over $\boldsymbol{k}$ which is isomorphic to B at all finite places of $\boldsymbol{k}$, and let G' be the corresponding inner form of G, defined by (0.2) with B' in place of B. Then $G'(\mathbb{Q}_p) \simeq G'_p$. Let K'_p be the stabilizer of a superspecial lattice $L_o \in X$, an Iwahori subgroup of G'_p – cf. (5.7) and Corollary 5.2. Then the set of superspecial points in $\mathcal{M}^{ss}(\mathbb{F})$ is in bijective correspondence with the double coset space, comp. (4.12),

$$G'(\mathbb{Q}) \setminus \left(G'(\mathbb{Q}_p)/K'_p \times G(\mathbb{A}_f^p)/K^p \right) \quad . \tag{7.5}$$

Let V' be the quadratic space over $\mathbb{Q}$ which is positive definite, and with

$$V'(\mathbb{Q}_p) = V'_p \quad \text{and} \quad V'(\mathbb{A}_f^p) = V(\mathbb{A}_f^p) \quad . \tag{7.6}$$

We note that $V'(\mathbb{Q}_p)$ and $V(\mathbb{Q}_p)$ have identical determinants and opposite Hasse invariants. Also, the quaternion algebra $C^+(V')$ associated in section 0 to V' is isomorphic to B'. Indeed, it suffices to check this locally, using the fact that $C^+(V') = B'_0 \otimes_{\mathbb{Q}} \boldsymbol{k}$ for the quaternion algebra B'_0 over $\mathbb{Q}$ given by Lemma 0.2. For the archimedean places the assertion now follows from the positive definiteness of V'. At the place p, both B' and $C^+(V')$ split since p is inert in $\boldsymbol{k}$. For the finite places not dividing p the claim is obvious.

The superspecial lattice L_o (base point) then defines a lattice in V'_p,

$$V'(\mathbb{Z}_p) = \mathrm{End}(L_o, F) \cap V'_p \quad . \tag{7.7}$$

Let Ω'_T be the fiber over T of the map defined by the quadratic form on V',

$$Q' : {V'}^3 \longrightarrow \mathrm{Sym}_3(\mathbb{Q}) \quad . \tag{7.8}$$

Then $G'(\mathbb{Q})$ acts transitively on $\Omega'_T(\mathbb{Q})$ and the stabilizer of a point $\mathbf{y} \in \Omega'_T(\mathbb{Q})$ is $Z'(\mathbb{Q})$, the kernel of the projection of $G'(\mathbb{Q})$ to $SO(V')$, comp. [**12**], Remark 7.4. The usual procedure therefore gives the following expression for the cardinality of $\mathcal{Z}(T,\omega)(\mathbb{F})$, cf. [**12**], Lemma 7.1 and Theorem 7.2.

Proposition 7.1. *Let* $K' = K'_p.K^p \subset G'(\mathbb{A}_f)$ *. Let*

$$\varphi_f^p = \mathrm{char}(\omega) \ , \quad \varphi'_p = \mathrm{char}(V'(\mathbb{Z}_p))^3$$

and

$$\varphi'_f = \varphi'_p \cdot \varphi_f^p \in S(V'(\mathbb{A}_f))^{K'} \quad .$$

For arbitrary choices of a base point $\mathbf{y} \in \Omega'_T(\mathbb{Q})$ *and of Haar measures on* $G'(\mathbb{A}_f)$ *and on* $Z'(\mathbb{A}_f)$ *), introduce the orbital integral*

$$O_T(\varphi'_f) = \int\limits_{Z'(\mathbb{A}_f)\backslash G'(\mathbb{A}_f)} \varphi'_f(g^{-1}\mathbf{y})dg.$$

Then

$$|\mathcal{Z}(T,\omega)(\mathbb{F})| = \mathrm{vol}(K')^{-1} \cdot \mathrm{vol}(Z'(\mathbb{Q}) \setminus Z'(\mathbb{A}_f)) \cdot O_T(\varphi'_f) \quad . \tag{7.9}$$

We therefore have obtained an explicit expression for (7.4).

We now want to compare this expression with the derivative of the T-th Fourier coefficient of a certain Eisenstein series which is associated to our data as follows, [**11**], [**12**]. (See Part I of [**11**] for a more extensive description of such incoherent Eisenstein series.) We let

$$\lambda_p : S(V(\mathbb{Q}_p)^3) \longrightarrow I_3(0, \chi_{V_p}) \tag{7.10}$$

be the usual map into the induced representation of $Sp_{6,\mathbb{Q}_p}$ defined by $\lambda_p(\varphi_p)(g) = (\omega(g)\varphi_p)(0)$ where $\omega = \omega_\psi$ denotes the action of $Sp_{6,\mathbb{Q}_p}$ on $S(V(\mathbb{Q}_p)^3)$ via the local Weil representation defined by the fixed additive character ψ. Let

$$\Phi_p = \lambda_p(\varphi_p) \ \text{ with } \varphi_p = \mathrm{char}\, V(\mathbb{Z}_p)^3 \quad . \tag{7.11}$$

Here $V(\mathbb{Z}_p) = \Lambda \otimes \mathbb{Z}_p$ in the notation of (2.1). We define $\Phi'_p = \lambda'_p(\varphi'_p)$ with $\varphi'_p = \mathrm{char}\, V'(\mathbb{Z}_p)^3$ in the analogous way. We complete Φ_p into an incoherent standard section

$$\Phi(s) = \Phi^2_\infty(s) \cdot \Phi^p_f(s) \cdot \Phi_p(s) \quad , \tag{7.12}$$

where $\Phi^2_\infty(s)$ is associated in the usual way to the Gaussian in $S(V'(\mathbb{R})^3)$ and where $\Phi^p_f(0) = \lambda^p_f(\varphi^p_f)$ with φ^p_f as in Proposition 7.1. Then for $h \in Sp_{6,\mathbb{R}}$ and each $T \in \mathrm{Sym}_3(\mathbb{Q})_{>0}$ which is represented by $V(\mathbb{A}_f^p)$ but not by $V(\mathbb{Q}_p)$ we have (comp. [**12**], section 8),

(7.13)
$$\begin{aligned} E'_T(h,0,\Phi) &= \mathrm{vol}(SO(V')(\mathbb{R})\cdot pr(K'))\cdot W_T^2(h) \\ &\qquad \times \frac{W'_{T,p}(e,0,\Phi_p)}{W_{T,p}(e,0,\Phi'_p)}\cdot \mathrm{vol}(K')^{-1}\mathrm{vol}(Z(\mathbb{Q})\setminus Z(\mathbb{A}_f))\cdot O_T(\varphi'_f)\ , \end{aligned}$$

provided that ω is *locally centrally symmetric,* i.e. invariant under the action of $\mu_2(\mathbb{A}_f^p)$ so that φ'_f is locally even. Here $pr(K')$ denotes the image of K' under the projection map $pr: G'(\mathbb{A}_f)\to SO(V')(\mathbb{A}_f)$. Also, $W_T^2(h)$ is an archimedean factor defined in analogy with $W_T^{\frac{5}{2}}(h)$ in [**12**], (8.24), comp. [**11**], section 7.

We will see in a moment that the value of the Whittaker functional in the denominator is non-zero.

Before stating the next proposition we recall that a nonsingular $T \in \mathrm{Sym}_3(\mathbb{Q}_p)$ is represented by precisely one of the quadratic spaces $V(\mathbb{Q}_p)$ and $V'_p = V'(\mathbb{Q}_p)$, [**11**], Proposition 1.3.

Proposition 7.2. *Let* $T \in \mathrm{Sym}_3(\mathbb{Q}_p)$ *with* $\det(T)\neq 0$.

(i) *If* $W'_{T,p}(e,0,\Phi_p)\neq 0$, *then* $T\in \mathrm{Sym}_3(\mathbb{Z}_p)$.

(ii) *If* $T \in \mathrm{Sym}_3(\mathbb{Z}_p)$ *and if* T *is represented by* $V(\mathbb{Q}_p)$ *and* $T\not\equiv 0 \bmod p$, *then*
$$W_{T,p}(e,0,\Phi'_p) = 2\cdot\gamma(V'_p)\cdot p^{-4}(p^2-1)\quad .$$

(iii) *If* $T\in\mathrm{Sym}_3(\mathbb{Z}_p)$ *is represented by* $V'_p = V'(\mathbb{Q}_p)$ *and* $T\not\equiv 0 \bmod p$, *then*
$$W'_{T,p}(e,0,\Phi_p) = \log p\cdot\gamma(V_p)\cdot(1+p^{-2})\cdot(1-p^{-2})\cdot e_p(T)\quad .$$

The factors $\gamma(V_p)$ *and* $\gamma(V'_p)$ *are eighth roots of unity given explicitly by Proposition A.4 of* [**11**]. *Moreover,*
$$\gamma(V_p)/\gamma(V'_p) = -1.$$

Remark. By (1.16) of [**11**], the quadratic space V'_p represents $T=\mathrm{diag}(\varepsilon_1 p^{a_1},\varepsilon_2 p^{a_2},\varepsilon_3 p^{a_3})$, with $0\le a_1\le a_2\le a_3$ and ε_i units precisely when:
$$-1 = (-1)^{a_1+a_2+a_3}\chi(-1)^{a_1+a_2+a_3+a_1a_2+a_2a_3+a_3a_1}\chi(\varepsilon_1)^{a_2+a_3}\chi(\varepsilon_2)^{a_1+a_3}\chi(\varepsilon_3)^{a_1+a_2}.$$

Here $\chi(x) = (x,p)_p$ so that $\chi(\Delta) = -1$.

Proof. We use the well-known relation, reviewed in the Appendix to [**11**], between the values of $W_{T,p}(e,s,\Phi_p)$ at integer values of s and representation densities. For a suitable basis the quadratic form on $V(\mathbb{Z}_p)$ has matrix

$$S = \mathrm{diag}(1,-1,1,-\Delta) \quad . \tag{7.14}$$

For $r \geq 0$, let $S_r = S \perp H_{2r}$, where H_{2r} denotes the split quadratic form of rank $2r$ over $\mathbb{Z}_p$. Then $W_{T,p}(e,r,\Phi_p) = 0$ for $T \in \mathrm{Sym}_3(\mathbb{Q}_p) \setminus \mathrm{Sym}_3(\mathbb{Z}_p)$ and

$$W_{T,p}(e,r,\Phi_p) = \gamma(V_p)\cdot\alpha_p(S_r,T) \quad , \tag{7.15}$$

where $\alpha_p(S_r,T)$ is the classical representation density of T by S_r and $\gamma(V_p)$ is the factor appearing in [**11**], Cor. A.1.5. Here we have used the fact that S is unimodular. There is a rational function $A_{S,T}(X)$ such that

$$\alpha_p(S_r,T) = A_{S,T}(p^{-r}) \quad , \tag{7.16}$$

i.e. $\alpha_p(S_r,T)$ is a rational function of $X = p^{-r}$. Taking (7.15) and (7.16) together we obtain

$$W'_{T,p}(e,0,\Phi_p) = -\log p\cdot\gamma(V_p)\cdot\frac{\partial}{\partial X}\{A_{S,T}(X)\}\Big|_{X=1} \quad . \tag{7.17}$$

To calculate $\alpha_p(S_r,T)$ we use Kitaoka's formulas in the form given in [**12**], section 10. We use repeatedly the standard reduction formula

$$\alpha_p(N\perp M, N\perp L) = \alpha_p(N\perp M, N)\cdot\alpha_p(M,L) \quad , \tag{7.18}$$

valid provided the quadratic form N is unimodular. In our case let

$$\tilde{T} = \mathrm{diag}(\varepsilon_2 p^{a_2}, \varepsilon_3 p^{a_3}), \qquad T_{\varepsilon_1} = \mathrm{diag}(\varepsilon_1, \varepsilon_2 p^{a_2}, \varepsilon_3 p^{a_3}) \quad . \tag{7.19}$$

Here $\varepsilon_i \in \mathbb{Z}_p^\times$, $a_i \geq 0$. Using the reduction formula (7.18) we obtain

$$\alpha_p(S_r, T_{\varepsilon_1}) = \alpha_p(S_r,\varepsilon_1)\cdot\alpha_p(\tilde{S}_r,\tilde{T}) \quad .$$

Here $\tilde{S} = \mathrm{diag}(1,-1,-\varepsilon_1\Delta)$ and $\tilde{S}_r = \tilde{S}\perp H_{2r}$ as before. Using the reduction formula again we obtain

$$\alpha_p(H_{2r+4}, T_{\varepsilon_1\Delta}) = \alpha_p(H_{2r+4},\varepsilon_1\Delta)\cdot\alpha_p(\tilde{S}_r,\tilde{T}) \quad ,$$

and hence

$$\alpha_p(S_r, T_{\varepsilon_1}) = \frac{\alpha_p(S_r, \varepsilon_1)}{\alpha_p(H_{2r+4}, \varepsilon_1\Delta)} \cdot \alpha_p(H_{2r+4}, T_{\varepsilon_1\Delta}) \quad , \tag{7.20}$$

and

$$A_{S,T_{\varepsilon_1}}(X) = \frac{A_{S,\varepsilon_1}(X)}{A_{H_4,\varepsilon_1\Delta}(X)} \cdot A_{H_4,T_{\varepsilon_1\Delta}}(X) \quad . \tag{7.21}$$

But we have [**19**], Theorem 3.2,

$$\alpha_p(H_{2r+4}, \varepsilon_1\Delta) = 1 - p^{-2}X \quad , \tag{7.22}$$

and

$$\alpha_p(S_r, \varepsilon_1) = 1 + p^{-2}X \quad , \tag{7.23}$$

with $X = p^{-r}$. For the other factor on the right hand side we use [**12**], Proposition 10.3, which is, in turn, a reformulation of a result of Kitaoka [**8**]:

Case a_2 **even:** Then

(7.24)

$$\frac{\alpha_p(H_{2r+4}, T_{\varepsilon_1\Delta})}{(1-p^{-2}X)(1-p^{-2}X^2)} = \sum_{\ell=0}^{\frac{a_2}{2}-1} p^\ell (X^{2\ell} + \chi(T)\cdot X^{a_2+a_3-2\ell}) + p^{\frac{a_2}{2}} \cdot X^{a_2} \cdot \sum_{j=0}^{a_3-a_2} (\sigma X)^j$$

where $\sigma = \chi(-\varepsilon_1\Delta\varepsilon_2)$ and where $\chi(T)$ is equal to 1 if a_3 is even and is equal to σ if a_3 is odd. Hence (comp. loc. cit.) the value of (7.24) at $X = 1$ is equal to zero if and only if $\chi(T) = \sigma = -1$. In this case the value of the derivative of (7.24) at $X = 1$ is equal to

$$-\sum_{\ell=0}^{\frac{a_2}{2}-1} p^\ell (a_2 + a_3 - 4\ell) - p^{\frac{a_2}{2}} \cdot \left(\frac{a_3 - a_2 + 1}{2}\right) = -e_p(T_{\varepsilon_1}) \quad ,$$

as a comparison with Proposition 6.2 shows.

Case a_2 **odd:** In this case

$$\frac{\alpha_p(H_{2r+4}, T_{\varepsilon_1\Delta})}{(1-p^{-2}X)\cdot(1-p^{-2}X^2)} = \sum_{\ell=0}^{(a_2-1)/2} p^\ell (X^{2\ell} + \chi(T)\cdot X^{a_2+a_3-2\ell}) \tag{7.25}$$

with $\chi(T)$ equal to $\chi(-\varepsilon_1\Delta\cdot\varepsilon_3)$ if a_3 is even and equal to $\chi(-\varepsilon_2\varepsilon_3)$ if a_3 is odd. This expression vanishes at $X = 1$ if and only if $\chi(T) = -1$ and in this case the value of the derivative of (7.25) at $X = 1$ is equal to

$$-\sum_{\ell=0}^{(a_2-1)/2} p^\ell (a_2 + a_3 - 4\ell) = -e_p(T_{\varepsilon_1}) \quad . \tag{7.26}$$

In both cases we therefore obtain

$$\frac{\partial}{\partial X}(A_{H_4,T_{\varepsilon_1\Delta}}(X))\Big|_{X=1} = -(1-p^{-2})^2 \cdot e_p(T_{\varepsilon_1}) \quad , \tag{7.27}$$

provided that T_{ε_1} is not represented by H_4. Taking into account (7.21)-(7.23) we obtain

$$\begin{aligned}\frac{\partial}{\partial X}\{A_{S,T_{\varepsilon_1}}(X)\}\Big|_{X=1} &= -\frac{1+p^{-2}}{1-p^{-2}} \cdot (1-p^{-2})^2 \cdot e_p(T_{\varepsilon_1}) \\ &= -(1+p^{-2})(1-p^{-2}) \cdot e_p(T_{\varepsilon_1}) \quad ,\end{aligned} \tag{7.28}$$

provided that T_{ε_1} is not represented by S, i.e. by $V(\mathbb{Q}_p)$. Taking into account (7.17) we obtain the formula in (iii).

We proceed in a similar way to prove (ii). For a suitable choice of a basis for $V'(\mathbb{Z}_p)$ the quadratic form has matrix

$$S' = \operatorname{diag}(1, -1, p, -p\Delta) \quad . \tag{7.29}$$

Using the reduction formula (7.18) twice we obtain

$$\alpha_p(S', T_{\varepsilon_1}) = \frac{\alpha_p(S', \varepsilon_1)}{\alpha_p(\tilde{S}', -\varepsilon_1\Delta)} \cdot \alpha_p(\tilde{S}', T_{-\varepsilon_1\Delta}) \quad , \tag{7.30}$$

where

$$\tilde{S}' = \operatorname{diag}(1, \Delta, p, -p\Delta) \tag{7.31}$$

is the quadratic form given by the norm form on the maximal order of the quaternion division algebra over $\mathbb{Q}_p$. Appealing to [**19**], Theorem 3.2 we have

$$\alpha_p(S', \varepsilon_1) = 1 - p^{-1}, \qquad \alpha_p(\tilde{S}', -\varepsilon_1\Delta) = 1 + \chi(-1)p^{-1} \tag{7.32}$$

and by [**3**], Proposition 6.10,

$$\alpha_p(\tilde{S}', T_{-\varepsilon_1\Delta}) = 2(1 + \chi(-1)p^{-1}) \cdot (p+1) \quad . \tag{7.33}$$

Inserting (7.32) and (7.33) into (7.30) we therefore obtain

$$\alpha_p(S', T_{\varepsilon_1}) = 2p^{-1} \cdot (p^2 - 1) \quad . \tag{7.34}$$

We have the relation

$$\begin{aligned}W_T(e, 0, \Phi'_p) &= \gamma(V'_p) \cdot |S'|^{\frac{3}{2}} \cdot \alpha_p(S', T) \\ &= 2 \cdot \gamma(V'_p) \cdot p^{-4} \cdot (p^2 - 1) \quad ,\end{aligned} \tag{7.35}$$

which proves (ii). $\square$

We now plug in the expressions obtained in Proposition 7.2 into (7.13) and obtain

(7.36)

$$\begin{aligned} E'_T(h,0,\Phi) = -\mathrm{vol}(SO(V')(\mathbb{R})\cdot pr(K'))\cdot W_T^2(h)\cdot \\ \cdot \log p \cdot \frac{(1+p^{-2})(1-p^{-2})}{2\cdot(p^2-1)}\cdot p^4\cdot e_p(T)\cdot \mathrm{vol}(K')^{-1}\cdot \\ \cdot \mathrm{vol}(Z(\mathbb{Q})\setminus Z(\mathbb{A}_f))\cdot O_T(\varphi'_f) \\ = -\mathrm{vol}(SO(V')(\mathbb{R})\cdot pr(K'))\cdot W_T^2(h)\cdot \frac{1}{2}\log p\cdot (p^2+1)\cdot \langle \mathcal{Z}(T,\omega)\rangle_p \end{aligned}$$

Taking into account the fact that the index of the Iwahori subgroup K'_p in the maximal compact subgroup K_p is equal to p^2+1, we finally obtain the following theorem.

Theorem 7.3. *Let $T\in \mathrm{Sym}_3(\mathbb{Q})_{>0}$ be represented by $V(\mathbb{A}_f^p)$ but not by $V(\mathbb{Q}_p)$. Also assume that ω is locally centrally symmetric.*

(i) *If $T\notin \mathrm{Sym}_3(\mathbb{Z}_{(p)})$, then $\mathcal{Z}(T,\omega)=\emptyset$ and $E'_T(h,0,\Phi)=0$.*

(ii) *If $T\in \mathrm{Sym}_3(\mathbb{Z}_{(p)})$ is not divisible by p, then $\mathcal{Z}(T,\omega)$ is zero-dimensional and*

$$E'_T(h,0,\Phi) = -\frac{1}{2}\mathrm{vol}(SO(V')(\mathbb{R}))\cdot W_T^2(h)\cdot \mathrm{vol}(pr(K))\cdot \log p\cdot \langle \mathcal{Z}(T,\omega)\rangle_p \ .$$

To apply this theorem to the intersection problem of special cycles consider the situation of the end of section 6. Let

$$W = W_1 + \ldots + W_r \tag{7.37}$$

be the decomposition of the standard 6-dimensional symplectic space into symplectic subspaces of dimension $2n_i$ compatible with the fixed symplectic basis, and let

$$\iota : H_{1,\mathbb{A}}\times \ldots \times H_{r,\mathbb{A}} \longrightarrow Sp_{6,\mathbb{A}} \tag{7.38}$$

be the corresponding homomorphism of metaplectic groups, covering the embedding

$$\iota : Sp(W_{1,\mathbb{A}})\times \ldots \times Sp(W_{r,\mathbb{A}}) \hookrightarrow Sp_{6,\mathbb{A}} \ . \tag{7.39}$$

Restricting to the archimedean place, for $(h_1,\ldots,h_r)\in H_{1,\mathbb{R}}\times\ldots\times H_{r,\mathbb{R}}$ we have

$$W_T^2(\iota(h_1,\ldots,h_r)) = W_{T_1}^2(h_1)\cdot \ldots \cdot W_{T_r}^2(h_r) \ , \tag{7.40}$$

where T has diagonal blocks $T_1,\ldots,T_r$.

Corollary 7.4. *We have the following identity, provided that $\omega_1, \dots, \omega_r$ are locally centrally symmetric.*

$$\sum_T E'_T(\iota(h_1, \dots, h_r), 0, \Phi) = -\frac{1}{2}\mathrm{vol}(SO(V')(\mathbb{R})) \cdot W^2_{T_1}(h_1) \dots W^2_{T_r}(h_1) \cdot$$
$$\cdot \mathrm{vol}(K) \cdot \log p \cdot \langle \mathcal{Z}(T_1, \omega_1), \dots, \mathcal{Z}(T_r, \omega_r)\rangle_p^{\mathrm{proper}}$$

Here the summation runs over $T \in \mathrm{Sym}_3(\mathbb{Z}_{(p)})_{>0}$ with diagonal blocks $T_1, \dots, T_r$, and such that (i) T is represented by $V(\mathbb{A}_f^p)$ but not by $V(\mathbb{Q}_p)$ and (ii) $T \not\equiv 0 \bmod p$. Also Φ is determined as in (7.12) with $\omega = \omega_1 \times \dots \times \omega_r$.

We note that as in [**11**] the left side of this expression is part of the Fourier coefficient corresponding to $(T_1, \dots, T_r)$ of the pullback of $E'(g, 0, \Phi)$ to $H_{1,\mathbb{A}} \times \dots \times H_{r,\mathbb{A}}$.

§8. Components of $\mathcal{Z}(T, \omega) \cap \mathcal{M}^{ss}$.

When $T \in \mathrm{Sym}_n(\mathbb{Z}_{(p)})_{>0}$ is divisible by p, i.e., when $\bar{T} = 0$, Theorem 6.1 says that $\big(\mathrm{image}(\mathcal{Z}(T, \omega)\big) \cap \mathcal{M}^{\mathrm{ss}}(\mathbb{F})$ is a union of certain components $\mathbb{P}_{L_i}$ of $\mathcal{M}^{\mathrm{ss}}(\mathbb{F})$. In this section, we describe the configurations of components which occur, in terms of the matrix T. We again consider the isocrystal $\mathcal{L}$ associated to a fixed base point $\xi = (A, \lambda, \iota, \overline{\eta}^p) \in \mathcal{M}^{\mathrm{ss}}(\mathbb{F})$.

Recall, from (5.1), that $V'_p = \{j \in \mathrm{End}(\mathcal{L}, F)^{(1)} \ ; \ j^* = j\}$ is the space of special endomorphisms of the graded, polarized isocrystal $(\mathcal{L}, <\ ,\ >, \iota)$. Suppose that $T \in p\mathrm{Sym}_n(\mathbb{Z}_{(p)})$ and that $\mathbf{j} \in (V'_p)^n$ with $Q'(\mathbf{j}) = T$. We want to determine the set

$$X(\mathbf{j}) := \{\ L \in X \ ; \ \mathbf{j} \in \big(N_L\big)^n\ \}. \tag{8.1}$$

Obviously this only depends on the $\mathbb{Z}_p$-submodule of V'_p spanned by the components of $\mathbf{j}$. Thus we can work with a $\mathbb{Z}_p$-basis for this submodule for which the restriction of the quadratic form Q' is diagonal.

We begin by determining, for $j \in V'_p$, the set

$$X(j) = \{L \in X \ ; \ \ j \in N_L\} \ . \tag{8.2}$$

We will always assume that j is not isotropic, i.e. $j^2 \neq 0$.

Recall from the end of section 4 the set

$$Y_i = \{\ L_i \subset \mathcal{L}_i \ ; \ \ F^2 L_i = pL_i\ \}, \tag{8.3}$$

and let

$$Y_i(j) = \{\ L_i \in Y_i\ ;\ \ j(L_i) \subset FL_i\}. \tag{8.4}$$

Then, combining Proposition 5.5 and Proposition 4.4, we have:

Proposition 8.1. *There is a natural surjection*

$$\coprod_{i=0,1} \coprod_{L_i \in Y_i(j)} \mathbb{P}_{L_i}(\mathbb{F}) \longrightarrow X(j).$$

Thus, we must give a description of the sets $Y_i(j)$, for $i = 0, 1$. These sets are invariant under homothety and $FY_0(j) = Y_1(j)$, since j commutes with F. Thus, it suffices to consider $Y_0(j)$.

Let

$$U = U_0 = \big(\mathcal{L}_0\big)^{p^{-1}F^2}, \tag{8.5}$$

be the 2-dimensional $\mathbb{Q}_{p^2}$-vector space of fixed points of the σ^2-linear endomorphism $p^{-1}F^2$. This space comes equipped with a nondegenerate alternating form $<\ ,\ >$ valued in $\mathbb{Q}_{p^2}$. Moreover, G'_p preserves U and restriction yields an isomorphism

$$G'_p \simeq \{g \in GL(U)\ ;\ \det(g) \in \mathbb{Q}_p^\times\},$$

cf. (5.7).

The set Y_0 is then precisely the set of W-lattices in $\mathcal{L}_0 = U \otimes_{\mathbb{Q}_{p^2}} \mathcal{K}$ of the form $L_0 = \Lambda \otimes_{\mathbb{Z}_{p^2}} W$, where $\Lambda = (L_0)^{p^{-1}F^2}$ is a $\mathbb{Z}_{p^2}$-lattice in U. Let $\mathcal{B}_0$ be the set of $\mathbb{Z}_{p^2}$-lattices Λ in U up to homothety, i.e., the set of vertices of the building $\mathcal{B}$ for the group $PGL(U)$. The natural surjection

$$pr : Y_0 \to \mathcal{B}_0 \tag{8.6}$$

is given by $L_0 = \Lambda \otimes W \mapsto [\Lambda]$.

Let $j \in V'_p$. The endomorphism $F^{-1}j$ of $\mathcal{L}$ has degree 0, is σ^{-1}-linear, and commutes with F. It therefore induces a σ-linear endomorphism of $U = \mathcal{L}_0^{p^{-1}F^2}$,

$$\beta = F^{-1}j|_U. \tag{8.7}$$

Obviously β conversely determines j. The condition $j^* = j$ is equivalent to $\beta = \beta^*$ where $*$ is defined by

$$(8.8) \qquad < \beta x, y > = < x, \beta^* y >^\sigma .$$

Indeed, for x and $y \in U$,

$$\begin{aligned} < \beta x, y > &= < F^{-1} jx, y > = p^{-1} < jx, Fy >^\sigma \\ &= < x, jp^{-1} Fy >^\sigma = < x, F^{-1} jy >^\sigma = < x, \beta y >^\sigma . \end{aligned}$$

We will refer to the elements of the space

$$(8.9) \qquad V_p'' := \{ \beta \in \mathrm{End}_{\mathbb{Q}_p}(U) \ ; \quad \beta \text{ is } \sigma\text{-linear, and } \beta^* = \beta \ \}$$

as *special endomorphisms* of $(U, < \ , \ >)$. Of course, as $\mathbb{Q}_p$-vector spaces, $V_p' \simeq V_p''$ via the map (8.7). Define a quadratic form Q'' on V_p'' by

$$(8.10) \qquad \beta^2 = Q''(\beta) \cdot 1_U,$$

and note that

$$(8.11) \qquad Q'(j) = pQ''(\beta),$$

since

$$\begin{aligned} Q''(\beta) < x, y >^\sigma &= < \beta x, \beta y > = < F^{-1} jx, F^{-1} jy > \\ &= p^{-1} < jx, jy >^\sigma = p^{-1} Q'(j) < x, y >^\sigma . \end{aligned}$$

If e_1, e_2 is a $\mathbb{Q}_{p^2}$-basis for U for which $< e_1, e_2 >= 1$, then a σ-linear endomorphism can be written in the form

$$\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sigma,$$

and

$$\beta^* = \begin{pmatrix} d^\sigma & -b^\sigma \\ -c^\sigma & a^\sigma \end{pmatrix} \sigma,$$

where $\sigma^2 = 1$ on U. Thus β is special if it has the form

$$(8.12) \qquad \beta = \begin{pmatrix} a & b \\ c & a^\sigma \end{pmatrix} \sigma,$$

for a, b, and $c \in \mathbb{Q}_{p^2}$ with $b^\sigma = -b$ and $c^\sigma = -c$. Then $Q''(\beta) = aa^\sigma - bc$.

If $\beta \in V_p''$ and $j \in V_p'$ are related by (8.7), then $L_0 = \Lambda \otimes_{\mathbb{Z}_{p^2}} W \in Y_0(j)$ if and only if $\beta(\Lambda) \subset \Lambda$. Our description of $Y_0(j)$ depends on the following simple observation which goes back to Kottwitz and Tate (comp. [**13**], Lemma 2.4). A special endomorphism β of U induces an involution on $\mathcal{B}$ (since we are always assuming that $\beta^2 = Q''(\beta) \cdot 1_U$ with $Q''(\beta) \neq 0$). Let $\mathcal{B}^\beta$ denote the fixed point set of β. Also let $d(x, y)$ denote the distance in the building.

Lemma 8.2. *If* β *is a special endomorphism of* U , *then*

$$\begin{aligned}\beta(\Lambda)\subset\Lambda \qquad &\iff \qquad d([\beta(\Lambda)],[\Lambda])\le \mathrm{ord}_p(Q''(\beta))=\mathrm{ord}_p(\det\beta)\\ &\iff \qquad d([\Lambda],\mathcal{B}^{\beta})\le \frac{1}{2}\cdot \mathrm{ord}_p(\mathbb{Q}''(\beta)) \quad .\end{aligned}$$

For a special endomorphism β of U , let

$$\mathcal{T}(\beta)=\{\ x\in\mathcal{B}\ \ ;\ \ d(x,\mathcal{B}^{\beta})\le\frac{1}{2}\mathrm{ord}_p(\det\beta)\ \} \tag{8.13}$$

be the closed tube of radius $\frac{1}{2}\cdot\mathrm{ord}_p(\det\beta)$ around the fixed point set. Let $\mathcal{T}(\beta)_0$ be the set of vertices in $\mathcal{T}(\beta)$.

Corollary 8.3. *For a special endomorphism* β *of* U *corresponding to a special endomorphism* $j\in V_p'$, *via (8.7),*

$$Y_0(j)=\mathrm{pr}^{-1}(\mathcal{T}(\beta)_0)\simeq\mathbb{Z}\times\mathcal{T}(\beta)_0,$$

where pr *is the projection of (8.6).* $\square$

Returning to our n-tuple of special endomorphisms $\mathbf{j}$, let $\boldsymbol{\beta}=(\beta_1,\ldots,\beta_n)$ be the corresponding tuple of special endomorphisms of U . As noted above, the sets $X(\mathbf{j})$ and $Y_0(\mathbf{j})$ depend only on the $\mathbb{Z}_p$-span of the components of $\mathbf{j}$. Thus, by changing the $\mathbb{Z}_p$-basis if necessary, we may assume that the matrix T is of the form

$$T=Q'(\mathbf{j})=\mathrm{diag}(\varepsilon_1p^{a_1},\ldots,\varepsilon_np^{a_n}) \tag{8.14}$$

with $1\le a_1\le\cdots\le a_n$. It follows by (8.11) that

$$Q''(\boldsymbol{\beta})=\mathrm{diag}(\varepsilon_1p^{r_1},\ldots,\varepsilon_np^{r_n})=p^{-1}T, \tag{8.15}$$

with $r_i=a_i-1$.

Corollary 8.4. *Let*

$$\mathcal{T}(\boldsymbol{\beta}):=\mathcal{T}(\beta_1)\cap\cdots\cap\mathcal{T}(\beta_n).$$

Then

$$Y_0(\mathbf{j})=\mathrm{pr}^{-1}(\mathcal{T}(\boldsymbol{\beta})_0)\simeq\mathbb{Z}\times\mathcal{T}(\boldsymbol{\beta})_0. \qquad \square$$

Let us now return to $X(j)$. Suppose we are given a path in the building $\mathcal{B}$ with consecutive vertices $x_0,\ x_1,\ \dots, x_r,\ \dots$. Let $\Lambda_0 \supset \Lambda_1 \supset \cdots \supset \Lambda_r \supset \dots$ be a corresponding sequence of lattices in U with $(\Lambda_r : \Lambda_{r+1}) = 1$. Construct a sequence of lattices

$$L_0 = \Lambda_0 \otimes W,\ L_1 = F^{-1}(\Lambda_1 \otimes W),\ \dots L_r = F^{-r}(\Lambda_r \otimes W)\dots, \tag{8.16}$$

where $L_r \in Y_0$ if r is even and $L_r \in Y_1$ if r is odd. Associated to this sequence of lattices is a 'chain' of $\mathbb{P}^1$'s

$$\mathbb{P}_{L_0},\ \mathbb{P}_{L_1},\ \dots, \mathbb{P}_{L_r},\ \dots \tag{8.17}$$

in X. These $\mathbb{P}^1$'s cross at the sequence of superspecial lattices:

$$\begin{aligned} L_0 \oplus L_1 &= \mathbb{P}_{L_0} \cap \mathbb{P}_{L_1}, \\ L_2 \oplus L_1 &= \mathbb{P}_{L_2} \cap \mathbb{P}_{L_1}, \\ L_2 \oplus L_3 &= \mathbb{P}_{L_2} \cap \mathbb{P}_{L_3}, \\ &\dots \\ L_{2t} \oplus L_{2t\pm1} &= \mathbb{P}_{L_{2t}} \cap \mathbb{P}_{L_{2t\pm1}}, \\ &\dots, \end{aligned} \tag{8.18}$$

which can be viewed as indexed by the edges in the path. Moreover, if the path lies in $\mathcal{T}(\beta)$, then the chain of $\mathbb{P}^1$'s lies in $X(j)$.
A given path also gives rise to additional chains of $\mathbb{P}^1$'s obtained from the first by applying powers of F. These chains are all disjoint (consider the index in a fixed lattice in U). More generally, any connected subset of $\mathcal{B}$ can be viewed as the dual graph to a connected curve of $\mathbb{P}^1$'s in X, uniquely determined up to the action of F.

Applying these considerations to $\mathcal{T}(\boldsymbol{\beta})$, we obtain the following.

Proposition 8.5. *The set $X(\mathbf{j})$ is a disjoint union of copies, indexed by $\mathbb{Z}$, of a connected union of $\mathbb{P}^1$'s. The dual graph of a connected component is naturally isomorphic to $\mathcal{T}(\boldsymbol{\beta})$, where $\boldsymbol{\beta}$ and $\mathbf{j}$ are related by (8.7). In particular, the number of $\mathbb{P}^1$'s in a connected component is $|\mathcal{T}(\boldsymbol{\beta})_0|$. The number of crossing points in a connected component is $|\mathcal{T}(\boldsymbol{\beta})_1|$, where $\mathcal{T}(\boldsymbol{\beta})_1$ denotes the set of edges contained in $\mathcal{T}(\boldsymbol{\beta})$.*

It thus remains to obtain a better understanding of the sets $\mathcal{T}(\boldsymbol{\beta})$. To this end we collect some facts on the action of special endomorphisms on the building $\mathcal{B}$. We first note that the building $\mathcal{B}$ has the property that every pair of points x and y is joined by a unique geodesic and that every path from x to y contains that geodesic.

Lemma 8.6. *(i) For a special endomorphism β of U, the fixed point set $\mathcal{B}^\beta$ is connected.*
(ii) If $x \in \mathcal{B}$, then the midpoint of the unique geodesic in $\mathcal{B}$ joining x and $\beta(x)$ is fixed by β and is the point of $\mathcal{B}^\beta$ nearest to x. If $x \in \mathcal{T}(\beta)$, then this geodesic lies in $\mathcal{T}(\beta)$.
(iii) The set $\mathcal{T}(\beta)$ contains the geodesic joining any two of its points.
(iv) If $\beta_1, \ldots, \beta_m$ are special endomorphisms of U, then $\mathcal{T}(\beta_1) \cap \cdots \cap \mathcal{T}(\beta_m)$ is connected.

Proof. If x and y are fixed points of β in $\mathcal{B}$, then the unique geodesic in $\mathcal{B}$ joining x and y is also fixed. This proves (i) since non-emptyness follows from (ii). The assertion (ii) follows from the fact that β induces an involution of $\mathcal{B}$ (comp. [**13**], Lemma 2.4.) and was in fact used in the proof of Lemma 8.2 above. The set $\mathcal{T}(\beta)$ is connected since each point of it is connected to $\mathcal{B}^\beta$ by a geodesic in $\mathcal{T}(\beta)$, and $\mathcal{B}^\beta$ is connected. Since the geodesic joining any pair of points is a subset of any path joining them, (iii) follows. Finally, (iv) is immediate from (iii). $\square$

Lemma 8.7. *Let β be a special endomorphism of U.*
(i) If $\mathrm{ord}_p(Q''(\beta))$ is even, then $\mathcal{B}^\beta \simeq \mathcal{B}(PGL_2(\mathbb{Q}_p))$.
(ii) If $\mathrm{ord}_p(Q''(\beta))$ is odd, then $\mathcal{B}^\beta$ is a single point, which is a midpoint of an edge.

Proof. The centralizer G_β in $GL_2(\mathbb{Q}_{p^2})$ of the σ-linear endomorphism β is a $\mathbb{Q}_p$-form of GL_2, and obviously $\mathcal{B}(PG_\beta) \subset \mathcal{B}^\beta$. If $\mathrm{ord}_p(Q''(\beta))$ is even, then $G_\beta \simeq GL_2(\mathbb{Q}_p)$, while, if $\mathrm{ord}_p(Q''(\beta))$ is odd, then $G_\beta \simeq \mathbb{B}_p^\times$. In the latter case $\mathcal{B}(PG_\beta) = \mathcal{B}^\beta$ is a midpoint. Let us show that also in the first case $\mathcal{B}(PG_\beta) = \mathcal{B}^\beta$. After correcting β by a power of p we may assume that $\beta^2 = 1_U$. If Λ is a representative of a vertex of $\mathcal{B}^\beta$, then β induces a σ-linear automorphism $\overline{\beta} : \Lambda/p\Lambda \to \Lambda/p\Lambda$. But $\overline{\beta}$ defines an $\mathbb{F}_p$-rational structure on $\mathbb{P}(\Lambda/p\Lambda)$ which cannot have more than $p+1$ $\mathbb{F}_p$-rational points. The assertion follows since $\mathcal{B}^\beta$ is connected, cf. Lemma 8.6, (i). $\square$

Lemma 8.8. *Suppose that $\beta_1, \dots, \beta_n$ is a collection of special endomorphisms which anticommute, i.e., such that $\beta_i\beta_j + \beta_j\beta_i = 0$ for $i \neq j$. The set of common fixed points $\mathcal{B}^{\beta_1} \cap \cdots \cap \mathcal{B}^{\beta_n}$ is nonempty and connected.*

Proof. The set $\mathcal{B}^{\beta_i}$ is preserved by β_j for each j. If $x_1 \in \mathcal{B}^{\beta_1}$, then the midpoint x_{12} of the geodesic from x_1 to $\beta_2(x_1)$ lies in $\mathcal{B}^{\beta_1} \cap \mathcal{B}^{\beta_2}$, by (i) and (ii) of Lemma 8.6. The midpoint x_{123} of the geodesic joining x_{12} and $\beta_3(x_{12})$ lies in $\mathcal{B}^{\beta_1} \cap \mathcal{B}^{\beta_2} \cap \mathcal{B}^{\beta_3}$, etc. $\square$

Lemma 8.9. *For a collection of special endomorphism β_i, as in Lemma 8.8, and a point $x \in \mathcal{B}$, let $x_i \in \mathcal{B}^{\beta_i}$ be the point closest to x, and let $d_i = d(x, x_i) = d(x, \mathcal{B}^{\beta_i})$. Assume that $d_1 \leq d_2 \leq \cdots \leq d_n$. Then*

$$\begin{aligned}
&x_1 \in \mathcal{B}^{\beta_1}\\
&x_2 \in \mathcal{B}^{\beta_1} \cap \mathcal{B}^{\beta_2}\\
&x_3 \in \mathcal{B}^{\beta_1} \cap \mathcal{B}^{\beta_2} \cap \mathcal{B}^{\beta_3}\\
&\quad \cdots\\
&x_n \in \mathcal{B}^{\beta_1} \cap \cdots \cap \mathcal{B}^{\beta_n}.
\end{aligned}$$

Moreover, if, for some i, $d_i = d_{i+1}$, then $x_i = x_{i+1}$. Finally, for any i with $1 \leq i \leq n$, the geodesic from x to x_i is

$$[x, x_i] = [x, x_1][x_1, x_2] \dots [x_{i-1}, x_i]. \qquad \square$$

Now return to the set $\mathcal{T}(\boldsymbol{\beta})$ of Corollary 8.4, where the β_i's anticommute and where $0 \leq r_1 \leq \cdots \leq r_n$, as in (8.14).

First suppose that r_1 is odd. Then $\mathcal{B}^{\beta_1}$ is the midpoint of an edge, and $\mathcal{B}^{\beta_1} \subset \mathcal{B}^{\beta_i}$ for all i. Since $r_1 \leq r_i$, for all i,

$$\mathcal{T}(\beta_1) \subset \mathcal{T}(\beta_i)$$

for all i, and hence $\mathcal{T}(\boldsymbol{\beta})$ is simply the ball of radius $r_1/2$ centered at the point $\mathcal{B}^{\beta_1}$.

Next suppose that $r_1 = 2t$ *is even.* There is a unit $u \in \mathbb{Z}_{p^2}^\times$ such that $\beta_0 := u^{-1}p^{-t}\beta_1$ satisfies $\beta_0^2 = 1$. Note however that β_0 is no longer a special endomorphism. We may then write $U = U^0 \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^2}$, where U^0 is the fixed point set of β_0. This gives a natural isomorphism $\mathcal{B}^{\beta_1} = \mathcal{B}^{\beta_0} \simeq \mathcal{B}(PGL(U^0))$, which is given on vertices by $[\Lambda] \mapsto [\Lambda^0]$, for $\Lambda = \Lambda^0 \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^2}$, where Λ^0 is a $\mathbb{Z}_p$-lattice in U^0.

We fix a basis e_1, e_2 for U^0 and write $\beta_1 = up^t \cdot \sigma = up^t \cdot \beta_0$. For $i \geq 2$, define the matrix $\gamma_i \in GL_2(\mathbb{Q}_{p^2})$ by $\beta_i = \gamma_i u\delta\sigma$. Then the relation $\beta_1\beta_i = -\beta_i\beta_1$ is equivalent to $\gamma_i \in GL_2(\mathbb{Q}_p)$. Next we claim that γ_i^2 is a scalar matrix. Indeed, we have

$$\beta_i^2 = (\gamma_i u\delta\sigma)(\gamma_i u\delta\sigma).$$

Since $\beta_1\beta_i = -\beta_i\beta_1$, we have

$$u\sigma(\gamma_i u\delta\sigma) = -(\gamma_i u\delta\sigma)u\sigma, \qquad i.e., \qquad u\gamma_i u^\sigma \delta^\sigma = -\gamma_i u\delta u^\sigma.$$

But $\delta = -\delta^\sigma \cdot (\text{scalar})$, and hence

$$u\gamma_i = \gamma_i u.$$

Therefore,

$$\beta_i^2 = \gamma_i^2 u\delta u^\sigma \delta^\sigma.$$

But $uu^\sigma = p^{-2t}\beta_i^2$ is a scalar matrix, hence, since

$$\beta_i^2 = \gamma_i^2 \cdot uu^\sigma(-\Delta)$$

is a scalar matrix, so is γ_i^2, as claimed.

Moreover,

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 0, \quad \text{for } i \neq j\text{, and} \qquad \gamma_i^2 = -\Delta^{-1}\varepsilon_1^{-1}\varepsilon_i p^{r_i}.$$

Under the isomorphism $\mathcal{B}^{\beta_1} \simeq \mathcal{B}(PGL(U_0)) =: \mathcal{B}^0$ we have

$$\mathcal{B}^{\beta_1} \cap \mathcal{B}^{\beta_i} \simeq \mathcal{B}^{0,\gamma_i},$$

where $\mathcal{B}^{0,\gamma_i}$ is the fixed point set of γ_i in $\mathcal{B}^0$.

Lemma 8.10. *Let* $\gamma \in GL_2(\mathbb{Q}_p)$ *with* $\mathrm{tr}(\gamma) = 0$ *. Then*

$$\mathcal{B}^{0,\gamma} = \begin{cases} \text{an apartment} & \text{if } \gamma^2 \in \mathbb{Q}_p^{\times,2}, \\ \text{a vertex} & \text{if } \gamma^2 \notin \mathbb{Q}_p^{\times,2}, \text{ and } \mathrm{ord}_p(\gamma^2) \text{ is even}, \\ \text{a midpoint} & \text{if } \gamma^2 \notin \mathbb{Q}_p^{\times,2}, \text{ and } \mathrm{ord}_p(\gamma^2) \text{ is odd}. \end{cases}$$

Furthermore γ *generates a split, unramified elliptic, or ramified elliptic Cartan subgroup of* $GL_2(\mathbb{Q}_p)$ *in the three cases respectively.*

Proof. The matrix γ has two eigenlines in $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$. In the first case, these lie in $\mathbb{P}^1(\mathbb{Q}_p)$, the boundary of $\mathcal{B}^0$, and the (infinite) geodesic joining them is the apartment fixed by γ . The other two cases are clear. $\square$

Corresponding to the cases we call β_i $(i \geq 2)$ split, unramified elliptic or ramified elliptic.

Lemma 8.11. *Assume that* r_1 *is even. If* β_i *is ramified elliptic (resp. unramified elliptic) for some* $i \geq 2$ *, then* β_j *is not unramified elliptic (resp. ramified elliptic) for any* $j \geq 2$ *.*

Proof. Otherwise one fixed point set would be a midpoint of an edge and the other a vertex in $\mathcal{B}^0$, contradicting Lemma 8.8. $\square$

Lemma 8.12. *If* $n = 3$ *or* 4 *, then* $\mathcal{B}^{\boldsymbol{\beta}} := \mathcal{B}^{\beta_1} \cap \cdots \cap \mathcal{B}^{\beta_n}$ *is a point. This point is a midpoint if at least one* r_i *is odd and is a vertex if all* r_i *are even.*

Proof. This is immediate if any of the r_i 's is odd, so suppose that each r_i is even. We may assume that $\mathcal{B}^{\beta_1} \cap \mathcal{B}^{\beta_2} \simeq \mathcal{B}^{0,\gamma_2}$ is an apartment associated to the two eigenlines of γ_2 in U^0 . Since $\gamma_2\gamma_3 = -\gamma_3\gamma_2$, these eigenlines are switched by γ_3 , so that γ_3 preserves $\mathcal{B}^{0,\gamma_2}$ and acts by a reflection through some unique fixed point. This point is a vertex $\mathcal{B}^{0,\gamma_2}$, since $\mathrm{ord}_p(\gamma_3^2)$ is even. If $n = 4$, γ_4 also fixes this vertex. $\square$

Proposition 8.13. *If* $n = 3$ *or 4, then* $\mathcal{T}(\boldsymbol{\beta})_0$ *consists of a single vertex if and only if* $r_1 = r_2 = 0$ *and if* r_i *is even for* $i \geq 3$ *and furthermore* $r_3 = 0$ *if* $\chi(-\varepsilon_1\varepsilon_2) = -1$ *. Equivalently, by Proposition 8.5, each connected component* *of* $\mathcal{Z}(T,\omega)\cap\mathcal{M}^{ss}$ *is irreducible of dimension 1 if and only if* T *is* $GL_3(\mathbb{Z}_p)$*-equivalent to* $\mathrm{diag}(p,\varepsilon_2 p,\varepsilon_3 p^{a_3})$ *with* a_3 *odd and with* $a_3=1$ *if* $\chi(-\varepsilon_2)=-1$ *resp.* $GL_4(\mathbb{Z}_p)$*-equivalent to* $\mathrm{diag}(p,\varepsilon_2 p,\varepsilon_3 p^{a_3},\varepsilon_4 p^{a_4})$ *with* a_3 *and* a_4 *odd and* $a_3=1$ *if* $\chi(-\varepsilon_2)=-1$ *.*

Proof. If there exists $i\geq 1$ with r_i odd, then $\mathcal{B}^{\boldsymbol{\beta}}$ consists of the midpoint of an edge and both vertices of this edge belong to $\mathcal{T}(\boldsymbol{\beta})_0$. Hence, if $\mathcal{T}(\boldsymbol{\beta})_0$ consists of a single vertex, all r_i have to be even. The unique vertex x_0 in $\mathcal{B}^{\boldsymbol{\beta}}$ belongs to $\mathcal{T}(\boldsymbol{\beta})_0$. If $r_1\geq 2$, then any vertex with distance 1 from x_0 in $\mathcal{B}$ also belongs to $\mathcal{T}(\boldsymbol{\beta})_0$, hence $r_1=0$. A similar argument applied to $\mathcal{B}^0=\mathcal{B}^{\beta_1}$ shows that $r_2=0$. However, in this case β_2 is unramified elliptic iff $\chi(-\Delta\varepsilon_1\varepsilon_2)=-1$. For such a β_2, $\mathcal{B}^{\beta_1}\cap\mathcal{B}^{\beta_2}=\mathcal{B}^{0,\gamma_2}$ consists of the single vertex x_0 and $(\mathcal{T}(\beta_1)\cap\mathcal{T}(\beta_2))_0=\{x_0\}$. If $\chi(-\Delta\varepsilon_1\varepsilon_2)=1$, then $\mathcal{B}^{\beta_1}\cap\mathcal{B}^{\beta_2}=\mathcal{B}^{0,\gamma_2}$ is an apartment in $\mathcal{B}^0$. To exclude the vertices on this apartment from $\mathcal{T}(\boldsymbol{\beta})_0$ we must have $r_3=0$. The result follows. □

It seems to us that the cases enumerated in Proposition 8.13 are the simplest to consider when one wants to determine the contribution of $\mathcal{Z}(T,\omega)$ to the intersection product of special cycles $\mathcal{Z}(T_1,\omega_1),\ldots,\mathcal{Z}(T_r,\omega_r)$ in the case of excess intersection. The hope would be to obtain a result similar to Corollary 7.4.

In general, the determination of the number of irreducible components within one connected component of $\mathcal{Z}(T,\omega)\cap\mathcal{M}^{ss}$, or equivalently of $|\mathcal{T}(\boldsymbol{\beta})_0|$ is a tedious exercise. Assume again that $r_1\leq r_2\leq\ldots\leq r_n$. If r_1 is odd, then $\mathcal{T}(\boldsymbol{\beta})$ is a ball of radius $r_1/2$ around the midpoint of an edge. Hence in this case

$$|\mathcal{T}(\boldsymbol{\beta})_0|=2(1+p^2+p^4+\ldots+p^{2(r_1-1)/2})\quad. \tag{8.19}$$

Next assume that r_1 is even. Assume that $n=3$.

By Lemma 8.11, there are then seven cases:

- **(1)** γ_2 is split, γ_3 is unramified elliptic, i.e., r_1, r_2 even, $\chi(-\varepsilon_1\varepsilon_2)=-1$, $\chi(-\varepsilon_1\varepsilon_3)=1$,
- **(2)** γ_2 is split, γ_3 is ramified elliptic, i.e., r_1 even, r_2 odd, $\chi(-\varepsilon_1\varepsilon_2)=-1$,
- **(3)** γ_2 is unramified elliptic, γ_3 is split, i.e., r_1, r_2 even, $\chi(-\varepsilon_1\varepsilon_2)=1$, $\chi(-\varepsilon_1\varepsilon_3)=-1$,
- **(4)** γ_2 is ramified elliptic, γ_3 is split, i.e., r_1 odd, r_2 even, $\chi(-\varepsilon_1\varepsilon_3)=-1$,

(5) γ_2 and γ_3 are unramified elliptic, i.e., r_1, r_2 even, $\chi(-\varepsilon_1\varepsilon_2) = 1$, $\chi(-\varepsilon_1\varepsilon_3) = 1$,

(6) γ_2 and γ_3 are ramified elliptic, i.e., r_1, r_2 odd,

(7) γ_2 and γ_3 are split, i.e., r_1, r_2 even, $\chi(-\varepsilon_1\varepsilon_2) = -1$, $\chi(-\varepsilon_1\varepsilon_3) = -1$.

Let us determine the cardinality $|\mathcal{T}(\boldsymbol{\beta})_0|$ in the case (1). At the fixed vertex $x_0 = \mathcal{B}^{\boldsymbol{\beta}}$, there are p^2+1 edges, p^2-p of which lie outside of $\mathcal{B}^0 = \mathcal{B}^{\beta_1}$ and $p+1$ of which lie in $\mathcal{B}^0$. If we move along a path beginning with an edge running out of $\mathcal{B}^0$, then our distances d_1, d_2, and d_3 from $\mathcal{B}^{\beta_1} = \mathcal{B}^0$, $\mathcal{B}^{\beta_2}$ and $\mathcal{B}^{\beta_3}$ increase at the same rate. Hence, we remain inside of $\mathcal{T}(\boldsymbol{\beta})$ if and only if we move a distance at most $r_1/2$. The number of vertices in $\mathcal{T}(\boldsymbol{\beta})$ reached in this fashion is therefore:

$$1 + (p^2-p)\big(1 + p^2 + p^4 + \cdots + p^{2(r_1/2-1)}\big). \tag{8.20}$$

The leading 1 in this expression is the contribution of the vertex x_0. Among the initial edges in $\mathcal{B}^0$, there are two which lie in the apartment $\mathcal{B}^{0,\gamma_2}$, and $p-1$ which lie outside of it. Suppose that we move a distance $j \geq 1$, beginning along one of these latter $p-1$ edges, and arrive at a vertex y. The point y has distances $d_1 = 0$, and $d_2 = d_3 = j$ from $\mathcal{B}^{\beta_1}$, $\mathcal{B}^{\beta_2}$ and $\mathcal{B}^{\beta_3}$ respectively. If we then move out of $\mathcal{B}^0$ along one of the p^2-p available initial edges at y, we may move at most an additional $\min(r_1/2, r_2/2-j)$ steps. There are

$$1 + (p^2-p)\big(1 + p^2 + p^4 + \cdots + p^{2(\min(r_1/2, r_2/2-j)-1)}\big) \tag{8.21}$$

points of $\mathcal{T}(\boldsymbol{\beta})$ reached in this fashion, and so the number of points of $\mathcal{T}(\boldsymbol{\beta})$ reached along the $p-1$ initial edges lying in $\mathcal{B}^0$ but outside of $\mathcal{B}^{0,\gamma_2}$ is

$$(p-1)\sum_{j=1}^{r_2/2} 1 + (p^2-p)\big(1 + p^2 + p^4 + \cdots + p^{2(\min(r_1/2, r_2/2-j)-1)}\big). \tag{8.22}$$

Finally, suppose that we begin along one of the two initial edges in the apartment $\mathcal{B}^{0,\gamma_2}$, and move a distance $k \geq 1$ in the apartment, arriving at a point y with distances $d_1 = d_2 = 0$ and $d_3 = k$. There are $p-1$ edges at y which lie in $\mathcal{B}^0$ but outside $\mathcal{B}^{0,\gamma_2}$, and p^2-p edges lying outside of $\mathcal{B}^0$. The number of vertices of $\mathcal{T}(\boldsymbol{\beta})$ reached along paths beginning on the $p-1$ initial edges is $p-1$ times

$$\sum_{j=1}^{\min(r_2/2, r_3/2-k)} 1 + (p^2-p)\big(1 + p^2 + p^4 + \cdots + p^{2(\min(r_1/2, r_2/2-j, r_3/2-k-j)-1)}\big). \tag{8.23}$$

The number of vertices of $\mathcal{T}(\boldsymbol{\beta})$ reached along paths beginning on the p^2-p 'vertical' initial edges is

$$1+(p^2-p)\big(1+p^2+p^4+\dots+p^{2(\min(r_1/2,r_3/2-k)-1)}\big). \tag{8.24}$$

Here the leading 1 counts the vertex y itself. Combining these contributions, we obtain

$$\begin{aligned}
|\mathcal{T}(\boldsymbol{\beta})_0| &= 1+(p^2-p)\big(1+p^2+p^4+\dots+p^{2(r_1/2-1)}\big)\\
&\quad+(p-1)\sum_{j=1}^{r_2/2} 1+(p^2-p)\big(1+p^2+p^4+\dots+p^{2(\min(r_1/2,r_2/2-j)-1)}\big)\\
&\quad+2\sum_{k=1}^{r_3/2} 1+(p^2-p)\big(1+p^2+p^4+\dots+p^{2(\min(r_1/2,r_3/2-k)-1)}\big)\\
&\quad+2(p-1)\sum_{k=1}^{r_3/2}\\
&\quad\sum_{j=1}^{\min(r_2/2,r_3/2-k)} 1+(p^2-p)\big(1+p^2+p^4+\dots+p^{2(\min(r_1/2,r_2/2-j,r_3/2-k-j)-1)}\big).
\end{aligned} \tag{8.25}$$

Similar explicit expressions may be obtained in the remaining 6 cases; we will not give them here.

We conclude this section by establishing a link between the quantity $|\mathcal{T}(\boldsymbol{\beta})_0|$ and the representation density of certain quadratic forms.

The group G'_p acts transitively on the set of $\mathbb{Z}_{p^2}$-lattices in U. Fix a $\mathbb{Z}_{p^2}$-lattice Λ_0 in U, and let K'_p be the stabilizer of Λ_0 in G'_p. The stabilizer of the point $[\Lambda_0]\in\mathcal{B}$ is $K'_pZ'_p$, where Z'_p is the group of scalar matrices in G'_p. Let

$$N_0=\{\beta\in V''_p;\ \beta(\Lambda_0)\subset\Lambda_0\}. \tag{8.26}$$

Note that, for $g\in G'_p$ and $\beta\in V''_p$,

$$\beta(g(\Lambda_0))\subset g(\Lambda_0) \iff g^{-1}\beta g\in N_0. \tag{8.27}$$

This condition depends only on the coset $gK'_pZ'_p$. Let $\varphi'^{,0}_p\in S(V''_p)$, the Schwartz space of V''_p, be the characteristic function of the lattice N_0, and let

$$\varphi'_p=\varphi'^{,0}_p\otimes\ldots\otimes\varphi'^{,0}_p\in S((V''_p)^n) \tag{8.28}$$

be the characteristic function of N_0^n. Let $\boldsymbol{\beta} \in (V_p'')^n$ be an n-tuple of special endomorphisms of U such that $Q''(\boldsymbol{\beta}) = p^{-1}T$. Then,

$$[\Lambda] = [g(\Lambda_0)] \in \mathcal{T}(\boldsymbol{\beta})_0 \iff \varphi_p'(g^{-1}\boldsymbol{\beta} g) = 1. \tag{8.29}$$

It follows that

$$\begin{aligned} |\mathcal{T}(\boldsymbol{\beta})_0| &= |\{gK_p'Z_p';\ \varphi_p'(g^{-1}\boldsymbol{\beta} g) = 1\}| \\ &= \int_{Z_p'\backslash G_p'} \varphi_p'(g^{-1}\boldsymbol{\beta} g)\, dg, \end{aligned} \tag{8.30}$$

where the measure on $Z_p'\backslash G_p'$ is taken so that $\mathrm{vol}(Z_p'\backslash K_p'Z_p') = 1$. Note that, if we write $\beta \in V_p''$ in the form

$$\beta = \beta_0\sigma = \begin{pmatrix} a & b \\ c & a^\sigma \end{pmatrix}\sigma, \tag{8.31}$$

as in (8.12) above, then $\beta_0 \in G_p'$, and the action of $g \in G_p' \subset GL_2(\mathbb{Q}_{p^2})$ is given by twisted conjugacy of β_0:

$$g^{-1}\beta g = g^{-1}\beta_0 g^\sigma \sigma. \tag{8.32}$$

Thus $|\mathcal{T}(\boldsymbol{\beta})_0|$ is given by a kind of twisted orbital integral. Of course, the integral in (8.30) will not be finite, in general, if $G_p'(\boldsymbol{\beta})$, the pointwise stabilizer of $\boldsymbol{\beta}$ is not compact modulo the center.

Lemma 8.14. *If $n = 3$ or 4, then $G_p'(\boldsymbol{\beta}) = Z_p'$, and the quantities in (8.30) are finite.*

Proof. Note that $G_p'/Z_p' \simeq SO(V_p'')$, and that, since $\det(T) \neq 0$, the components of $\boldsymbol{\beta}$ span an n-dimensional non-degenerate subspace of V_p''. The group $G_p'(\boldsymbol{\beta})/Z_p' \simeq SO(V_p'')(\boldsymbol{\beta})$ acts trivially on this span, and hence is trivial (via, $\det = 1$ when $n = 3$). Thus, by Witt's Theorem, the map $g \mapsto g^{-1}\boldsymbol{\beta} g$ gives a bijection – in fact a homeomorphism – of G_p'/Z_p to the hyperboloid

$$\Omega_T = \{x \in (V_p'')^n;\ Q''(x) = p^{-1}T\}. \tag{8.33}$$

Since Ω_T is closed in $(V_p'')^n$ and since N_0^n is compact, the support of the function $g \mapsto \varphi_p'(g^{-1}\boldsymbol{\beta} g)$ is compact in G_p'/Z_p', and the integral in (8.30) is finite. $\square$

Finally, let S be the matrix for the quadratic form Q'' on the rank 4 $\mathbb{Z}_p$-lattice N_0, with respect to some $\mathbb{Z}_p$-basis. Then the classical representation density of $p^{-1}T$ by S is

$$
\begin{aligned}
(8.34)\qquad \alpha(S,p^{-1}T) &= \lim_{t\to\infty} p^{t(\frac{n(n+1)}{2}-4n)} \int\limits_{\substack{x\in U^n\\ S[x]-T\,\equiv\,0\,(\mathrm{mod}\ p^t)}} \varphi_p'(x)\,dx\\
&= \int_{\Omega_T} \varphi_p'(x)\,d\mu_T(x)\\
&= \int_{G_p'/Z_p'} \varphi_p'(g^{-1}\boldsymbol{\beta} g)\,d_Tg.
\end{aligned}
$$

Here dx is the Haar measure on V_p'' which is self dual with respect to the bilinear form associated to Q'', and $d\mu_T$ is the measure on Ω_T which is defined by the limit in the first line. Note that $d\mu_T$ is invariant under $G_p'/Z_p' = SO(V_p'')$. Finally, d_Tg is the Haar measure on G_p'/Z_p' coming from $d\mu_T$ under the homeomorphism $G_p'/Z_p \simeq \Omega_T$. It only remains, then, to determine $\mathrm{vol}(K_p'Z_p'/Z_p', d_Tg)$.

Consider the case $n=3$. Let ω be a top degree, translation invariant, differential form on $(V_p'')^3$ whose associated measure dx is self dual with respect to the bilinear pairing coming from Q''. Let η be a top degree differential form on Sym_3 such that the volume of $\mathrm{Sym}_3(\mathbb{Z}_p)$ for the associated measure is 1. Let $\mu : (V_p'')^3 \to \mathrm{Sym}_3$ be the map taking x to $Q''(x)$ or, rather, its algebro-geometric counterpart. Note that μ is equivariant for the action of $GL(3)$, and that, for $h \in GL(3)$, $h^*(\omega) = \det(h)^4\omega$, and $h^*(\eta) = \det(h)^4\eta$. There is a differential form τ on $(V_p'')^3$ of degree 6 such that

$$
(8.35)\qquad \omega = \tau \wedge \mu^*(\eta),
$$

and $h^*(\tau) = \tau$. Moreover, τ can be taken to be invariant under the action of $SO(V_p'')$. Then, for any $T \in \mu(V_p'')^3 \subset \mathrm{Sym}_3(\mathbb{Q}_p)$ with $\det(T) \neq 0$, the measure $d\mu_T$ on Ω_T, defined above, is precisely the measure defined by the restriction of τ to Ω_T. The invariance properties of τ imply that the pullback $i_x^*(\tau)$ to $SO(V)$ via a map $i_x : SO(V) \to (V_p'')^3$, $g \mapsto g\cdot x$, where $\det(\mu(x)) \neq 0$, is independent of x. Thus, the measure d_Tg and the volume $\kappa_p^{-1} := \mathrm{vol}(K_p'Z_p'/Z_p, d_Tg)$ are independent of T.

Theorem 8.15. *For any* $T \in p\mathrm{Sym}_3(\mathbb{Z}_p)$ *with* $\det(T) \neq 0$, *and for any* $\boldsymbol{\beta} \in (V_p'')^3$ *with* $T = pQ''(\boldsymbol{\beta})$,

$$
|\mathcal{T}(\boldsymbol{\beta})_0| = (1-p^{-4})^{-1}\cdot \alpha_p(S,p^{-1}T),
$$

where $\alpha_p(S, p^{-1}T)$ *is the representation density of the quadratic form* $p^{-1}T$ *by the quadratic form* Q'' *on* N_0 *.*

Proof. By what was said above, both sides of this identity differ by a multiplicative constant independent of T. To evaluate this constant we take $T = \mathrm{diag}(p, p, p)$. By Proposition 8.13 we have $|\mathcal{T}(\boldsymbol{\beta})_0| = 1$. On the other hand, putting $T' = p^{-1}T = \mathrm{diag}(1, 1, 1)$ we have $\alpha_p(S, T') = 1 - p^{-4}$, as is checked easily using the reduction formulae.

Remark 8.16. For a suitable basis for the lattice N_0 in the 4-dimensional $\mathbb{Q}_p$-vector space V_p'', we have

$$S = \mathrm{diag}(1, 1, 1, \Delta), \tag{8.36}$$

where $\Delta \in \mathbb{Z}_p^\times$ is a *nonsquare.* In the case in which S is a hyperbolic space of dimension 4, i.e., if Δ were a square, explicit formulas for the representation densities $\alpha_p(S, T')$ where $T' = \mathrm{diag}(\varepsilon_1 p^{r_1}, \varepsilon_2 p^{r_2}, \varepsilon_3 p^{r_3})$ with $0 \leq r_1 \leq r_2 \leq r_3$, were found by Kitaoka, comp. section 7. In our case, Theorem 8.15 reduces the computation of the densities, to the problem of counting the number of vertices $|\mathcal{T}(\boldsymbol{\beta})_0|$. As explained above (at least in one of seven cases) this can be done explicitly. The use of these explicit computations in the building is, however, limited since one wants to determine more generally the representation densities $\alpha_p(S_r, T')$ where $S_r = S \perp H_{2r}$ for any $r \geq 0$, comp. section 7 above. In these more general cases this combinatorial method seems difficult to handle. However, just as in the case of the usual (twisted) orbital integrals [**15**] it should not be forgotten.

We end this section with a global result. Let us fix $T \in \mathrm{Sym}_3(\mathbb{Z}_{(p)})_{>0}$ and $\omega \in V(\mathbb{A}_f^p)^3$. For $i = 0, 1$ let us consider the following subset $I(T, \omega)$ of

$$\Omega_T'(\mathbb{Q}) \times Y_i \times G'(\mathbb{A}_f^p)/K^p \quad . \tag{8.37}$$

Here G' is the usual inner form of G (comp. beginning of section 7) and Ω_T' is the hyperbola defined in (7.8). The subset consists of the triples $(\underline{x}, L, gK^p)$ such that

(i) $g^{-1}\underline{x}g \in \omega$
(ii) $L \in Y_i(\underline{j})$, where $\underline{j} = \underline{x}(p)$ is the p-component of $\underline{x}$.

The set of irreducible components of the supersingular locus $\mathcal{Z}(T, \omega) \cap \mathcal{M}^{ss}$ breaks up into two disjoint subsets, one for $i = 0$ and one for $i = 1$, which are interchanged by the Frobenius in $\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$ (comp. end of section 4). For each subset we have

$$\mathrm{Irred}(\mathcal{Z}(T,\omega)^{ss})_i = G'(\mathbb{Q}) \setminus I(T,\omega) \quad . \tag{8.38}$$

The group $G'(\mathbb{Q})/Z'(\mathbb{Q})$ acts simply transitively on $\Omega'_T(\mathbb{Q})$. Let us fix a base point $\mathbf{x}$ and let us put $Y_i(\mathbf{j}) = Y_i(\mathbf{x}(p))$. Let us also set

$$I(\mathbf{x},\omega) = \{g \in G'(\mathbb{A}_f^p)/K^p;\ g\mathbf{x}g^{-1} \in \omega\} \quad . \tag{8.39}$$

Then we may write

$$\mathrm{Irred}(\mathcal{Z}(T,\omega)^{ss})_i = Z'(\mathbb{Q}) \setminus [Y_i(\mathbf{j}) \times I(\mathbf{x},\omega)] \quad . \tag{8.40}$$

Let

$$Z'(\mathbb{Q})^0 = \{z \in Z'(\mathbb{Q});\ \mathrm{ord}_p(z) = 0\} \quad . \tag{8.41}$$

Then $Z'(\mathbb{Q})^0$ acts trivially on $Y_i(\mathbf{j})$. We have

$$Z'(\mathbb{Q}) = < p > \times Z'(\mathbb{Q})^0 \quad , \tag{8.42}$$

and p acts faithfully on $Y_i(\mathbf{j})$. We may identify the quotient space for this action with $\mathcal{T}(\mathbf{j})_0$, comp. Corollary 8.4. Hence we obtain

$$\mathrm{Irred}(\mathcal{Z}(T,\omega)^{ss})_i = \mathcal{T}(\mathbf{j})_0 \times (Z'(\mathbb{Q})^0 \setminus I(\mathbf{x},\omega)) \quad . \tag{8.43}$$

The cardinality of the second factor in (8.43) is given by an orbital integral, comp. Prop. 7.1,

$$|Z'(\mathbb{Q})^0 \setminus I(\mathbf{x},\omega)| = \mathrm{vol}(K^p)^{-1} \cdot O_T(\varphi_f^p) \quad . \tag{8.44}$$

Similarly, the cardinality of the first factor is given by a kind of twisted orbital integral. For this fix a base lattice $\Lambda_0 \subset U$ and introduce the lattice of V''_p,

$$V''(\mathbb{Z}_p) = \{\beta \in V''_p;\ \beta(\Lambda_0) \subset \Lambda_0\} \quad . \tag{8.45}$$

Let

$$\varphi''_p = \mathrm{char}\, V''(\mathbb{Z}_p)^3 \quad . \tag{8.46}$$

Then, setting $\boldsymbol{\beta} = F^{-1}\mathbf{j}$, we have

$$|\mathcal{T}(\mathbf{j})_0| = \mathrm{vol}(K_p)^{-1} \cdot \int\limits_{Z'(\mathbb{Q}_p)\setminus G'(\mathbb{Q}_p)} \varphi''_p(g^{-1}\boldsymbol{\beta} g)dg \quad . \tag{8.47}$$

Taking (8.44) and (8.47) together we obtain the following formula.

Proposition 8.17. *The number of irreducible components of the supersingular locus* $\mathcal{Z}(T,\omega)\cap\mathcal{M}^{ss}$ *is equal to*

$$2\cdot \mathrm{vol}(K)^{-1}\cdot TO_T(\varphi_p'')\cdot O_T(\varphi_f^p)\quad .$$

Here $TO_T(\varphi_p'')$ *denotes the integral appearing in (8.47).*

Something analogous can, of course, also be done to write the cardinality of the set of superspecial points of $\mathcal{Z}(T,\omega)$ as a product of an orbital integral and a kind of twisted orbital integral. We do not know whether these expressions have a global significance, i.e. are related to automorphic forms (Eisenstein series?) or some (relative?) trace formula.

§9. Components in the Siegel case.

The purpose of this section is to show that the method of the previous section may also be applied to the case considered in the companion paper [**12**], with similar results. The notations in this section will be different from those used in the rest of the paper. As mentioned in the introduction, this section benefitted from the corrections and suggestions of Ch. Kaiser. Also, it is based on his results in [**6**].

We begin by briefly recalling the situation considered in sections 4 and 5 of [**12**]. Recall from loc. cit. the polarized isocrystal $(\mathcal{L},<\ ,\ >)$. Then $\dim_{\mathcal{K}}\mathcal{L}=4$ and all slopes of F are equal to $1/2$. Let

$$U=(\mathcal{L})^{p^{-1}F^2}\quad . \tag{9.1}$$

Then U is a $\mathbb{Q}_{p^2}$-form of $\mathcal{L}$ and is equipped with the symplectic form $<\ ,\ >$. Furthermore, U is a B_p'-vector space of dimension 2. Here

$$B_p'=\mathbb{Q}_{p^2}+\mathbb{Q}_{p^2}\Pi,\quad \Pi^2=p,\quad \Pi a=a^\sigma\Pi \tag{9.2}$$

is the quaternion division algebra over $\mathbb{Q}_p$ which acts on U via $\Pi\mapsto F$. On U we have the B_p'-hermitian form (with respect to the main involution of B_p'),

$$(\ ,\ ):U\times U\longrightarrow B_p' \tag{9.3}$$

defined by

$$(x,y)=-<x,\Pi y>\delta+<x,y>\cdot\delta\Pi\quad ,$$

for our fixed element $\delta \in \mathbb{Z}_{p^2}^\times$ for which $\delta^\sigma = -\delta$. The twisted form of the symplectic group may be identified with

$$G'(\mathbb{Q}_p) = \{g \in GL_{B'_p}(U);\ (gx, gy) = \nu(g) \cdot (x, y),\ \nu(g) \in \mathbb{Q}_p^\times\} \quad .$$

Recall the set of *distinguished lattices* in $\mathcal{L}$,

$$\tilde{X} = \{\tilde{L} \subset \mathcal{L};\ \tilde{L} \supset F\tilde{L} \supset p\tilde{L},\ F\tilde{L} = c \cdot \tilde{L}^\perp \text{ for some } c \in \mathcal{K}^\times\} \quad . \tag{9.4}$$

A distinguished lattice $\tilde{L}$ is stable under $p^{-1}F^2$ and defines therefore a lattice $\tilde{\Lambda} = (\tilde{L})^{p^{-1}F^2}$ in U. This lattice is stable under the maximal order $\mathcal{O}'_p = \mathbb{Z}_{p^2} + \mathbb{Z}_{p^2}\Pi$ in B'_p. In this way we obtain a one-to-one correspondence between $\tilde{X}$ and the set of $\mathcal{O}'_p$-lattices $\tilde{\Lambda}$ in U such that

$$\tilde{\Lambda}^\perp = c \cdot \Pi \cdot \tilde{\Lambda} \quad , \quad \text{some } c \in \mathbb{Q}_{p^2}^\times \quad . \tag{9.5}$$

Here

$$\tilde{\Lambda}^\perp = \{x \in U;\ (x, \tilde{\Lambda}) \subset \mathcal{O}'_p\} = \{x \in U;\ < x, \tilde{\Lambda} > \subset \mathbb{Z}_p\} \quad . \tag{9.6}$$

Recall the set of special endomorphisms of $\mathcal{L}$,

$$V'_p = \{j \in \mathrm{End}(\mathcal{L}, F);\ j^* = j,\ \mathrm{tr}(j) = 0\} \quad . \tag{9.7}$$

It is equipped with the quadratic form $q(j) = j^2$. For $\tilde{L} \in \tilde{X}$ we consider

$$N_{\tilde{L}} = \{j \in V'_p;\ j(\tilde{L}) \subset \tilde{L}\} = \{j \in V'_p;\ j(\tilde{\Lambda}) \subset \tilde{\Lambda}\} \quad . \tag{9.8}$$

Here again $\tilde{\Lambda} = (\tilde{L})^{p^{-1}F^2}$. As in the previous section we fix $T \in \mathrm{Sym}_n(\mathbb{Z}_{(p)})$ with $\det(T) \neq 0$ and $\mathbf{j} \in (V_p)^n$ with $q(\mathbf{j}) = T$. Consider the set

$$\tilde{X}(\mathbf{j}) = \{\tilde{L} \in \tilde{X};\ \mathbf{j} \in N_{\tilde{L}}^n\} \quad . \tag{9.9}$$

Then, by theorems 5.12 and 5.13 of [**12**], there is a one-to-one correspondence between $\tilde{X}(\mathbf{j})$ and those projective lines $\mathbb{P}_{\tilde{L}}$ of special lattices whose projection to the supersingular locus $\mathcal{M}^{ss}$ lies entirely in the image of the special cycle $\mathcal{Z}(T, \omega)$, cf. [**12**], section 5. Therefore it suffices to determine $\tilde{X}(\mathbf{j})$ resp. for a single $j \in V'_p$ the corresponding set $X(j)$. Consider the building $\mathcal{B}$ of $G'_{ad}(\mathbb{Q}_p)$. It is a tree. Its vertices correspond to B'_p-homothety classes $[\Lambda]$ of $\mathcal{O}'_p$-lattices in U which are fixed under the involution induced by $\Lambda \mapsto \Lambda^\perp$, comp. [**6**], section 5. There are two types of vertices. One of them comes from a distinguished lattice, i.e. is represented by a lattice $\tilde{\Lambda}$ satisfying (9.5). We denote by $\mathcal{B}_0$ the set of vertices of this type. The other is represented by a lattice Λ with

$$\Lambda^\perp = c \cdot \Lambda \quad , \quad \text{some } c \in \mathbb{Q}_{p^2}^\times \quad , \tag{9.10}$$

i.e. $\Lambda = (L)^{p^{-1}F^2}$ comes from a *superspecial lattice* $L \subset \mathcal{L}$ in the sense of section 4 of [**12**]. An edge in $\mathcal{B}$ is represented by two lattices of opposite types with

$$\Pi\Lambda \subset \Lambda' \subset \Lambda \quad . \tag{9.11}$$

Any $j \in V_p'$, with $q(j) \neq 0$ induces an involution of $\mathcal{B}$. Since we have a theory of elementary divisors for $\mathcal{O}_p'$-lattices and since $\mathcal{B}$ is a tree the proof of [**13**], Lemma 2.4 (i.e. of Kottwitz and Tate) applies and gives the following result.

Lemma 9.1. *Let $j \in V_p'$ with $q(j) \neq 0$. For a $\mathcal{O}_p'$-lattice Λ in U representing a vertex $[\Lambda] \in \mathcal{B}$ we have*

$$\begin{aligned} j(\Lambda) \subset \Lambda &\Longleftrightarrow d([j(\Lambda)], [\Lambda]) \leq 2 \ \mathrm{ord}_p q(j) \\ &\Longleftrightarrow d([\Lambda], \mathcal{B}^j) \leq \mathrm{ord}_p q(j) \quad . \end{aligned}$$

We used here that $q(j)^2 = Nm^0(j)$ when j is considered as an element of $GL_{B_p'}(U) \cong GL_2(B_p^{'\mathrm{op}})$.

For $j \in V_p'$ we introduce

$$\mathcal{T}(j) = \{x \in \mathcal{B};\ d(x, \mathcal{B}^j) \leq \mathrm{ord}_p q(j)\} \tag{9.12}$$

and

$$\mathcal{T}(j)_0 = \mathcal{T}(j) \cap \mathcal{B}_0 \tag{9.13}$$

Denoting by

$$\mathrm{pr} : \tilde{X} \longrightarrow \mathcal{B}_0 \quad , \quad \tilde{L} \longmapsto [\tilde{\Lambda}]$$

the natural projection we therefore obtain the following result.

Corollary 9.2. *For $j \in V_p'$ with $q(j) \neq 0$ we have*

$$\tilde{X}(j) = \mathrm{pr}^{-1}(\mathcal{T}(j)_0) \simeq \mathbb{Z} \times \mathcal{T}(j)_0 \quad .$$

On the right hand side the first factor $\mathbb{Z}$ enumerates the connected components of $\tilde{X}(j)$.
Similarly, if $\mathbf{j} \in (V_p')^n$ with non-singular $T = q(\mathbf{j})$ we may change the basis of the $\mathbb{Z}_p$-span of the components such that the components j_i satisfy $q(j_i) \neq 0$. Then we put

$$\mathcal{T}(\mathbf{j}) = \mathcal{T}(j_1) \cap \ldots \cap \mathcal{T}(j_n) \quad \text{and} \quad \mathcal{T}(\mathbf{j})_0 = \mathcal{T}(\mathbf{j}) \cap \mathcal{B}_0 \tag{9.14}$$

By Lemma 9.1 this is independent of the choice of basis.

Corollary 9.3.

$$\tilde{X}(\mathbf{j}) = \mathrm{pr}^{-1}(\mathcal{T}(\mathbf{j})_0) = \mathbb{Z} \times \mathcal{T}(\mathbf{j})_0 \quad .$$

It therefore remains to obtain a better understanding of the sets $\mathcal{T}(\mathbf{j})$, especially of the fixed point sets in $\mathcal{B}$ of special endomorphisms.

We review the results of Kaiser [**6**]. To state these results we will call $j \in V_p'$ with $q(j) \neq 0$

split, if $q(j) \in \mathbb{Q}_p^{\times,2}$
unramified elliptic, if $q(j) \in \mathbb{Q}_p^{\times} - \mathbb{Q}_p^{\times,2}$ and $\mathrm{ord}_p q(j)$ even
ramified elliptic, if $q(j) \in \mathbb{Q}_p^{\times} - \mathbb{Q}_p^{\times,2}$ and $\mathrm{ord}_p q(j)$ odd.

We will also need the following notation. Let E be a finite extension of $\mathbb{Q}_p$. Then we will denote by $\mathcal{B}(PGL_2(E))^+$ the simplicial complex which is the first barycentric subdivision of the building of $PGL_2(E)$. Thus there are two kinds of vertices in $\mathcal{B}(PGL_2(E))^+$: those that come from $\mathcal{B}(PGL_2(E))$ which lie on $q+1$ edges, where q is the cardinality of the residue field, and those that come from midpoints which lie on 2 edges. The first type will be called *primary* and the second type *secondary.*

Lemma 9.4. (Kaiser): *Case by case we have for the fixed point set $\mathcal{B}^j$ of a special endomorphism j :*
if j is split, *then $\mathcal{B}^j$ consists of a single superspecial vertex, cf. (9.10).*
if j is unramified elliptic, *the $\mathcal{B}^j$ is isomorphic to $\mathcal{B}(PGL_2(\mathbb{Q}_{p^2}))^+$ by an isomorphism which takes primary vertices to distinguished vertices (cf. (9.5)) and secondary vertices to superspecial vertices (cf. (9.10)).*

if j is ramified elliptic, *then* $\mathcal{B}^j$ *is isomorphic to* $\mathcal{B}(PGL_2(\mathbb{Q}_p(j)))$, *where* $\mathbb{Q}_p(j)$ *is the ramified extension generated by* j .

Here the isomorphisms are equivariant for the action of $G'(\mathbb{Q}_p)_j \cong GL_2(\mathbb{Q}_p(j))$. The Lemma follows from [**6**], Lemmas 5.2.2 and 5.2.3. together with Lemma 5.3.2., at least in the elliptic cases. The split case is analogous. Let us give our version of Kaiser's isomorphisms.

Let j *be split.* Correcting j by an element of $\mathbb{Q}_p^\times$ (which does not change the fixed point set) we may assume $j^2 = 1$. The corresponding eigenspace decomposition of U has the form

$$U = U_1 \oplus U_{-1} \quad . \tag{9.15}$$

Here U_1 and U_{-1} are both B'_p -vector spaces of dimension 1 which are orthogonal to each other for the symplectic form $<\ ,\ >$, hence each of them is equipped with a symplectic form. A $\mathcal{O}'_p$ -lattice Λ in U is fixed by j iff

$$\Lambda = \Lambda_1 \oplus \Lambda_{-1} \quad , \quad \text{where } \Lambda_1 = \Lambda \cap U_1 \quad , \quad \Lambda_{-1} = \Lambda \cap U_{-1} \quad . \tag{9.16}$$

Since $\dim_{B'_p} U_1 = \dim_{B'_p} U_{-1} = 1$, we may enumerate the lattices on the right hand side of (9.16) by $\mathbb{Z}^2$ in such a way that passage to the dual lattice corresponds to multiplication by -1 ,

$$\Lambda \longmapsto \Lambda^\perp \Longleftrightarrow (n,m) \longmapsto (-n,-m) \quad . \tag{9.17}$$

The identity $\Lambda^\perp = \Pi^r \Lambda$ for $[\Lambda] \in \mathcal{B}^j$ therefore implies that

$$(-n,-m) = (n+r, m+r) \quad , \text{ i.e. } \ r = -2n = -2m \quad .$$

Hence the homothety class of Λ is uniquely determined and, since r is even, $[\Lambda]$ is superspecial.

Next let j *be unramified elliptic.* Correcting j by an element of $\mathbb{Q}_p^\times$ we may assume $j^2 = \Delta$. We obtain the corresponding eigenspace decomposition,

$$U = U_1 \oplus U_{-1} \quad , \tag{9.18}$$

where

$$U_1 = \{x \in U;\ j(x) = \delta x\} \ , \ U_{-1} = \{x \in U;\ j(x) = -\delta x\} \quad .$$

Each of these subspaces is a 2-dimensional vector space over the unramified quadratic extension $E = \mathbb{Q}_p(j)$, they are orthogonal to each other, and $\deg \Pi = 1$ for this $\mathbb{Z}/2$-grading of U. The isomorphism

$$\mathcal{B}(PGL_E(U_1))^+ \xrightarrow{\sim} \mathcal{B}^j \tag{9.19}$$

sends now

$$[pN \subseteq M \subseteq N] \;\text{ to }\; [N \oplus \Pi^{-1} M] \quad .$$

Here the brackets on the left indicate the class for the equivalence relation for which

$$\begin{aligned} pN \subseteq M \subseteq N \sim pN' \subseteq M' \subseteq N' \Leftrightarrow \exists\, \alpha \in E^\times : (M', N') &= (\alpha M, \alpha N) \\ \text{or} \quad (M', N') &= (\alpha p N, \alpha M) \quad . \end{aligned}$$

Finally, let j be ramified elliptic. After correcting j by an element in $\mathbb{Q}_p^\times$ we may assume that

$$j^2 = \begin{cases} p \\ \Delta p \end{cases} \tag{9.20}$$

Put

$$\tilde{j} = \begin{cases} \Pi^{-1} j \\ \varepsilon \cdot \delta^{-1} \Pi^{-1} j \end{cases} \quad . \tag{9.21}$$

Here $\varepsilon \in \mathbb{Q}_{p^2}$ is a fixed element with $Nm_{\mathbb{Q}_{p^2}/\mathbb{Q}_p}(\varepsilon) = -1$. Then $\tilde{j}$ induces a σ-linear automorphism of the $\mathbb{Q}_{p^2}$-vector space U with $\tilde{j}^2 = 1$. We put

$$U_1 = U^{\tilde{j}} \quad , \quad U = U_1 \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^2} \quad . \tag{9.22}$$

Then U_1 is a vector space of dimension 2 over $E = \mathbb{Q}_p(j)$ and the isomorphism

$$\mathcal{B}(PGL_E(U_1)) \xrightarrow{\sim} \mathcal{B}^j \tag{9.23}$$

sends

$$[M] \;\text{ to }\; [M \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^2}] \quad .$$

We next investigate the relative position of the fixed point sets of two special endomorphisms j, j' which are orthogonal to each other, i.e., $jj' = -j'j$. We distinguish the following mutually exclusive cases.

First case: *One of j, j' is split.* Then $\mathcal{B}^{j,j'}$ is a single superspecial vertex, cf. Lemma 9.4 above.

Second case: *One of j, j' is unramified elliptic.* Let us assume that j is unramified elliptic, and let us consider the corresponding eigenspace decomposition (9.18). With respect to this decomposition we have $\deg \Pi^{-1} j' = 0$ and the induced endomorphism

$$\Pi^{-1} j' : \ U_1 \longrightarrow U_1 \tag{9.24}$$

is τ-linear, where τ is the generator of $\mathrm{Gal}(E/\mathbb{Q}_p) = \mathrm{Gal}(\mathbb{Q}_p(j)/\mathbb{Q}_p)$. Furthermore we have a bijection between fixed point sets

$$(\mathcal{B}(PGL_E(U_1))^+)^{\Pi^{-1} j'} \xrightarrow{\sim} \mathcal{B}^{j,j'} \quad . \tag{9.25}$$

Indeed, if $pN \subseteq M \subseteq N$ is a representative of a vertex of $\mathcal{B}(PGL_E(U_1))^+$, then the corresponding vertex $[N \oplus \Pi^{-1} M]$ of $\mathcal{B}$ is fixed under j' iff

$$\begin{aligned}[][\Pi^{-1} j'(M) \oplus j'(N)] =& [N \oplus \Pi^{-1} M] \Leftrightarrow \\ & [p \cdot \Pi^{-1} j'(M) \subseteq p \cdot \Pi^{-1} j'(N) \subseteq \Pi^{-1} j'(M)] = [pN \subseteq M \subseteq N] \ .\end{aligned}$$

We also note that

$$(\Pi^{-1} j')^2 = p^{-1} q(j') \quad . \tag{9.26}$$

Hence we see that if j' is unramified elliptic, then $\Pi^{-1} j'$ fixes a unique secondary vertex of $\mathcal{B}(PGL_E(U_1))^+$, hence $\mathcal{B}^{j,j'}$ consists of a single superspecial vertex of $\mathcal{B}$. If j' is ramified elliptic, then after correcting j' by an element in $\mathbb{Q}_p^\times$ we may assume that j'^2 is equal to p or to Δp, comp. (9.20). As in (9.21) we put

$$\tilde{j}' = \begin{cases} \Pi^{-1} j' & \text{if } j'^2 = p \\ \varepsilon \cdot \delta^{-1} \Pi^{-1} j' & \text{if } j'^2 = \Delta p \quad . \end{cases} \tag{9.27}$$

Then $\tilde{j}'$ is a τ-linear automorphism of U_1 with $\tilde{j}'^2 = 1$. We may thus write

$$U_1 = U_{1,1} \otimes_{\mathbb{Q}_p} E \quad , \quad U_{1,1} = U_1^{\tilde{j}'} \quad . \tag{9.28}$$

Since the extension E of $\mathbb{Q}_p$ is unramified we have bijections (unramified descent)

$$\mathcal{B}(PGL_{\mathbb{Q}_p}(U_{1,1}))^+ \xrightarrow{\sim} (\mathcal{B}(PGL_E(U_1))^+)^{\tilde{j}} \xrightarrow{\sim} \mathcal{B}^{j,j'} \quad . \tag{9.29}$$

Hence in this case the common fixed point set of j, j' is isomorphic to $\mathcal{B}(PGL_2(\mathbb{Q}_p))^+$. Under this isomorphism primary vertices correspond to distinguished vertices and secondary vertices to superspecial vertices.

Third case: *Both j, j' are ramified elliptic.* After correcting j and j' by elements of $\mathbb{Q}_p^\times$, we may assume that $q(j), q(j') \in \{p, \Delta p\}$. Let us define $\tilde{j}$ as in (9.21) and put $U_1 = U^{\tilde{j}}$. Then U_1 is a 2-dimensional vector space over $E = \mathbb{Q}_p(j)$. Define now

$$\tilde{j}' = \begin{cases} \delta^{-1}\Pi^{-1}j' & \text{if } q(j) = p \\ \varepsilon \cdot \Pi^{-1}j' & \text{if } q(j) = \Delta p \end{cases} . \tag{9.30}$$

Then

$$\tilde{j} \cdot \tilde{j}' = \tilde{j}' \cdot \tilde{j} \tag{9.31}$$

Hence $\tilde{j}'$ induces an automorphism of U_1 which is τ-linear ($<\tau> = \mathrm{Gal}(E/\mathbb{Q}_p)$) and such that

$$\tilde{j}'^2 = \begin{cases} -\Delta^{-1} & \text{if } q(j) = q(j') = p \\ -1 & \text{if } q(j) = p,\ q(j') = \Delta p \\ -1 & \text{if } q(j) = \Delta p,\ q(j') = p \\ -\Delta & \text{if } q(j) = \Delta p,\ q(j') = \Delta p \end{cases} \tag{9.32}$$

For fixed j, precisely one possible j' has the property that $\tilde{j}'^2 \equiv 1 \bmod \mathbb{Q}_p^{\times,2}$ and for the other possibility for j' we have $\tilde{j}'^2 \notin Nm_{E/\mathbb{Q}_p}(E^\times)$. For $\tilde{j}'^2 \in \mathbb{Q}_p^{\times,2}$, we can correct $\tilde{j}'$ by an element in $\mathbb{Q}_p^\times$, so as to obtain a τ-linear automorphism $\tilde{j}''$ of U_1 with $(\tilde{j}'')^2 = 1$. Hence

$$U_1 = U_{1,1} \otimes_{\mathbb{Q}_p} E \quad , \quad U_{1,1} = U_1^{\tilde{j}''} . \tag{9.33}$$

In this case we obtain bijections

$$\mathcal{B}(PGL_{\mathbb{Q}_p}(U_{1,1}))^+ \xrightarrow{\sim} \mathcal{B}(PGL_E(U_1))^{\tilde{j}''} \xrightarrow{\sim} \mathcal{B}^{j,j'} \tag{9.34}$$

(note that we are dealing here with a *ramified* descent from E to $\mathbb{Q}_p$, which explains the barycentric subdivision). Under this identification primary vertices correspond to superspecial vertices and secondary vertices correspond to distinguished vertices. Indeed, $\mathcal{B}^{j,j'}$ is contained in the fixed point locus of $E_1^\times$ where $E_1 = \mathbb{Q}_p(jj')$ is the unramified quadratic field extension of $\mathbb{Q}_p$ generated by jj', hence the assertion follows from [**6**], Lemma 5.3.2. a). Since $\chi((jj')^2) = -\chi(\tilde{j}'^2) = -1$, the element jj' indeed generates a field extension.
Now assume that $\tilde{j}'^2 \notin \mathbb{Q}_p^{\times,2}$, i.e. that $(jj')^2 = c^2 \in \mathbb{Q}_p^{\times,2}$. Put $j_0 = c^{-1}(jj')$. We consider the eigenspace decomposition corresponding to j_0,

$$U = U_1 \oplus U_{-1} \tag{9.35}$$

This time, in contrast to (9.15), since $j_0^* = -j_0$, both U_1 and U_{-1} are isotropic and $(\ , \ )$ induces a perfect pairing between these one-dimensional B'_p-vector spaces. A $\mathcal{O}_p$-lattice Λ in U is fixed by j_0 iff

(9.36) $$\Lambda = \Lambda_1 \oplus \Lambda_{-1} \quad , \quad \text{with } \Lambda_1 = U_1 \cap \Lambda \text{ and } \Lambda_{-1} = U_{-1} \cap \Lambda \quad .$$

Let $M \subset U_1$ be a $\mathcal{O}_{B'_p}$-lattice and let $M^\perp \subset U_{-1}$ be the dual lattice. Then

(9.37) $$\begin{aligned} \mathcal{B}^{j_0} &= \{\Pi^{\mathbb{Z}} \cdot M\} \times \{\Pi^{\mathbb{Z}} \cdot M^\perp\}/B_p^\times \\ &= (\mathbb{Z} \oplus \mathbb{Z})/\mathbb{Z} \quad , \end{aligned}$$

with the diagonal action of $\mathbb{Z}$. The action of j on $\mathcal{B}^{j_0}$ is induced by

(9.38) $$j : (m, n) \mapsto (n + i, m - i + 2) \quad ,$$

since $\mathrm{ord}_p(j^2) = 1$. The integer i is independent of the choice of M. It follows that

(9.39) $$\mathcal{B}^{j,j'} = \mathcal{B}^{j_0,j} = \{(i-1, 0)\} \in (\mathbb{Z} \oplus \mathbb{Z})/\mathbb{Z} \quad .$$

Furthermore, this vertex is superspecial if $i - 1$ is even and distinguished if $i - 1$ is odd. Consider now the hermitian form on U_1,

(9.40) $$\{x, y\} = (x, j(y)) \quad .$$

Now any hermitian form on a one-dimensional B'_p-vector space admits a selfdual $O_{B'_p}$-lattice. If $M' = \Pi^k M$ is selfdual, then

$$O_{B'_p} = \{M', M'\} = (M', j(M')) = (\Pi^k M, \Pi^{k-i+2} \cdot M^\perp) = \Pi^{2k-i+2} \cdot O_{B'_p} \quad .$$

It follows that i is even and hence $\mathcal{B}^{j,j'}$ consists in this case of a single distinguished vertex.

Let us consider now a 4-tuple of special endomorphisms $j_1, \ldots, j_4$ with $q(j_i) \in \mathbb{Q}_p^\times$ and which are pairwise orthogonal to each other. The following Lemma is the analogue of Lemma 8.12.

Lemma 9.5. *$\mathcal{B}^{\mathbf{j}} := \mathcal{B}^{j_1} \cap \ldots \cap \mathcal{B}^{j_4}$ is a vertex. More precisely, $\mathcal{B}^{\mathbf{j}}$ is a superspecial vertex if at least one j_i is split or at least two j_i are unramified elliptic and $\mathcal{B}^{\mathbf{j}}$ is a distinguished vertex if at least three j_i are ramified elliptic.*

Proof. The cases where at least one j_i is split, or at least two j_i are unramified elliptic have already been taken care of. In these cases the common fixed point is a superspecial vertex.

Let us now assume that j_1, j_2, j_3 are all ramified elliptic. We proceed as in the third case above, starting with $j = j_1$ and $j' = j_2$. Therefore we introduce $\tilde{j} = \tilde{j}_1$ as in (9.21) and $U_1 = U^{\tilde{j}}$. We also introduce $\tilde{j}' = \tilde{j}_2$ as in (9.30). If $\tilde{j}_2^2 \notin \mathbb{Q}_p^{\times,2}$, then the common fixed point set of j_1, j_2 is a distinguished vertex which is then equal to $\mathcal{B}^{\mathbf{j}}$. If $\tilde{j}_2^2 \in \mathbb{Q}_p^{\times,2}$ we rescale $\tilde{j}_2$ to obtain $\tilde{j}_2''$ with $(\tilde{j}_2'')^2 = 1$, and write $U_1 = U_{1,1} \otimes_{\mathbb{Q}_p} E$, for $U_{1,1} = U_1^{\tilde{j}_2''}$ and $E = \mathbb{Q}_p(j_1)$. Finally we introduce $\tilde{j}_3$ as in (9.30). Then $\tilde{j}_3$ commutes with $\tilde{j}_1$ and anticommutes with $\tilde{j}_2''$. Now j_1 acts on U_1; we set $U_{1,-1} = j_1(U_{1,1})$. Then we obtain a $\mathbb{Z}/2$-grading of U_1,

$$U_1 = U_{1,1} \oplus U_{1,-1} \quad . \tag{9.41}$$

Then $U_{1,-1}$ ist the (-1)-eigenspace of the $\mathbb{Q}_p$-linear endomorphism $\tilde{j}_2''$ of U_1. Since $\tilde{j}_3$ anticommutes with $\tilde{j}_2''$, it has degree one with respect to this grading. The primary vertices of $\mathcal{B}(PGL_{\mathbb{Q}_p}(U_{1,1})) = \mathcal{B}^{j_1,j_2}$ are represented by lattices $M \subset U_{1,1}$ and such a lattice is represented by the lattice

$$M \oplus j_1(M) = M \otimes_{\mathbb{Z}_p} O_E \tag{9.42}$$

in U_1. Now

$$\tilde{j}_3(M \oplus j_1(M)) = \tilde{j}_3 j_1(M) \oplus \tilde{j}_3(M) = j_1\tilde{j}_3(M) \oplus j_1(j_1\tilde{j}_3(M)) \quad .$$

Hence since $j_1\tilde{j}_3$ acts on $U_{1,1}$ with $\mathrm{ord}_p((j_1\tilde{j}_3)^2) = 1$, we see that $\mathcal{B}^{j_1,j_2,j_3} = (\mathcal{B}(PGL_{\mathbb{Q}_p}(U_{1,1}))^+)^{j_1,\tilde{j}_3}$ consists of a single secondary vertex. By the third case above, this corresponds to a distinguished vertex of $\mathcal{B}$. $\square$

Remark 9.6. We thus see that in most cases it is sufficient to intersect the fixed point loci of 3 special endomorphisms to get down to a vertex. The only case when 4 are needed is if one of them is unramified elliptic and the others are ramified elliptic. Let us discuss the case when j_1 is unramified elliptic and j_2, j_3, j_4 are ramified elliptic. We introduce the δ-eigenspace U_1 for j_1 which is a 2-dimensional vector space over $E = \mathbb{Q}_p(j_1)$. The special endomorphism j_2 induces the τ-linear endomorphism of U_1

$$\tilde{j}_2 := \begin{cases} \Pi^{-1} j_2 & \text{if } q(j_2) = p \\ \varepsilon\delta^{-1}\Pi^{-1} j_2 & \text{if } q(j_2) = \Delta p \end{cases} \quad . \tag{9.43}$$

Of course, here we again scaled j_2 by an element in $\mathbb{Q}_p^\times$. As in (9.28) we have $U_1 = U_{1,1} \otimes_{\mathbb{Q}_p} E$ with $U_{1,1} = U_1^{\tilde{j}_2}$. As in (9.30) above we put for $i = 3, 4$

$$\tilde{j}_i = \begin{cases} \delta^{-1}\Pi^{-1} j_i & \text{if } q(j_2) = p \\ \varepsilon \cdot \Pi^{-1} j_i & \text{if } q(j_2) = \Delta p \end{cases} \quad . \tag{9.44}$$

Then $\tilde{j}_3$ and $\tilde{j}_4$ are endomorphisms of the 2-dimensional $\mathbb{Q}_p$-vector space $U_{1,1}$ which anticommute and whose squares are scalar matrices of even valuation. Furthermore, we have

$$\tilde{j}_i^2 = \begin{cases} -\Delta^{-1}p^{-1}q(j_i) \\ \\ -p^{-1}q(j_i) \end{cases} \equiv -\Delta \cdot q(j_2) \cdot q(j_i) \bmod \mathbb{Q}_p^{\times,2} \quad . \tag{9.45}$$

Hence $\tilde{j}_i$ is unramified elliptic if $\chi(-q(j_2)q(j_i)) = 1$ and is split if $\chi(-q(j_2)q(j_i)) = -1$. Two possibilities occur: either one of $\tilde{j}_3$, $\tilde{j}_4$ is unramified elliptic in which case this involution of $\mathcal{B}(PGL_2(\mathbb{Q}_p)) = \mathcal{B}(PGL_{\mathbb{Q}_p}(U_{1,1}))$ has a primary vertex as its fixed point set, or both $\tilde{j}_3$, $\tilde{j}_4$ are split and have apartments as their fixed point sets which intersect in a primary vertex. In either case the common fixed point set is a primary vertex which by the second case preceding Lemma 9.5. is mapped to a distinguished vertex of $\mathcal{B}$.

Remark 9.7: Note that very subtle relations hold between T and $\mathcal{T}(\mathbf{j})_0$. For instance we see that if $\varepsilon_1 = 1$ and $a_1 = 0$, then $\mathcal{T}(\mathbf{j})_0$ is empty as it is contained in the ball of radius 0 around a superspecial vertex. Of course, we already knew this from [**12**] via Corollary 9.3 above, since if T represents 1, then $\tilde{X}(\mathbf{j}) = \emptyset$ (only isolated points in the special cycle $\mathcal{Z}(T,\omega)$). A more surprising conclusion is that the following combination is excluded:
$\chi(\varepsilon_1) = 1$, a_1 even
$\chi(\varepsilon_2) = -1$, a_2 even
a_3 and a_4 odd and $\chi(-\varepsilon_3\varepsilon_4) = 1$.
Indeed, the fixed point set of j_1 would be a superspecial vertex and the common fixed point set of j_2, j_3, j_4 would be a distinguished vertex, a contradiction to the fact that $\mathcal{B}^{\mathbf{j}}$ is non-empty. It seems, however, that all these cases can already be excluded on the basis of the following remark. If $T \in \mathrm{Sym}_4(\mathbb{Q}_p)$ with $\det(T) \neq 0$ is represented by V_p' then

$$-1 = \varepsilon(V_p') = \varepsilon(T) \cdot (\det(T), -\det(V_p'))_p = \varepsilon(T) \cdot (\det(T), -1)_p \quad ,$$

([**11**], (1.16)) where $\varepsilon(V_p')$ resp. $\varepsilon(T)$ denotes the Hasse invariant of V_p' resp. T. This remark is to be compared with Remark 2.10 in [**13**] (in this case, too, the exclusion principle by types of vertices is implied by the exclusion principle by the representability of T by the relevant quadratic form).

Proposition 9.8. *Let $T \in \mathrm{Sym}_4(\mathbb{Z}_{(p)})_{>0}$ be $GL_4(\mathbb{Z}_p)$ - equivalent to*

$$T \sim \mathrm{diag}(\varepsilon_1 p^{a_1}, \varepsilon_2 p^{a_2}, \varepsilon_3 p^{a_3}, \varepsilon_4 p^{a_4}) \quad , \quad 0 \leq a_1 \leq \ldots \leq a_4 \quad .$$

Then each connected component of the supersingular locus in the image of $\mathcal{Z}(T,\omega)$ *is irreducible of dimension one, or equivalently* $|\mathcal{T}(\mathbf{j})_0| = 1$ *for any* $\mathbf{j} \in V_p^4$ *with* $q(\mathbf{j}) = T$ *, if and only if the following conditions hold:*
a) Assume that a_1 *is even. Then*

(i) $a_1 = 0$ *and* $\chi(\varepsilon_1) = -1$
(ii) a_2, a_3, a_4 *are all odd*
(iii) *if* $\chi(-\varepsilon_2\varepsilon_3) = 1$ *then* $a_3 = 1$ *and if* $\chi(-\varepsilon_2\varepsilon_3) = -1$ *then* $a_4 = 1$ *.*

b) Assume that a_1 *is odd. Then*

(i) $a_1 = 1$
(ii) *if* $\chi(-\varepsilon_1\varepsilon_2) = 1$ *then* $a_2 = 1$ *and if* $\chi(-\varepsilon_1, \varepsilon_2) = -1$ *then* $a_3 = 1$ *.*

Proof. a) Let y be the common fixed vertex of $j_1, \dots, j_4$. If $a_1 \geq 2$, then the ball with radius 2 around y is contained in $\mathcal{T}(\mathbf{j})$. Since this ball contains more than one distinguished vertex we see that if $|\mathcal{T}(\mathbf{j})_0| = 1$, we must have $a_1 \leq 1$. If $a_1 = 0$ and $\chi(\varepsilon_1) = 1$, then $\mathcal{T}(\mathbf{j}) = \emptyset$. Hence if $|\mathcal{T}(\mathbf{j})_0| = 1$, then condition (i) holds. Also (ii) holds. Indeed, otherwise by our previous results y would be a superspecial vertex which corresponds to the midpoint of an edge in $\mathcal{B}^{j_1} \simeq \mathcal{B}(PGL_2(\mathbb{Q}_{p^2}))^+$. The two primary vertices x_1, x_2 of this edge have distance 1 to y. Let $\mathcal{T}(\mathbf{j})_0 = \{x\}$. Since $a_1 = 0$, we have $x \in \mathcal{B}^{j_1}$ and $d(x_1, \mathcal{B}^{j_i}) = d(x_2, \mathcal{B}^{j_i}) \leq d(x, \mathcal{B}^{j_i})$ for any $i \geq 2$ and hence $\{x_1, x_2\} \subset \mathcal{T}(\mathbf{j})_0$, a contradiction. Hence (ii) holds and y is a distinguished vertex, and $\mathcal{B}^{j_1,j_2} \simeq \mathcal{B}(PGL_2(\mathbb{Q}_p))^+$. Furthermore, the fixed point set of j_3 in $\mathcal{B}^{j_1,j_2}$ is either $\{y\}$ or an apartment in $\mathcal{B}(PGL_2(\mathbb{Q}_p))^+$ depending on whether $\chi(-\varepsilon_2\varepsilon_3) = 1$ or $\chi(-\varepsilon_2\varepsilon_3) = -1$, cf. third case before Lemma 9.5 (comp. Remark 9.6 above). In the first case $a_3 \leq 1$ because otherwise all distinguished vertices of distance 2 from y in $\mathcal{B}(PGL_2(\mathbb{Q}_p))^+$ would be contained in $\mathcal{T}(\mathbf{j})_0$. In the second case, the fixed point set of j_4 in $\mathcal{B}^{j_1,j_2}$ is an apartment in $\mathcal{B}(PGL_2(\mathbb{Q}_p))^+$ which intersects the previous apartment in y and hence $a_4 \leq 1$. We have therefore shown that $|\mathcal{T}(\mathbf{j})_0| = 1$ implies all conditions (i) – (iii).
Conversely, if conditions (i) – (iii) hold, then $\mathcal{B}^{\mathbf{j}}$ is a distinguished vertex y and $y \in \mathcal{T}(\mathbf{j})_0$. Let $x \in \mathcal{T}(\mathbf{j})_0$. Since $a_1 = 0$, we have $x \in \mathcal{B}^{j_1}$. For any $i \geq 2$, looking back at the second case before Lemma 9.5, the closest point to x in $\mathcal{B}^{j_i}$ cannot be a midpoint of an edge in $\mathcal{B}^{j_1} \cong \mathcal{B}(PGL_2(\mathbb{Q}_{p^2}))^+$, hence either $x \in \mathcal{B}^{j_1,j_i}$ or $d(x, \mathcal{B}^{j_i}) \geq 2$. The first possibility cannot hold for all i if $x \neq y$. But for i where the second possibility holds, this implies $a_i \geq 2$. The condition (iii) implies that we are in the first case of (iii) and $i = 4$. But then $x \in \mathcal{B}^{j_1,j_2,j_3} = \{y\}$.

b) Let again y be the common fixed vertex of $j_1, \dots, j_4$. As before the case

$a_1 \geq 3$ is excluded since otherwise the ball with radius 3 would be contained in $\mathcal{T}(\mathbf{j})$ and hence $\mathcal{T}(\mathbf{j})_0$ would have more than one element. For the same reason y has to be distinguished and $\mathcal{T}(\mathbf{j})_0 = \{y\}$. Consider $\mathcal{B}^{j_1} \simeq \mathcal{B}(PGL_2(\mathbb{Q}_p(j_1))$. Considering the number of neighbours of a vertex contained in $\mathcal{B}^{j_1}$ in $\mathcal{B}$ resp. $\mathcal{B}^{j_1}$ we see that all neighbours of a superspecial vertex in $\mathcal{B}^{j_1}$ have to lie in $\mathcal{B}^{j_1}$. Hence $\mathcal{T}(j_1)_0 = \mathcal{B}_0^{j_1}$ (set of distinguished vertices in $\mathcal{B}^{j_1}$). Since $\mathcal{T}(\mathbf{j})_0 = \{y\}$, it follows that $a_2 = 1$, since otherwise the distinguished vertices in the ball with radius 2 around y would be contained in $\mathcal{T}(\mathbf{j})_0$. It follows now by our previous results that if $\chi(-\varepsilon_1\varepsilon_2) = 1$ then $\mathcal{B}^{j_1,j_2} = \{y\}$ and if $\chi(-\varepsilon_1\varepsilon_2) = 1$ then $\mathcal{B}^{j_1,j_2} \cong \mathcal{B}(PGL_2(\mathbb{Q}_p))^+$ where primary vertices correspond to superspecial vertices in $\mathcal{B}$ and secondary vertices correspond to distinguished vertices in $\mathcal{B}$. In the first case $\mathcal{T}(j_1)_0 \cap \mathcal{T}(j_2)_0 = \mathcal{B}_0^{j_1,j_2} = \{y\}$ and we are in the case of the first clause of (ii). In the second case it follows that $a_3 = 1$ because otherwise the secondary vertices in the ball of radius 2 around y in $\mathcal{B}(PGL_2(\mathbb{Q}_p))^+$ would all be contained in $\mathcal{T}(\mathbf{j})_0$. In this case $\mathcal{T}(j_1)_0 \cap \mathcal{T}(j_2)_0 \cap \mathcal{T}(j_3)_0 = \mathcal{B}_0^{j_1,j_2,j_3} = \{y\}$. This proves the necessity of conditions (i) and (ii), and the sufficiency is proved in a similar way. $\square$

Remark: The cases enumerated in Proposition 9.8 should be considered first when investigating cases of excess intersection of special cycles.

We finally remark that, as in Theorem 8.15, there is a close relation between $|\mathcal{T}(\mathbf{j})_0|$ and the representation density of T by V_p'. Also the number of irreducible components of the image of $\mathcal{Z}(T,\omega)$ in $\mathcal{M}^{ss}$ may be expressed as a product of two orbital integrals, one prime to p, the other p-adic. As in Proposition 8.16, the global significance of such an expression is unclear.

§10. Special cycles on the special fibre.

So far most of our detailed results concerned the intersection of the supersingular locus in the special fibre with (the image of) a special cycle $\mathcal{Z}(T,\omega)$. In this section we will take a more global view and also consider the intersection of $\mathcal{Z}(T,\omega)$ with the ordinary locus.

We will use the following notations. By $\mathcal{M}^{\mathrm{ord}}$ resp. $\mathcal{Z}(T,\omega)^{\mathrm{ord}}$ we denote the open complement of the supersingular locus. Also let

(10.1) $$\overline{\mathcal{M}} = \mathcal{M} \times_{\mathrm{Spec}\,\mathbb{Z}_{(p)}} \mathrm{Spec}\,\mathbb{F}_p \quad , \quad \overline{\mathcal{Z}}(T,\omega) = \mathcal{Z}(T,\omega) \times_{\mathrm{Spec}\,\mathbb{Z}_{(p)}} \mathrm{Spec}\,\mathbb{F}_p$$

and $\overline{\mathcal{M}}^{\mathrm{ord}}$, $\overline{\mathcal{Z}}(T,\omega)^{\mathrm{ord}}$ denote the special fibres.

Let $(A, \lambda, \iota, \overline{\eta}^p) \in \mathcal{M}^{\mathrm{ord}}(\mathbb{F})$. The action of $\mathcal{O}_C \otimes \mathbb{Z}_{p^2} = M_4(\mathbb{Z}_{p^2})$ on the p-divisible group $A(p)$ gives a decomposition $A_0(p) = \mathcal{A}^4$, where $\mathcal{A}$ is a p-divisible group of dimension 2 and height 4, comp. section 4. Let M be the Dieudonné module of $\mathcal{A}$. It comes equipped with the action of $\mathbb{Z}_{p^2}$,

$$\iota_0 : \mathbb{Z}_{p^2} \to \mathrm{End}(M) \quad . \tag{10.2}$$

We consider the decomposition of M into its étale and its infinitesimal component,

$$M = M^0 \oplus M^1 \quad . \tag{10.3}$$

Each summand is $\mathbb{Z}_{p^2}$ -stable and in fact

$$M^0 \cong (\mathbb{Z}_{p^2} \otimes W, \mathrm{id} \otimes \sigma_W) \quad , \quad M^1 \cong (\mathbb{Z}_{p^2} \otimes W, p \cdot \mathrm{id} \otimes \sigma_W) \quad . \tag{10.4}$$

The alternating form $< \, , \, >$ induced by the polarization identifies M^1 with the Serre-dual of M^0 .

Now let j be a special endomorphism of $(A, \lambda, \iota, \overline{\eta}^p)$. It induces an endomorphism of M of the form

$$j = j^0 \oplus j^1 \quad , \tag{10.5}$$

where j^1 is the dual to j^0 and where

$$j^0 : M^0 \to M^0 \tag{10.6}$$

is linear for the W -action but antilinear for the $\mathbb{Z}_{p^2}$ -action. Denoting as usual by σ the non-trivial automorphism of $\mathbb{Z}_{p^2}$, and using the identification (10.4) we may write

$$j^0 = a\sigma \quad , \quad a \in \mathbb{Z}_{p^2} \quad , \tag{10.7}$$

and then

$$Q(j) = Nm(a) \quad . \tag{10.8}$$

We may therefore state the following fact.

Lemma 10.1. *Let $T \in \mathrm{Sym}_n(\mathbb{Z}_{(p)})_{>0}$ with $\overline{\mathcal{Z}}(T,\omega)^{\mathrm{ord}} \neq \emptyset$. Then $n \leq 2$ and T is represented by the norm form on $\mathbb{Z}_{p^2}$. In particular, the p-adic ordinal of each diagonal entry of T is even.* $\square$

We next use the theory of Serre-Tate canonical coordinates to investigate the infinitesimal deformation of $\mathcal{Z}(T,\omega)^{\mathrm{ord}}$ at $(A,\lambda,\iota,\overline{\eta}^p)$. The p-divisible group $\mathcal{A}$ is an extension

$$0 \to L \otimes \hat{\mathbb{G}}_m \to \mathcal{A} \to L' \otimes \mathbb{Q}_p/\mathbb{Z}_p \to 0 \tag{10.9}$$

where L and L' are free $\mathbb{Z}_{p^2}$-modules of rank one. The polarization allows us to identify L' with the $\mathbb{Z}_p$-dual of L. By Serre-Tate theory we may identify the formal deformation space of $\mathcal{A}$ with

$$\mathrm{Hom}(L', L) \otimes \hat{\mathbb{G}}_m \quad . \tag{10.10}$$

Inside this formal torus of relative dimension 4 the locus where the action ι_0 and the polarization lifts is given by the formal subtorus of relative dimension 2 (we identify L' with the dual of L),

$$\mathcal{T} = \mathrm{Sym}^2_{\mathbb{Z}_{p^2}}(L) \otimes \hat{\mathbb{G}}_m \quad . \tag{10.11}$$

Fix a generator e of the $\mathbb{Z}_{p^2}$-module L. Then $e \otimes e$ is a generator of the $\mathbb{Z}_{p^2}$-module $\mathrm{Sym}^2_{\mathbb{Z}_{p^2}}(L)$ and to any multiple $B(e\otimes e) \in \mathrm{Sym}^2_{\mathbb{Z}_{p^2}}(L) \otimes \mathbb{G}_m$ corresponds the symmetric bilinear form on L with values in $\hat{\mathbb{G}}_m$,

$$\varphi_B(x,y) = \mathrm{tr}_{\mathbb{Z}_{p^2}/\mathbb{Z}_p}(xBy) \quad . \tag{10.12}$$

The condition that an endomorphism j of the special fibre $\mathcal{A}$ extends to the deformation corresponding to $\varphi : L' \to L \otimes \mathbb{G}_m$ (cf. (10.10)) is

$$j \circ \varphi = \varphi \circ j \quad , \tag{10.13}$$

where j denotes the induced maps on L resp. L'. For a deformation lying in $\mathcal{T}$ and corresponding to $B(e \otimes e)$ this translates into

$$\varphi_B(j(z_1), z_2) = \varphi_B(z_1, j^*(z_2)), \quad z_1, z_2 \in L \quad . \tag{10.14}$$

Let us now consider a special endomorphism j of $(A,\lambda,\iota,\overline{\eta}^p)$. Then j induces an endomorphism of L which sends

$$xe \longmapsto A \cdot x^\sigma e \quad , \tag{10.15}$$

for some well-defined scalar $A \in \mathbb{Z}_{p^2}$. Then since $j^* = j$, the locus in $\mathcal{T}$ where j deforms is given by the condition

$$\varphi_B(Ax^\sigma, y) = \varphi_B(x, Ay^\sigma) \quad , \quad \text{i.e.}$$

$$(AB)^\sigma = AB \quad , \quad \text{i.e.} \quad AB \in \hat{\mathbb{G}}_m \quad . \tag{10.16}$$

Let us write

$$B = a + b\delta \quad , \quad A = \alpha + \beta\delta \quad \text{with } a, b \in \hat{\mathbb{G}}_m \quad , \quad \alpha, \beta \in \mathbb{Z}_p \quad .$$

Then the identity (10.16) is equivalent to

$$a\beta + b\alpha = 0 \quad . \tag{10.17}$$

Let us choose multiplicative coordinates on the formal torus $\mathcal{T}$ dual to the entries a and b of B. Then, by (10.17) the locus inside $\mathcal{T}$ to which j deforms is given by the equation

$$q_b^\alpha = q_a^{-\beta} \quad , \tag{10.18}$$

which we wish to consider locally around the base point $q_a = q_b = 1$.

Proposition 10.2. *Let* $t \in \mathbb{Z}_{(p)}$ *with* $\operatorname{ord}_p(t) = 0$. *Then* $\mathcal{Z}(t, \omega)^{\text{ord}}$ *is a smooth relative curve over* $\operatorname{Spec} \mathbb{Z}_{(p)}$.

Proof. It suffices to check this locally around a point $(A, \lambda, \iota, \overline{\eta}^p) \in \mathcal{Z}(t, \omega)^{\text{ord}}(\mathbb{F})$. Let us introduce local coordinates T_a and T_b around the origin of $\mathcal{T}$,

$$q_a = 1 + T_a \quad , \quad q_b = 1 + T_b \quad . \tag{10.19}$$

Then the leading term in the deformation equation (10.18) of j with $Q(j) = t$ is

$$\alpha T_b + \beta T_a = 0 \quad . \tag{10.20}$$

On the other hand,

$$Q(j) = \operatorname{Nm} A = \alpha^2 - \Delta\beta^2 \quad . \tag{10.21}$$

Hence the hypothesis implies that (10.20) defines a non-trivial linear equation in the tangent space of $\mathcal{T}$ and the claim follows. $\square$

Corollary 10.3. *Let* $t \in \mathbb{Z}_{(p)}$ *with* $\mathrm{ord}_p t = 2s \geq 0$ *. Then* $\mathcal{Z}(t,\omega)^{\mathrm{ord}}$ *is a relative divisor over* $\mathrm{Spec}\,\mathbb{Z}_{(p)}$ *of the form* $p^{2s} \cdot \mathcal{Z}(t_0,\omega)^{\mathrm{ord}}$ *where* $\mathcal{Z}(t_0,\omega)^{\mathrm{ord}}$ *is a smooth relative divisor over* $\mathrm{Spec}\,\mathbb{Z}_{(p)}$ *.*

Proof. Let $(\alpha,\beta) \in \mathbb{Z}_p^2$ be the vector corresponding to the special endomorphism j with $Q(j) = t$. By assumption $t = p^{2s} t_0$ with $\mathrm{ord}_p t_0 = 0$ and hence

$$(\alpha,\beta) = p^s \cdot (\alpha_0,\beta_0) \quad , \quad (\alpha_0,\beta_0) \in \mathbb{Z}_p^2 \quad . \tag{10.22}$$

By the previous Proposition, the locus inside $\mathcal{T}$ corresponding to (α_0,β_0) is a formal subtorus of relative dimension 1 of $\mathcal{T}$. The locus inside $\mathcal{T}$ corresponding to (α,β) is given by (10.18), i.e. is the inverse image of the locus corresponding to (α_0,β_0) under the isogeny of degree p^{2s} given by multiplication by p^s,

$$p^s : \mathcal{T} \longrightarrow \mathcal{T} \quad . \tag{10.23}$$

The result follows. $\square$

We next consider the intersection of special divisors. By Corollary 3.8 we know that for $n \geq 3$ $\overline{\mathcal{Z}}(T,\omega)^{\mathrm{ord}} = \emptyset$ and that for $n = 2$ the cycle $\overline{\mathcal{Z}}(T,\omega)^{\mathrm{ord}}$ consists of finitely many points which are all in one and the same isogeny class of type I_1. Let $T \in \mathrm{Sym}_2(\mathbb{Z}_{(p)})_{>0}$ be $GL_2(\mathbb{Z}_p)$-equivalent to

$$T' = \mathrm{diag}\,(\varepsilon_1 p^{2s_1}, \varepsilon_2 p^{2s_2}) \quad . \tag{10.24}$$

Let j_1, j_2 be special endomorphisms of $(A,\lambda,\iota,\overline{\eta}^p) \in \mathcal{M}^{\mathrm{ord}}(\mathbb{F})$ with $Q(j_1,j_2) = T'$. As before, we associate to j_1 and j_2 scalars in $\mathbb{Z}_{p^2}$,

$$A_1 = \alpha_1 + \beta_1\delta \quad , \quad A_2 = \alpha_2 + \beta_2\delta \quad . \tag{10.25}$$

The condition $j_1 j_2 = -j_2 j_1$ translates into

$$A_1 A_2^\sigma = -A_2 A_1^\sigma \Leftrightarrow \alpha_1\alpha_2 - \Delta\beta_1\beta_2 = 0 \quad . \tag{10.26}$$

Let us first assume that $s_1 = s_2 = 0$. Then the local equations for $\mathcal{Z}(T,\omega)$ in $\mathcal{T}$ in terms of the multiplicative coordinates q_a, q_b resp. the local coordinates T_a, T_b at the base point are given by

$$\alpha_1 T_a + \beta_1 T_b = 0 \quad , \quad \alpha_2 T_a + \beta_2 T_b = 0 \quad . \tag{10.27}$$

The determinant of this system of linear equations is equal to $\alpha_1\beta_2 - \alpha_2\beta_1$. This is a unit because otherwise $A_1A_2^\sigma$ would be divisible by p which is excluded by our hypothesis. It follows that the determinant is a unit, hence the loci inside $\mathcal{T}$ where j_1 resp. j_2 deform intersect transversally.

Combined with Corollary 10.3 this gives the following statement. Here we content ourselves with a statement concerning the special fibre.

Proposition 10.4. *Let* $T \in \mathrm{Sym}_2(\mathbb{Z}_{(p)})_{>0}$. *Then* $\overline{\mathcal{Z}}(T,\omega)^{\mathrm{ord}}$ *is an artinian scheme of length* $p^{\mathrm{ord}_p(\det T)}$ *at each of its points.* $\square$

Corollary 10.5. *Assume that* $\xi \in \overline{\mathcal{Z}}(t_1,\omega_1) \times_{\overline{\mathcal{M}}} \overline{\mathcal{Z}}(t_2,\omega_2)(\mathbb{F})$ *is an isolated ordinary point. Then the intersection multiplicity of the divisors* $\overline{\mathcal{Z}}(t_1,\omega_1)^{\mathrm{ord}}$ *and* $\overline{\mathcal{Z}}(t_2,\omega_2)^{\mathrm{ord}}$ *at* ξ *is equal to*

$$e(\xi) = p^{\mathrm{ord}_p(\det T_\xi)} \quad ,$$

where $T_\xi \in \mathrm{Sym}_2(\mathbb{Z}_{(p)})_{>0}$ *is the fundamental matrix of* ξ , *comp. end of section 6.* $\square$

Remark. Let $t \in \mathbb{Z}_{(p)}$ with $\mathrm{ord}_p t$ even and > 0 . Combining Theorem 6.1 with Corollary 10.3 we see that, outside of the supersingular locus, the corresponding cycle $\overline{\mathcal{Z}}(t,\omega)$ has a p-power multiplicity and furthermore $\overline{\mathcal{Z}}(t,\omega)$ contains some projective lines inside the supersingular locus. If $\mathrm{ord}_p t = 0$, then, outside the supersingular locus, $\overline{\mathcal{Z}}(t,\omega)$ is a smooth divisor and $\overline{\mathcal{Z}}(t,\omega)$ cuts the supersingular locus in finitely many superspecial points.

We conclude this section with one remark of a global nature.

Proposition 10.6. *Let* $T \in \mathrm{Sym}_n(\mathbb{Z}_{(p)})_{>0}$. *The special cycle* $\mathcal{Z}(T,\omega)$ *is proper over* $\mathrm{Spec}\,\mathbb{Z}_{(p)}$, *unless*

a) $B = M_2(\boldsymbol{k})$
b) $n = 1$
c) $T = t \in \mathrm{Nm}\ \boldsymbol{k}^\times$.

Proof. If B is a division algebra, then $\mathcal{M}$ is proper over $\mathrm{Spec}\,\mathbb{Z}_{(p)}$ and hence so is $\mathcal{Z}(T,\omega)$ which maps by a finite morphism to $\mathcal{M}$. Hence we may assume a) and hence

$$C(V) \otimes \boldsymbol{k} \simeq M_4(\boldsymbol{k}) \quad . \tag{10.28}$$

Therefore, if $(A, \lambda, \iota, \overline{\eta}^p)$ is a point of $\mathcal{M}$ the abelian variety A is up to isogeny a product A_0^4 where A_0 is an abelian variety of dimension 2 equipped with an action

$$i_0 : \boldsymbol{k} \hookrightarrow \mathrm{End}^0(A_0) \quad . \tag{10.29}$$

Furthermore the polarization λ defines a $\boldsymbol{k}$-linear polarization λ_0 on A_0. A special endomorphism j of $(A, \lambda, \iota, \overline{\eta}^p)$ induces an endomorphism $j_0 : A_0 \to A_0$ up to isogeny with

$$j_0^* = j_0 \quad \text{and} \quad j_0 \circ \iota_0(a) = \iota_0(a^\sigma) \circ j_0 \ , \quad a \in \boldsymbol{k} \quad . \tag{10.30}$$

We now wish to check the valuative criterion of properness. Thus we assume that the point $(A, \lambda, \iota, \overline{\eta}^p)$ of $\mathcal{M}$ is over the generic point of a complete discrete valuation ring with algebraically closed residue field. We also assume that A and hence A_0 has semi-stable *bad*, i.e., multiplicative reduction. We wish to show that b) and c) hold. We may write the associated rigid-analytic abelian variety as a quotient [**17**]

$$A_0^{an} = L \otimes \mathbb{G}_m^{an} / i(L') \quad . \tag{10.31}$$

The polarization λ_0 identifies $L' \otimes \mathbb{Q}$ with the dual of $L \otimes \mathbb{Q}$ and the embedding $i : L' \to L \otimes \mathbb{G}_m^{an}$ defines a pairing

$$L' \times L^* \longrightarrow \mathbb{G}_m^{an} \xrightarrow{\mathrm{val}} \mathbb{Z}$$

which, combined with the previous identification, defines a *symmetric* bilinear form

$$(\ ,\ ) \ : \ L \otimes \mathbb{Q} \times L \otimes \mathbb{Q} \longrightarrow \mathbb{Q} \tag{10.32}$$

which is *positive-definite* and satisfies

$$(\iota_0(a)x, y) = (x, \iota_0(a)y) \quad , \quad a \in \boldsymbol{k} \quad .$$

After choosing a generator of $L \otimes \mathbb{Q}$ the form (10.32) is therefore of the form

$$(x, y) = \mathrm{tr}(bxy) \tag{10.33}$$

for a totally positive element b of k .

A special endomorphism j defines an endomorphism of L which sends x to $a \cdot x^\sigma$ for a fixed scalar $a \in \boldsymbol{k}$ and which satisfies

$$(ab)^\sigma = ab \quad , \tag{10.34}$$

comp. (10.14). But equation (10.34) determines a up to a scalar in $\mathbb{Q}$, hence there cannot be a non-trivial special endomorphism of $(A, \lambda, \iota, \overline{\eta}^p)$ anticommuting with j . Hence b) is proved and c) follows from the identity

$$Q(j) = \mathrm{Nm}(a) \quad . \qquad \square$$

The results given in this section can be regarded as first steps towards preparing the stage for a theory of Hilbert-Blumenthal surfaces in characteristic p, analogous to the classical theory of Hirzebruch and Zagier, e.g. [**5**], [**2**]. Namely, one might ask whether the intersection numbers of the special cycles $\overline{\mathcal{Z}}(t, \omega)$ (for $t \in \mathbb{Z}_{(p)>0}$) are related to the Fourier coefficients of some sort of modular form. For this a better understanding of the singularities of $\overline{\mathcal{Z}}(t, \omega)$ at supersingular points seems to be necessary. Similarly, one might investigate whether the special curves $\overline{\mathcal{Z}}(t, \omega)$ can be used to determine the position of $\overline{\mathcal{M}}$ (suitably compactified if $B \simeq M_2(\boldsymbol{k})$) in the classification of algebraic surfaces. For this it would seem necessary to better understand the irreducible components of special curves. One contrast to the situation in characteristic zero is that on $\overline{\mathcal{M}}$ it is easy to find effective ample divisors (to be on the safe side, assume B to be a division algebra). Indeed the supersingular locus $\overline{\mathcal{M}}^{ss}$ splits into a sum of two divisors (sum of the *even* resp. *odd* projective lines $\mathbb{P}^1_L$) which are exchanged by the Frobenius in $\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$ and each of which is ample. Indeed they span a 2-dimensional submodule of the ℓ-adic cohomology of $\overline{\mathcal{M}} \times_{\mathrm{Spec}\,\mathbb{F}_p} \mathrm{Spec}\,\mathbb{F}$ which is stable under the actions of the Galois group and of the Hecke algebra prime to p; by [**4**] this submodule must therefore lie in the subspace cut out by a one-dimensional automorphic representation of $G(\mathbb{A})$ and all effective divisor classes in such a subspace are ample.

§11. The case of a split prime.

In this section we briefly sketch the case when p is split in $\boldsymbol{k}$, and we assume, as in section 2, that B is split at each prime $\wp$ of $\boldsymbol{k}$ dividing p. Note that the case

$\boldsymbol{k} = \mathbb{Q} \oplus \mathbb{Q}$ is allowed. For p split in $\boldsymbol{k}$, $\mathcal{O}_C \otimes \mathbb{Z}_p \simeq M_4(\mathbb{Z}_p)$ as before, but now $\mathcal{O}_{\boldsymbol{k}} \otimes \mathbb{Z}_p \simeq \mathbb{Z}_p \oplus \mathbb{Z}_p$.

Fix a point $\xi = (A, \lambda, \iota, \overline{\eta}^p) \in \mathcal{M}(\mathbb{F})$. Let $A(p)$ be the p-divisible group of A. The action of $\mathcal{O}_C \otimes \mathbb{Z}_p \simeq M_4(\mathbb{Z}_p)$ on $A(p)$ allows us to write as before $A(p) = \mathcal{A}^4$. On the p-divisible group $\mathcal{A}$ there is an action of $\mathcal{O}_{\boldsymbol{k}} \otimes \mathbb{Z}_p \simeq \mathbb{Z}_p \oplus \mathbb{Z}_p$ which allows us to write further

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \quad , \tag{11.1}$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are p-divisible groups of dimension 1 and height 2. The principal quasi-polarization of $\mathcal{A}$ induced by λ comes from a pair of principal quasi-polarizations λ_1 and λ_2 of $\mathcal{A}_1$ and $\mathcal{A}_2$. By the Serre-Tate theorem, deforming ξ is equivalent to deforming $(\mathcal{A}_1, \lambda_1)$ and $(\mathcal{A}_2, \lambda_2)$. Since λ_i deforms automatically with $\mathcal{A}_i$, the deformation space of $(\mathcal{A}_i, \lambda_i)$ is the same as that of $\mathcal{A}_i$, i.e. of a suitable point on the moduli space of elliptic curves. In particular, we deduce the following result.

Proposition 11.1. *The supersingular locus $\mathcal{M}^{ss}$ consists of a finite number of isolated points.* $\square$

We next give an estimate on the space V_ξ of special endomorphisms of $\xi = (A, \lambda, \iota, \overline{\eta}^p)$. We have an inclusion

$$V_\xi \subset \{ j : \mathcal{A} \to \mathcal{A},\ \deg j = 1,\ j^* = j \} \quad . \tag{11.2}$$

Here $\deg j$ refers to the $\mathbb{Z}/2$-grading of $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$. Therefore j corresponds to a pair of homomorphisms of p-divisible groups

$$j_1 : \mathcal{A}_1 \to \mathcal{A}_2 \quad , \quad j_2 : \mathcal{A}_2 \to \mathcal{A}_1 \quad . \tag{11.3}$$

Using the polarizations λ_1 and λ_2, the condition $j^* = j$ is equivalent to

$$j_2 = j_1^* \quad , \quad j_1 = j_2^* \quad . \tag{11.4}$$

The quadratic form is given by the degree of j_1 (or j_2),

$$Q(j) \cdot \mathrm{id}_{\mathcal{A}_1} = j_1^* j_1 \quad . \tag{11.5}$$

Decomposing $\mathcal{A}_1$ and $\mathcal{A}_2$ up to isogeny into simple p-divisible groups we now obtain the following statement.

Proposition 11.2. *We have case by case for the rank of the* $\mathbb{Z}_{(p)}$ *-module* V_ξ *of special endomorphisms of* $\xi = (A, \lambda, \iota, \overline{\eta}^p)$.

(i) *If* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are both supersingular, then* $\operatorname{rk} V_\xi = 4$.
(ii) *If* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are both ordinary, then* $\operatorname{rk} V_\xi \leq 2$.
(iii) *If one of* $\mathcal{A}_1, \mathcal{A}_2$ *is supersingular and the other ordinary, then* $V_\xi = (0)$. $\square$

We now fix $T \in \mathrm{Sym}_3(\mathbb{Z}_{(p)})_{>0}$ and $\omega \subset V(\mathbb{A}_f^p)^3$ and consider the special cycle $\mathcal{Z}(T, \omega)$. By the previous propositions we know that $\mathcal{Z}(T, \omega)$ is a finite set of points lying over the supersingular set $\mathcal{M}^{ss}$. Again invoking the Serre-Tate theorem we have an identification for $\xi = (A, \lambda, \iota, \overline{\eta}^p, \mathbf{j}) \in \mathcal{Z}(T, \omega)$

$$\hat{\mathcal{Z}}(T, \omega)_\xi = \mathrm{Def}(\mathcal{A}_1, \mathcal{A}_2; \mathbf{j}_1) = \mathrm{Def}(\mathcal{A}_1, \mathcal{A}_2; M_1) \quad . \tag{11.6}$$

Here $\mathbf{j}_1$ is the triple consisting of the “first component” $j_1 : \mathcal{A}_1 \to \mathcal{A}_2$ of the three members j of $\mathbf{j}$ and M_1 is the $\mathbb{Z}_p$ -module spanned by them. The length of the Artinian scheme on the RHS of (11.6) was determined by Gross and Keating [**3**] and we therefore obtain the following analogue of Proposition 6.2.

Proposition 11.3. *The length of the local ring* $\mathcal{O}_{\mathcal{Z}(T,\omega),\xi}$ *only depends on the* $GL_3(\mathbb{Z}_p)$ *-equivalence class of* T *and is equal to* $e_p(T)$ *, given in Proposition 6.2 above.* $\square$

As in section 7 we introduce in the setting of (6.19)-(6.23)

$$\langle \mathcal{Z}(T_1, \omega_1), \ldots, \mathcal{Z}(T_r, \omega_r) \rangle_p^{\mathrm{proper}} = \sum_\xi e(\xi) \quad . \tag{11.7}$$

Here ξ runs over the isolated points of $\mathcal{Z}(T_1, \omega_1) \times_{\mathcal{M}} \ldots \times_{\mathcal{M}} \mathcal{Z}(T_r, \omega_r)$ which by Propositions 11.1 and 11.2 are the points ξ where the fundamental matrix $T_\xi \in \mathrm{Sym}_3(\mathbb{Q})$ is non-singular. In the special case when $r = 1$, the cycle $\mathcal{Z}(T, \omega)$ lies over the supersingular locus and is finite (without any further condition on the divisibility by p of T). In this case we put $\langle \mathcal{Z}(T, \omega) \rangle_p = \langle \mathcal{Z}(T, \omega) \rangle_p^{\mathrm{proper}}$, i.e.

$$\langle \mathcal{Z}(T, \omega) \rangle_p = \sum_{\xi \in \mathcal{Z}(T,\omega)} e(\xi) \quad . \tag{11.8}$$

Appealing to Proposition 11.3 we have that (11.8) equals

$$\langle \mathcal{Z}(T, \omega) \rangle_p = e_p(T) \cdot |\mathcal{Z}(T, \omega)(\mathbb{F})| \quad . \tag{11.9}$$

As in the inert case we introduce the twisted quadratic space V' over $\mathbb{Q}$ which is positive-definite, is isomorphic to V at all finite places $\neq p$ and at p has the same determinant as $V(\mathbb{Q}_p)$ but opposite Hasse invariant. The quaternion algebra $B' = C^+(V')$ over $\boldsymbol{k}$ is ramified at the two infinite places, is isomorphic to B at all finite places not over p and is ramified at the two places $\wp_1$ and $\wp_2$ over p. Let G' be the corresponding inner form of G. Then we have a bijection

$$\mathcal{M}^{ss}(\mathbb{F}) = G'(\mathbb{Q}) \setminus \left(G'(\mathbb{Q}_p)/K'_p \times G(\mathbb{A}_f^p)/K^p \right) \quad . \tag{11.10}$$

Here K'_p is the unique maximal compact subgroup of $G'(\mathbb{Q}_p) = B'^{,\times}_{\wp_1} \times B'^{,\times}_{\wp_2}$.

After identifying $V'(\mathbb{Q}_p)$ with the ramified quaternion algebra $\mathbb{B}_p$ over $\mathbb{Q}_p$ equipped with its norm form, we have a natural lattice $V'(\mathbb{Z}_p)$ in $V'(\mathbb{Q}_p)$, namely the ring of integers in $\mathbb{B}_p$. The usual procedure now yields the following expression for the cardinality of $\mathcal{Z}(T,\omega)(\mathbb{F})$, comp. Proposition 7.1.

Proposition 11.4. *Let* $K' = K'_p \cdot K^p \subset G'(\mathbb{A}_f)$. *Let*

$$\varphi_f^p = \mathrm{char}(\omega) \quad , \quad \varphi'_p = \mathrm{char}(V'(\mathbb{Z}_p)^3)$$

and

$$\varphi'_f = \varphi'_p \cdot \varphi_f^p \in S(V'(\mathbb{A}_f)^3)^{K'} \quad .$$

Then

$$|\mathcal{Z}(T,\omega)(\mathbb{F})| = \mathrm{vol}(K')^{-1} \cdot \mathrm{vol}(Z'(\mathbb{Q}) \setminus Z'(\mathbb{A}_f)) \cdot O_T(\varphi'_f) \quad ,$$

where the orbital integral on the right is defined as in Proposition 7.1.

The relationship to the Eisenstein series is now established just as in section 7 above. We define Φ_p and Φ'_p as in (7.11) and complete Φ_p into an incoherent standard section as before. Then we obtain formula (7.13) for the present case. The next proposition gives the values and derivatives of the relevant Whittaker functions at $s=0$.

Proposition 11.5. *Let* $T \in \mathrm{Sym}_3(\mathbb{Q}_p)$ *with* $\det(T) \neq 0$.
(i) If $W_{T,p}(e,0,\Phi'_p) \neq 0$, *then* $T \in \mathrm{Sym}_3(\mathbb{Z}_p)$.
(ii) If $T \in \mathrm{Sym}_3(\mathbb{Z}_p)$ *is represented by* $V'(\mathbb{Q}_p)$, *then*

$$W_{T,p}(e,0,\Phi_p) = \gamma(V'_p) \cdot 2p^{-4}(p+1)^2.$$

(iii) If $T \in \mathrm{Sym}_3(\mathbb{Z}_p)$ *is represented by* $V'(\mathbb{Q}_p)$ *, then*

$$W'_{T,p}(e, 0, \Phi_p) = \gamma(V_p) \cdot \log(p) \cdot (1 - p^{-2})^2 \cdot e_p(T).$$

Proof. We again use the relation (7.15) between the Whittaker functional and the representation densities and obtain the analogue of (7.17)

$$W'_{T,p}(e, 0, \Phi_p) = -\log(p) \cdot \gamma(V_p) \frac{\partial}{\partial X} \left\{ A_{S,T}(X) \right\} \Big|_{X=1}.$$

Here S is the matrix for the quadratic form on $V(\mathbb{Q}_p)$, i.e.,

$$S = \mathrm{diag}(1, -1, 1, -1) = H_4,$$

a hyperbolic form of dimension 4. By the same argument which yields Corollary 10.6 of [**12**], we obtain

$$\frac{\partial}{\partial X} \left\{ A_{S,T}(X) \right\} \Big|_{X=1} = -(1 - p^{-2})^2 \cdot e_p(T)$$

where the factor $(1-p^{-4})$ in Corollary 10.6 of [**12**] has been changed to $(1-p^{-2})$ since the reduction formula (10.11) of [**12**] and hence the factor $(1+p^{-2})$ of (10.14) there does not arise in our present situation. This proves (iii).

Finally, (ii) follows from Proposition 6.10 of [**3**], cf. also Lemma 7.10 of [**12**], where we note that

$$S' = \mathrm{diag}(1, -\beta, -p, p\beta)$$

has determinant p^2 , and that

$$W_{T,p}(e, 0, \Phi'_p) = \gamma(V'_p) \cdot |\det(S')|^{3/2} \cdot \alpha_p(S', T).$$

□

Using these values in (7.13) we obtain the analogue of Theorem 7.3 in the case of a split prime.

Theorem 11.5. *Let* $T \in \mathrm{Sym}_3(\mathbb{Q})_{>0}$ *be represented by* $V(\mathbb{A}_f^p)$ *, but not by* $V(\mathbb{Q}_p)$ *. Also assume that* ω *is locally centrally symmetric.*
(i) If $T \notin \mathrm{Sym}_3(\mathbb{Z}_{(p)})$ *, then* $\mathcal{Z}(T, \omega)$ *is empty and* $E'_T(h, 0, \Phi) = 0$ *.*

(ii) For any such $T \in \mathrm{Sym}_3(\mathbb{Z}_{(p)})_{>0}$ *, the cycle* $\mathcal{Z}(T,\omega)$ *is either empty or zero dimensional and*

$$E'_T(h,0,\Phi) = -\frac{1}{2}\mathrm{vol}(SO(V')(\mathbb{R})\cdot \mathrm{pr}(K))\cdot W_T^2(h)\cdot \log(p)\cdot \langle \mathcal{Z}(T,\omega)\rangle_p .$$

Proof. Using Proposition 11.4 and (7.13), we have

$$\begin{aligned} E'_T(h,0,\Phi) &= \mathrm{vol}(SO(V')(\mathbb{R})\cdot \mathrm{pr}(K'))\cdot W_T^2(h) \\ &\qquad \times \Big(-\frac{1}{2}\log(p)\cdot (p-1)^2\cdot e_p(T)\cdot |\mathcal{Z}(T,\omega)(\mathbb{F})|\Big) \\ &= -\frac{1}{2}\mathrm{vol}(SO(V')(\mathbb{R})\cdot \mathrm{pr}(K))\cdot W_T^2(h)\cdot \langle \mathcal{Z}(T,\omega)\rangle_p , \end{aligned}$$

where we have used the fact that

$$\frac{\mathrm{vol}(K')}{\mathrm{vol}(K)} = (p-1)^2.$$

□

§12. A global model and generating series.

In this section we formulate a moduli problem which is represented by a scheme over $\mathrm{Spec}\,\mathbb{Z}\left[\frac{1}{N}\right]$ for some specified integer N and whose base change to $\mathrm{Spec}\,\mathbb{Z}_{(p)}$, for any $p\nmid N$, coincides with the moduli space introduced in section 2. We similarly extend the special cycles. We then use the results of sections 7 and 11 to give a partial identification of the generating series for the arithmetic degrees of these special cycles.

As in section 1, we start with the quadratic space (V,Q) over $\mathbb{Q}$ of signature $(2,2)$ and the element $\tau \in B = C^+(V)$ with $\tau = -\tau^\iota$ and $\tau^2 < 0$. On the left $C(V)$ -module $U = C(V)$, we have the nondegenerate alternating form

$$< x,y > = \mathrm{tr}^0(y^\iota \tau x). \tag{12.1}$$

We also have the open compact subgroup K of $G(\mathbb{A}_f)$ which, as always, is assumed to be sufficiently small, i.e., the image $\mathrm{pr}(K)\subset SO(V)(\mathbb{A}_f)$ is neat.

Fix a $\mathbb{Z}$-lattice Λ in $V(\mathbb{Q})$. Let $\mathcal{O}_C = C(\Lambda)$ be its Clifford algebra and let $U_\mathbb{Z} = \mathcal{O}_C$ be the corresponding lattice in $U(\mathbb{Q})$. There exists an integer N such that for any prime p with $p\nmid N$ all of the following conditions are satisfied.

(i) $p \neq 2$.

(ii) $\Lambda \otimes \mathbb{Z}_p$ is a self dual $\mathbb{Z}_p$-lattice in $V(\mathbb{Q}_p)$.

(iii) $\tau \in (\mathcal{O}_C \otimes \mathbb{Z}_p)^\times$ and the form $<\ ,\ >$ is perfect on the lattice $U(\mathbb{Z}_p) = U_\mathbb{Z} \otimes \mathbb{Z}_p$. In particular, $\mathcal{O}_C \otimes \mathbb{Z}_p$ is invariant under the involution $x \mapsto x^* = \tau x^\iota \tau^{-1}$.

(iv) $K = K_p K^p$ with $K^p \subset G(\mathbb{A}_f^p)$ and

$$K_p = \mathrm{GSpin}(V)(\mathbb{Q}_p) \cap (\mathcal{O}_C \otimes \mathbb{Z}_p)^\times.$$

We now fix such an integer N and formulate a moduli problem over $\mathrm{Spec}\,\mathbb{Z}\left[\frac{1}{N}\right]$. To do so, we introduce the category of abelian varieties *up to N-primary isogeny.* As in the case of the cateory of abelian varieties up to isogeny prime to p (as used in section 2), this category is obtained from that of abelian varieties by formally inverting all isogenies whose degrees involve only prime factors of N. The moduli problem associates to $S \in \mathrm{Sch}/\mathbb{Z}\left[\frac{1}{N}\right]$ the set of isomorphism classes of 4-tuples $(A, \lambda, \iota, \bar{\eta}_N)$ where

(i) A is an abelian scheme over S up to N-primary isogeny.

(ii) $\lambda : A \to \hat{A}$ is a $\mathbb{Z}\left[\frac{1}{N}\right]^\times$-class of principal polarizations on A.

(iii) $\iota : \mathcal{O}_C \otimes \mathcal{O}_{\boldsymbol{k}} \longrightarrow \mathrm{End}(A) \otimes \mathbb{Z}\left[\frac{1}{N}\right]$ is a homomorphism such that

$$\iota(c \otimes a)^* = \iota(c^* \otimes a),$$

for the Rosati involution of $\mathrm{End}^0(A)$ determined by λ and the involution $*$ of $C(V)$.

(iv) $\bar{\eta}_N$ is a K_N-equivalence class of $\mathcal{O}_C \otimes \mathcal{O}_{\boldsymbol{k}}$-linear isomorphisms

$$\eta_N : V_N(A) \xrightarrow{\sim} U(\mathbb{A}_N)$$

which preserves the symplectic forms up to a scalar in $\mathbb{A}_n^\times$.

Here we have used the notation $\mathbb{A}_N = \prod_{p|N} \mathbb{Q}_p$ and $V_N(A) = \prod_{p|N} V_p(A)$ and have written K as

$$K = K_N \cdot \prod_{p|N} K_p,$$

cf. (iv) in the conditions on N above. As before we impose the determinant condition (1.1).

Theorem 12.1. *The moduli problem defined above is representable by a quasi-projective scheme $\mathcal{M}$ over $\mathbb{Z}\left[\frac{1}{N}\right]$ whose base change to $\mathrm{Spec}\,\mathbb{Z}_{(p)}$ is the moduli* *scheme introduced in section 2 for any* $p \nmid N$ *. In particular, its generic fiber is the moduli scheme* M *introduced in section 1.*

We shall content ourselves with showing how to establish a bijection $\mathcal{M}(\operatorname{Spec} k) \simeq M(\operatorname{Spec} k)$ for an algebraically closed field k of characteristic 0. Let $(A, \lambda, \iota, \bar{\eta}) \in M(k)$ and fix $\eta \in \bar{\eta}$. We obtain a $\hat{\mathbb{Z}}$-lattice in $\hat{V}(A)$ defined by

$$T_\eta(A) := \eta^{-1}(U(\hat{\mathbb{Z}})), \tag{12.2}$$

where $U(\hat{\mathbb{Z}}) = U_{\mathbb{Z}} \otimes \hat{\mathbb{Z}}$. Then, by our choice of N, for any $p \nmid N$, the lattice $T_\eta(A) \otimes \mathbb{Z}_p$ in $V_p(A)$ is the unique $\mathcal{O}_C \otimes \mathcal{O}_{\boldsymbol{k}}$-stable lattice which is self-dual for the form $<\ ,\ >_\eta = \eta^*(<\ ,\ >)$ induced by η. In particular, for all $p \nmid N$, $T_\eta(A) \otimes \mathbb{Z}_p$ is independent of the choice of η. Let B be the abelian variety in the isogeny class of A with

$$\hat{T}(B) = T_\eta(A). \tag{12.3}$$

Then B is equipped with an action of $\mathcal{O}_C \otimes \mathcal{O}_{\boldsymbol{k}}$ and, by the above, it is unique up to N-primary isogeny. Furthermore, there exists a polarization $\lambda_0 \in \lambda$ and a trivialization of the roots of unity of k such that, for the pairing $<\ ,\ >_{\lambda_0}$ on $\hat{V}(A)$ thus determined,

$$<\ ,\ >_{\lambda_0} = <\ ,\ >_\eta\ . \tag{12.4}$$

Then λ_0 defines a polarization of B which is principal in the category of abelian varieties varieties up to N-primary isogeny. Finally, $\bar{\eta}_N$ is obtained by simply ignoring the components of $\bar{\eta}$ prime to N. Then $(B, \lambda_0, \iota, \bar{\eta}_N) \in \mathcal{M}(k)$ and this procedure defines the map $M(k) \to \mathcal{M}(k)$. The map in the opposite direction is constructed in a similar way.

We next turn to special cycles. The definition of special endomorphism, Definition 1.2, carries over in the obvious way to $(A, \lambda, \iota, \bar{\eta}_N) \in \mathcal{M}(S)$. If S is connected, the special endomorphisms form a quadratic space over $\mathbb{Z}\left[\frac{1}{N}\right]$. For the remainder of this section, we fix a compact open subset $\omega_N \subset V(\mathbb{A}_N)^3$, stable under the action of K_N. For any $T \in \operatorname{Sym}_3(\mathbb{Z}\left[\frac{1}{N}\right])_{>0}$, we define $\mathcal{Z}(T) = \mathcal{Z}(T, \omega_N)$ as the scheme representing the functor of 5-tuples $(A, \lambda, \iota, \bar{\eta}_N; \mathbf{j})$, where $\xi = (A, \lambda, \iota, \bar{\eta}_N) \in \mathcal{M}(S)$, and $\mathbf{j} = (j_1, j_2, j_3)$ is a triple of special endomorphisms with $Q(\mathbf{j}) = T$ and where $\eta \circ \mathbf{j} \circ \eta^{-1} \in \omega_N$ for one, and hence for all, $\eta \in \bar{\eta}_N$. We note that the generic fiber of $\mathcal{Z}(T,\omega_N)$ is the special cycle associated in section 1 to T and

$$\omega = \omega_N \times \prod_{p \nmid N} \omega_p, \tag{12.5}$$

where $\omega_p = (\Lambda \otimes \mathbb{Z}_p)^3$. Similarly, the base change of $\mathcal{Z}(T,\omega_N)$ to $\operatorname{Spec}\mathbb{Z}_{(p)}$ for $p \nmid N$ is the special cycle associated in section 2 to T and

$$\omega^p = \omega_N \times \prod_{\substack{\ell \nmid N \\ \ell \neq p}} \omega_\ell, \tag{12.6}$$

where $\omega_\ell = (\Lambda \otimes \mathbb{Z}_\ell)^3$.

We next introduce the Eisenstein series associated to ω_N. Let $\varphi_f \in S(V(\mathbb{A}_f)^3)$ be the characteristic function of the set ω defined in (12.5) and let Φ_f be the standard section of the induced representation with $\Phi_f(0) = \lambda_f(\varphi_f)$, where $\lambda_f : S(V(\mathbb{A}_f)^3) \to I_f(0,\chi)$. We again complete Φ_f into an incoherent section $\Phi = \Phi_\infty \cdot \Phi_f$ with $\Phi_\infty = \Phi^2_\infty$ as usual and we form the corresponding Eisenstein series $E(g,s,\Phi)$.

For $\tau = u + iv \in \mathfrak{H}_3$, the Siegel half space of genus 3, we let

$$h_\tau = \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & {}^t a^{-1} \end{pmatrix} \in Sp_6(\mathbb{R}) \tag{12.7}$$

where $a \in GL_3(\mathbb{R})$ with $\det(a) > 0$ and $v = a\,{}^t a$. Note that $h_\tau(i) = \tau \in \mathfrak{H}_3$. Then

$$\phi(\tau) := \det(v)^{-1} \cdot E'(h_\tau, 0, \Phi) \tag{12.8}$$

is a (non-holomorphic) Siegel modular form of weight 2 with respect to the arithmetic subgroup $\Gamma_H = Sp_6(\mathbb{Q}) \cap K_H$ where $K_H \subset Sp_6(\mathbb{A}_f)$ is an open compact subgroup fixing Φ_f in the induced representation.

It will be convenient to introduce the following definition.

Definition 12.2. An element $T \in \mathrm{Sym}_3(\mathbb{Z}\left[\frac{1}{N}\right])_{>0}$ is *regular* if (i) $\mathrm{Diff}(T) = \{p\}$, i.e., if T is represented by $V(\mathbb{Q}_\ell)$ for all primes $\ell \neq p$ but is not represented by $V(\mathbb{Q}_p)$, (ii) $p \nmid N$, and (iii) if p is inert in $\boldsymbol{k}$, then $p \nmid T$.

If T is regular with associated prime p, we can introduce the degree

$$\widehat{\deg}(\mathcal{Z}(T,\omega_N)) = \log(p)\cdot < \mathcal{Z}(T,\omega_N) \otimes \mathbb{Z}_{(p)} >_p, \tag{12.9}$$

as in sections 7 and 11.

The next result provides a partial analogue for the case of Hilbert-Blumenthal surfaces of the main results of [**11**], and [**14**].

Theorem 12.3. *The modular form* $\phi(\tau)$ *has a Fourier expansion of the form*

$$C^{-1}\cdot\phi(\tau) = \sum_{T \text{ regular}} \widehat{\deg}(\mathcal{Z}(T,\omega_N))\cdot q^T + \sum_{p<\infty} \sum_{\substack{T\in \mathrm{Sym}_3(\mathbb{Q})_{>0} \\ \mathrm{Diff}(T)=\{p\} \\ T \text{ not regular}}} a_T\cdot q^T$$

$$+ \sum_{\substack{T\in \mathrm{Sym}_3(\mathbb{Q}) \\ \mathrm{sig}(T)=(2,1)}} a_T(v)\cdot q^T + \sum_{\substack{T\in \mathrm{Sym}_3(\mathbb{Q}) \\ \det(T)=0}} a_T(v)\cdot q^T,$$

where

$$C = -\frac{1}{2}\cdot \mathrm{vol}(SO(V')(\mathbb{R})\mathrm{pr}(K)),$$

and where $q^T = e^{2\pi i \mathrm{tr}(T\tau)}$.

Proof. First recall that the only nonsingular $T\in \mathrm{Sym}_3(\mathbb{Q})$ which contribute to the Fourier expansion of $\phi(\tau)$ are those T for which $|\mathrm{Diff}(T)|=1$, [**11**], Theorem 6.1. Furthermore, if $\mathrm{Diff}(T)=\{p\}$ for a finite prime p, then the associated term in the Fourier expansion is holomorphic and hence has the form $a_T\cdot q^T$. This follows from (iii) of Theorem 6.1 of [**11**], since, in our present situation, the quadratic space $V^{(p)}$ is positive definite and hence the theta integral whose T-th Fourier coefficient contributes the τ-dependence to the T-th Fourier coefficient of ϕ is holomorphic. Suppose that, in addition, T is regular, so that $p\nmid N$ and, if p is inert in $\boldsymbol{k}$, then $p\nmid T$. Then Theorems 7.3 and 11.5 provide the claimed expression for the Fourier coefficient. Note that

$$q^T = \det(v)^{-1}\cdot W_T^2(h_\tau). \tag{12.10}$$

□

Remark 12.4. Obviously one would like to extend this result to include an arithmetic interpretation of all of the Fourier coefficients of $\phi(\tau)$. For example, consider the terms in the second sum for T's with $\mathrm{Diff}(T) = \{p\}$ with $p \nmid N$ but with p inert in $\boldsymbol{k}$ and $p \mid T$. For such T's one would like to identify a_T with some kind of arithmetic degree of the *one dimensional* cycle $\mathcal{Z}(T, \omega^p)$, where ω^p is as in (12.6), whose structure is described in section 8. For p dividing N, one has to deal with problems of bad reduction.

§13. On the hereditary nature of special cycles.

In this series of papers [**11**], [**12**], [**13**], [**14**] we have investigated special cycles on Shimura varieties associated to orthogonal groups of signature $(r, 2)$ for $r = 3, 2, 1, 0$. In this section we make some remarks on the relations among these cycles in the various cases.

Analytically speaking, this relation is fairly obvious. Quite generally, let (V, q) be a quadratic space of signature $(r, 2)$ over Q. Let $V' \subset V$ be a subspace of signature $(r - n, 2)$ of the form $V' =< x >^{\perp}$ for some $x \in V(\mathbb{Q})^n$. Denoting by G resp. G' the groups of spinorial similitudes and by $\mathcal{D}$ resp. $\mathcal{D}'$ the spaces of oriented negative 2-planes of $V(\mathbb{R})$ resp. $V'(\mathbb{R})$ we obtain injections

$$(13.1) \qquad G' \hookrightarrow G \quad , \quad \mathcal{D} \hookrightarrow \mathcal{D} \quad .$$

For $K' = K \cap G'(\mathbb{A}_f)$ we also obtain a (generically 1-1) morphism of Shimura varieties

$$(13.2) \qquad Sh(G', \mathcal{D})_{K'} \hookrightarrow Sh(G, \mathcal{D})_K \quad .$$

Under this map, a special cycle for $Sh(G', \mathcal{D}')_{K'}$ defined as in [**10**] is mapped into a certain disjoint sum of connected components of some special cycles on $Sh(G, \mathcal{D})_K$, comp. Proposition 1.4 above. In this sense the special cycles are hereditary for the various Shimura varieties, cf. also section 9 of [**10**].

For $r = 0, \ldots, 3$ we have a moduli-theoretic definition of the corresponding Shimura varieties. In each of these cases the special cycles were defined by imposing *special endomorphisms* on the abelian varieties with additional structure parametrized by the model in question. However, in each of the cases the notion of special endomorphisms depended on the moduli problem and the hereditary nature of these notions is hardly transparent.

Our aim in this section is more modest. We only wish to compare the infinitesimal deformation theory of a supersingular point of the moduli problem with values in $\mathbb{F}$ and of the special cycles passing through this point, for various r. By the theorem of Serre-Tate this leads case by case to the following deformation problems.

r=3 *(good reduction case):* $\mathrm{Def}(\mathcal{A}, \lambda)$, where $\mathcal{A}$ is a supersingular formal group of dimension 2 and height 4 and where λ is a principal quasi-polarization of $\mathcal{A}$. A *special endomorphism* is an element $j \in \mathrm{End}(\mathcal{A})$ such that $j^* = j$ and $\mathrm{tr}^0(j) = 0$.

r=2 *(good reduction case):* $\mathrm{Def}(\mathcal{A}, \lambda, \iota)$, where $\mathcal{A}$ is a supersingular formal group of dimension 2 and height 4 and where $\iota : \mathcal{O} \to \mathrm{End}(\mathcal{A})$ is an action of $\mathcal{O} = \mathbb{Z}_{p^2}$ or $\mathcal{O} = \mathbb{Z}_p \oplus \mathbb{Z}_p$ satisfying the *special condition* and where λ is a principal quasi-polarization which is ι-linear. A *special endomorphism* is an element $j \in \mathrm{End}(\mathcal{A})$ such that $j^* = j$ and $j \circ \iota(a) = \iota(\overline{a}) \circ j$, $a \in \mathcal{O}$. In this case, automatically, $\mathrm{tr}^0(j) = 0$.

r=1 *(good reduction case):* $\mathrm{Def}(\mathcal{A})$, where $\mathcal{A}$ is a supersingular formal group of dimension 1 and height 2. A *special endomorphism* is an element $j \in \mathrm{End}(\mathcal{A})$ such that $\mathrm{tr}^0(j) = 0$.

r=1 *(p-adic uniformization case):* $\mathrm{Def}(\mathcal{A}, \iota)$, where $(\mathcal{A}, \iota)$ is a s.f. O_{B_p}-module (here B_p is the quaternion division algebra over $\mathbb{Q}_p$). A *special endomorphism* is an element $j \in \mathrm{End}(\mathcal{A}, \iota)$ such that $\mathrm{tr}^0(j) = 0$.

r=0 *(good reduction case):* $\mathrm{Def}(\mathcal{A}, \iota)$ where $\mathcal{A}$ is a supersingular formal group of dimension 1 and height 2 and where $\iota : \mathcal{O} \to \mathrm{End}(\mathcal{A})$ satisfies the special condition. Here $\mathcal{O}$ is the ring of integers in a quadratic field extension of $\mathbb{Q}_p$. A *special endomorphism* is an element $j \in \mathrm{End}(\mathcal{A})$ such that $j \circ \iota(a) = \iota(\overline{a}) \circ j$, $a \in \mathcal{O}$.

To illustrate the hereditary nature of these definitions we now start with an object of the deformation problem for a given r and an n-tuple $\mathbf{j} = (j_1, \ldots, j_n)$ of special endomorphisms and construct under certain hypotheses an object of the deformation problem for $r-1$ and an $(n-1)$-tuple $\mathbf{j}' = (j'_1, \ldots, j'_n)$ of special endomorphisms of the deformation problem for $r-1$.

r=3. Suppose that we are given $(\mathcal{A}, \lambda)$ and a collection $\mathbf{j}$ of special endomorphisms with $q(j_1) = 1$, or Δ. Then j_1 defines a *special* action $\iota : \mathcal{O} \to \mathrm{End}(\mathcal{A})$, where $\mathcal{O} = \mathbb{Z}_p \oplus \mathbb{Z}_p$ or $\mathcal{O} = \mathbb{Z}_{p^2}$, and $\mathbf{j}' = (j_2, \dots, j_n)$ is an $(n-1)$-tuple of special endomorphisms of $(\mathcal{A}, \lambda, \iota)$.

r=2. Suppose that we are given $(\mathcal{A}, \lambda, \iota)$ where we assume $\mathcal{O} = \mathbb{Z}_{p^2}$. A collection $\mathbf{j}$ of special endomorphisms with $q(j_1) = 1$ (resp. $q(j_1) = \Delta$) determines idempotents via (6.7) (resp. (6.13)) and hence an isomorphism

$$(\mathcal{A}, \lambda) \simeq (\mathcal{A}_1, \lambda_1) \times (\mathcal{A}_2, \lambda_2).$$

We also have a natural isomorphism from $\mathcal{A}_1$ to $\mathcal{A}_2$, namely $\underline{\delta}$ (resp. ψ_1'') in the notation of (6.5) (resp. (6.11)). Composing this natural isomorphism with $j_2, \dots, j_n$ we obtain

$$j_2', \dots, j_n' : \mathcal{A}_1 \longrightarrow \mathcal{A}_1 \quad .$$

Since $j_i'^{*} = -j_i'$, it follows that $\mathrm{tr}^0(j_i') = 0$ and hence that $(j_2', \dots, j_n')$ is an $(n-1)$-tuple of special endomorphisms of $\mathcal{A}_1$.

r=1. Let $\mathcal{A}$. Let $\mathbf{j}$ with $q(j_1) = \Delta$ or p or Δp. Then j_1 defines a special action $\iota : \mathcal{O} \to \mathrm{End}(\mathcal{A})$, where $\mathcal{O} = \mathbb{Z}_p[j_1]$. Furthermore $(j_2, \dots, j_n)$ is an $(n-1)$-tuple of special endomorphisms of $(\mathcal{A}, \iota)$.

We remark that the hereditary nature of these deformation problems is a weak support of our conjecture on the singularities of the intersections of special cycles in the case of isolated intersections, comp. [**12**], Conjecture 6.3. In our present context of arithmetic Hirzebruch-Zagier cycles, the conjecture states that for an isolated intersection point ξ of the special cycles $\mathcal{Z}(T_1, \omega_1), \dots, \mathcal{Z}(T_r, \omega_r)$ we have

$$(\mathcal{O}_{\mathcal{Z}(T_1,\omega_1)} \otimes^{\mathbb{L}} \dots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(T_r,\omega_r)})_\xi = (\mathcal{O}_{\mathcal{Z}(T_1,\omega_1)} \otimes \dots \otimes \mathcal{O}_{\mathcal{Z}(T_r,\omega_r)})_\xi \quad . \tag{13.3}$$

Hence the entity $e(\xi)$ in Proposition 6.2 would be the intersection multiplicity in the sense of Serre.

We close this section with a curiosity. Let us start with an object $(\mathcal{A}, \lambda, \iota)$ of the deformation problem for $r = 2$ where $\mathcal{O} = \mathbb{Z}_{p^2}$. Let $\mathbf{j}$ with $j_1^2 = p$ (the case where $j_1^2 = \Delta p$ is analogous). Then $\mathbb{Z}_{p^2}[j_1] = O_{B_p'}$ and ι and j_1 define the structure $(\mathcal{A}, \iota')$ of a s.f. $O_{B_p'}$-module. However, for $i \geq 2$, the special endomorphism j_i *anticommutes* with the $O_{B_p'}$-action (in the sense of the main involution of $O_{B_p'}$). Hence j_i is *not* a special endomorphism in the sense of **r=1** *(p-adic uniformization case)* of the s.f. $O_{B_p'}$-module $(\mathcal{A}, \iota')$. We have no explanation for this seeming perplexity, except to point out that this remark seems to exclude a simple method of creating bad singularities in the special cycles in the good reduction cases considered above. Note that, in the case **r=1** *(p-adic uniformization case)*, the special cycles can have bad singularities, e.g. have embedded components; comp. [**13**].

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