On the Equi-Consistency of the Failure of the GAP-1 Transfer Property and an Inaccessible Cardinal

T h o r a l f   R ä s c h

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Dissertation — Summary

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Constructing models of a given consistent theory is often done in logic. In the easiest cases we can use the well-known Löwenheim-Skolem theorem:

Theorem (Löwenheim-Skolem). *Every consistent theory T in an infinite language $\mathcal{L}$ has a model of power at most $|\mathcal{L}|$. Moreover, if T has infinite models, then it has infinite models of any given power greater than $|\mathcal{L}|$.*

By this theorem we know that every theory will fail to distinguish between infinite cardinals. So we can head for the question what will happen when we ask for pairs of infinite cardinals.

To start with, consider the language $\mathcal{L} = \{\dot{A}, \dots\}$ where $\dot{A}$ is a unary predicate. Call an $\mathcal{L}$-model $\mathfrak{A} = \langle\, \mathbb{A}; A, \dots \,\rangle$ a (κ, λ)-*model*, if $|\mathbb{A}| = \kappa$ and $|A| = \lambda$.

Now define for infinite cardinals α, β, κ and λ the following notion of a general Transfer Property

$$(\alpha, \beta) \longrightarrow (\kappa, \lambda),$$

meaning that if a theory T has an (α, β)-model, then it has also a (κ, λ)-model.

In 1962, Morley and Vaught proved for infinite cardinals $\alpha < \beta$ that

$$(\beta, \alpha) \longrightarrow (\aleph_1, \aleph_0).$$

In fact, using homogenous models they showed that for a consistent and countable theory, having a (β, α)-model, they can construct an elementary chain of length $\aleph_1$ of models $\mathfrak{A}_\nu$ such that for arbitrary $\bar{\nu} < \nu < \aleph_1$ we always have $A^{\mathfrak{A}_{\bar{\nu}}} = A^{\mathfrak{A}_\nu}$ and $\mathfrak{A}_{\bar{\nu}}$ is an elementary submodel of $\mathfrak{A}_\nu$, being a proper subset. Moreover, $A^{\mathfrak{A}_\nu}$ and $\mathbb{A}^{\mathfrak{A}_\nu}$ are both countable for all $\nu < \aleph_1$. Then the union of this chain of models is obviously an $(\aleph_1, \aleph_0)$-model.

Using the Löwenheim-Skolem Theorem stated above we can find a generalized version of it, providing that for arbitrary infinite cardinals $\alpha < \beta$ we always have

$$(\beta, \alpha) \longrightarrow (\alpha^+, \alpha).$$

Furthermore, Chang has proved for all infinite cardinals $\alpha < \beta$ and regular δ such that $2^{<\delta} = \delta$ that the following holds

$$(\beta, \alpha) \longrightarrow (\delta^+, \delta).$$

And so a natural question arises given by the so-called *gap-one conjecture* or *gap-one two cardinal problem* asserting that every theory T of a countable language $\mathcal{L}$ which has an (α^+, α)-model, also has a (β^+, β)-model for infinite cardinals α, β.

Chang's result stated above shows the gap-one conjecture, where β is a regular cardinal, follows from GCH. Jensen, adding $\square_\kappa$ to the hypothesis, proved that $(\aleph_1, \aleph_0) \longrightarrow (\kappa^+, \kappa)$ when κ is a singular cardinal. In fact, Jensen has proved that the full (and very strong) gap-one conjecture already follows from the axiom of constructibility.

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Let us now look at a special version of the gap-one two cardinal problem — sometimes also called *Chang's Transfer Property*, in fact, on the following

Question 1. *Under what circumstances can the following transfer property fail:*

$$(\aleph_1, \aleph_0) \longrightarrow (\aleph_2, \aleph_1)?$$

More precisely, we are going to answer the following questions:

Question 2. *What is the consistency strength of the failure of the above mentioned Chang's Transfer Property:* $(\aleph_1, \aleph_0) \longrightarrow (\aleph_2, \aleph_1)$*?*

Question 3. *What extensions of* ZFC *are consistent with the failure of Chang's or even more general transfer properties?*

Chang's Transfer Property is closely related to a combinatorical problem of the existence of the following tree: For an infinite cardinal κ we call a tree T a κ^+-*Aronszajn tree* if T has height κ^+ such that every branch and every level has cardinality at most κ. Let a *special Aronszajn tree* be an Aronszajn tree T whose nodes are one-to-one functions from ordinals less than κ^+ into κ, ordered by inclusion. Or equivalently, there is a function $\sigma : T \longrightarrow \kappa$ such that $\sigma(x) \neq \sigma(y)$ for all tree elements $x <_T y$.

It is well-known that we can easily construct an $\aleph_1$-Aronszajn tree and, moreover, under GCH we can also construct a special κ^+-Aronszajn tree for every regular κ. We will, in fact, remind the reader of the proof in the appendix.

The connection now between special Aronszajn trees and the gap-one conjecture is given by the following statement:

Theorem. *There is a sentence φ in a finite language such that for all infinite cardinals κ, φ has a (κ^+, κ)-model if and only if there exists a special κ^+-Aronszajn tree.*

With this theorem in mind, a canonical counterexample to Chang's Transfer Property stated above involves the absence of a special $\aleph_2$-Aronszajn tree.

In 1972, Mitchell shows that it is consistent with ZFC that there is no special Aronszajn tree if and only if it is consistent that there exists a Mahlo cardinal. As a corollary, Mitchell shows that if it is consistent that there is a Mahlo cardinal, then it is consistent that Chang's Transfer Property, $(\aleph_1, \aleph_0) \not\longrightarrow (\aleph_2, \aleph_1)$, fails.

Theorem (Mitchell)**.** *The theory "ZFC and $\exists\tau(\tau$ is Mahlo$)$" is equi-consistent to the theory*

"ZFC and there is no special $\aleph_2$-Aronszajn trees"

and implies the consistency of "ZFC and $(\aleph_1, \aleph_0) \not\longrightarrow (\aleph_2, \aleph_1)$".

Mitchell's counterexample for the failure of the transfer property stated above, in fact, is given by the formula saying that there is a special $\aleph_2$-Aronszajn tree. This is sufficient for his theorem because there is always an Aronszajn tree of height $\aleph_1$.

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We will now improve the last statement, trying to get the failure of Chang's Transfer Property not only from a Mahlo but from an inaccessible cardinal, providing the existence of a special $\aleph_2$-Aronszajn tree. So, we have to take another suitable theory which will have enough (γ^+, γ)-models apart from the case $\gamma = \aleph_1$.

Furthermore, we know by the result of Chang we have mentioned above that we cannot expect to find the desired counterexample in an universe where GCH holds, in fact, where just $\aleph_1 = 2^{<\aleph_1} = 2^{\aleph_0}$ holds. However, we will find a model of set theory, proving the existence of the counterexample for the failure of Chang's Transfer Property such that $2^\kappa = \kappa^+$ holds for all uncountable κ and $2^{\aleph_0} = \aleph_2$ and so, GCH only minimally fails.

In fact, we are going to prove the following statement:

Theorem. *The theory*

$$\mathsf{ZFC} \;+\; \text{“}\exists\tau\,(\,\tau \textit{ is inaccessible}\,)\text{”}$$

is equi-consistent to the theory

$$\mathsf{ZFC} \;+\; \text{“}(\aleph_1, \aleph_0) \not\longrightarrow (\aleph_2, \aleph_1)\text{”}.$$

This statement obviously improves Mitchell's theorem above and will follow from the next two theorems we are going to prove.

Theorem. *Suppose there is a model of* ZFC *with an inaccessible cardinal τ. Moreover, let $\theta < \kappa$ be two regular cardinals below τ. Then there is a forcing extension of* $\mathbf{L}$ *that is a model of the following:*

$$\begin{aligned} \mathsf{ZFC} \quad &+ \quad 2^\theta = \kappa^+ \quad + \quad \text{“there is a special } \kappa^+\text{-Aronszajn tree”} \\ &+ \quad \text{“}2^\alpha = \alpha^+ \text{ for all infinite cardinals } \alpha < \theta \text{ or } \alpha \geqslant \kappa\text{”} \\ &+ \quad \text{“}(\gamma^+, \gamma) \not\longrightarrow (\kappa^+, \kappa) \text{ for all regular cardinals } \gamma \neq \kappa\text{”}. \end{aligned}$$

And moreover:

Theorem. *Suppose there is a model of set theory* ZFC *such that*

$$(\gamma', \gamma) \not\longrightarrow (\kappa^+, \kappa)$$

holds for a given pair of cardinals $\gamma' > \gamma \geqslant \omega$ and an uncountable regular cardinal κ. Then the following theory is consistent

$$\mathsf{ZFC} \;+\; \text{“}\exists\tau\,(\,\tau \textit{ is inaccessible}\,)\text{”}.$$

It is enough to prove the last two theorems: Considering the first theorem, starting from an appropiate ground model that has an inaccessible cardinal we will consider a suitable notion of forcing, due to Mitchell. Working then in the generic forcing extension, we will consider a theory $\mathfrak{T}$ and show the failure of the above stated transfer property by constructing a counterexample. Moreover, in the forcing extension we will have a special κ^+-Aronszajn tree and –as desired– sufficiently small powers of cardinals. And so the proof will be done.

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Moreover, we are going to look at the proof of the first main theorem more closely. We are able to find the desired counterexample to the considered transfer property even with a much weaker theory.

However, considering the new theory, the main tool within the proof –the morass structure– is getting slightly more complex. Fortunately, the main idea of the old proof is preserved. In fact, we will consider the following theory:

$$\boxed{\begin{aligned} \mathsf{ZFC}^- \quad + \quad & \mathbf{V} = \mathbf{L}[C] \text{ for } C \subseteq \mathrm{On} \ + \ 2^{<A} = A \\ + \quad & A \text{ is the largest cardinal } \ + \ A \text{ regular,} \end{aligned}}$$

and even this theory will have (γ^+, γ)-models for arbitrary regular cardinals $\gamma < \theta$ or $\gamma > \kappa$, working within the forcing extension we will have constructed by then.

More important, we will be able to construct the desired model of set theory –such that the transfer property above fails– as a forcing extension of a model of GCH as the following statement promises:

Theorem. *Assuming* GCH, *let τ be inaccessible. Moreover, consider two more regular cardinals $\theta < \kappa$ below τ. Then there is a forcing extension such that within this model of set theory we have $2^\theta = 2^\kappa = \kappa^+ = \tau$. Furthermore, we have for all regular cardinals $\gamma < \theta$ or $\gamma > \kappa$ the following failure of the transfer property:*

$$(\gamma^+, \gamma) \not\longrightarrow (\kappa^+, \kappa).$$

Of course, it is always possible to get a special κ^+-Aronszajn tree within the forcing extension by choosing τ appropiate as we will see. Moreover, the last theorem gives us many possibilities to get nice independent statements for the failure of Chang's Transfer Property with respect to large cardinals.

Having a large cardinal, say a measurable one or even a larger cardinal –just providing there is an inaccessible cardinal below to work with– starting from a suitable model satisfying GCH, we then can apply the forcing of the last theorem and we get the desired failure of the transfer property in a universe where we still have the existence property of that large cardinal we have started from.

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Finally, the proof of the second theorem will use the proof idea of Chang's statement that we mentioned above a few times. Working in a suitable $\mathbf{L}[D]$ by choosing the predicate D carefully, we will be close enough to the universe $\mathbf{V}$ to have sufficient consistency preservation between $\mathbf{L}[D]$ and $\mathbf{V}$ and even close enough to the constructible universe to get sufficient fitting properties on powers of cardinals to be able to apply Chang's proof idea.

In the appendix we will remind the reader of a well-known and often used theorem of Jensen—he never has published but mentioned, giving a characterization of a weak version of the square principle with special Aronszajn trees.

This finishes the survey of my thesis.