Institut für Mathematik — Mathematische Logik

# On the Equi-Consistency of the Failure of the GAP-1 Transfer Property and an Inaccessible Cardinal

— Dissertation —

zur Erlangung des akademischen Grades
"doctor rerum naturalium"
(Dr. rer. nat.)
in der Wissenschaftsdisziplin "Mathematische Logik"

eingereicht an der
Mathematisch-Naturwissenschaftlichen Falkultät
der Universität Potsdam

von
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Potsdam, den 10. August 2005

*dedicated in memoriam to my grandfather*

*April 19, 1920 — August 10, 2004*

*"Es ist schwierig und oft unmöglich, den Wert eines Problems im Voraus richtig zu beurteilen; denn schließlich entscheidet der Gewinn, den die Wissenschaft dem Problem verdankt."*

*"Ein mathematisches Problem sei ferner schwierig, damit es uns reizt, und dennoch nicht völlig unzugänglich, damit es unserer Anstrengung nicht spotte; es sei uns ein Wahrzeichen auf den verschlungenen Pfaden zu verborgenen Wahrheiten — uns hernach lohnend mit der Freude über die gelungene Lösung."*

*"Unermeßlich ist die Fülle von Problemen in der Mathematik, und sobald ein Problem gelöst ist, tauchen an dessen Stelle zahllose neue Probleme auf."*

David Hilbert
"Mathematische Probleme"

Vortrag, gehalten auf dem internationalen
Mathematischen Kongress zu Paris, 1900

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# Contents

# Preface

*F i n a l l y, i t i s d o n e .*

**How to read the thesis?** It is devided into eight chapters and an appendix. Each of them can be read almost independently from the other ones. Just the Chapters 3–5 are connected somehow and for Chapter 8 the reader needs to know what we have done in the previous chapters.

The *first chapter* gives a short introduction to the whole thesis, where the reader can see what we are heading to and, maybe not less important, why this goal is interesting, giving the development of the theory and statements that have been proved so far in the past.

The *second chapter* is providing the fundamental set theory we are using. The reader will find basic statements, either cited to standard text books and papers, or, on the other hand, we will give proofs to technical facts that we have extracted from the upcoming proof—not to fog the idea of forthcoming statements.

The *third chapter* gives a short introduction to the notion of forcing we are mainly using in this thesis. In fact, as in the second chapter, we are going to cite proven facts as often as possible as far as we are going to use them. Otherwise the reader will find a short proof.

The *fourth chapter* defines and explains the main tool we are going to use, the so-called coarse morass. People who are familiar with the theory of morasses, are easily able to understand what is going on there.

Finally in the *fifth chapter*, we give a proof of the first main theorem—strongly using details of the third and fourth chapter.

In the *sixth chapter* we consider another theory which will be weaker than the one we have used in the fifth chapter. This gives us the possibility to find a nice counterexample—as we will see. Although the main idea of the given proof survives, we need to define an even general version of the morass structure.

In the *seventh chapter* we consider and prove the second main theorem. This will be possible almost independently from the former chapters.

In the *last chapter*, we finally put both theorems together and to top the theory off, giving a few more remarks on it we have considered during the proofs. Additionally, we discuss general versions of the given main theorems.

In the *appendix*, we give a proof of a well-known theorem of Jensen. The proof is written very basically and completely independent from the chapters of the main part of this thesis. The reason and context to this statement will be provided in the first chapter.

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**Acknowledgments.** There are a lot of people I really would like to mention here—not only people who have influenced me directly, preparing the thesis. However, writing the preface for the thesis and not my memoirs, I am going to mention just the two people with the strongest connection to my thesis—my two supervisors.

*Professor Ronald B. Jensen.* Now, almost a decade later that I met Professor Jensen the first time, I owe him a lot—more than I could summarize in some lines. He would not like to read this anyway. However, this might be the right place to say: Thank you, Ronald, for your understanding, support and motivation in all of the past years so far.

*Professor Martin Weese.* My deep thank I would like to give also to Professor Weese. Being a second supervisor for my thesis, a boss for me during the past years in my position at university, and moreover a friend. With his calmness and experience he often has guided me in the past. I would like to thank him especially for his deep trust in me and my work, and also in my (seldom easy) way of doing. I would like to stress the financial support he was giving to me the whole time—starting with supporting my attendings of several conferences in logic, or even accepting my needs on some technical equipment.

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Special thank I would like to give here to my better half, Karen, just for being there—being at my side with much patience, especially during the hardest time in the last months.

Last but not least, I would like to thank Anna-Luise and Marc Messerschmidt for reading the words and patiently killing typos during the last weeks.

Potsdam August 2005

CHAPTER 1

# Introduction

Constructing models of a given consistent theory is often done in logic.[1] In the easiest cases we can use the well-known Löwenheim-Skolem theorem:

**Theorem** (Löwenheim-Skolem). *Every consistent theory $T$ in an infinite language $\mathcal{L}$ has a model of power at most $|\mathcal{L}|$. Moreover, if $T$ has infinite models, then it has infinite models of any given power greater than $|\mathcal{L}|$.*

By this theorem we know that every theory will fail to distinguish between infinite cardinals. So we can head for the question what will happen when we ask for pairs of infinite cardinals.

To start with, consider the language $\mathcal{L} = \{\dot{A}, \dots\}$ where $\dot{A}$ is a unary predicate. Call an $\mathcal{L}$-model $\mathfrak{A} = \langle \mathbb{A}; A, \dots \rangle$ a $(\kappa, \lambda)$-*model*, if $|\mathbb{A}| = \kappa$ and $|A| = \lambda$.

Now define for infinite cardinals $\alpha$, $\beta$, $\kappa$ and $\lambda$ the following notion of a general Transfer Property

$$(\alpha, \beta) \longrightarrow (\kappa, \lambda),$$

meaning that if a theory $T$ has an $(\alpha, \beta)$-model, then it has also a $(\kappa, \lambda)$-model.

In [MorVau62], Morley and Vaught proved for infinite cardinals $\alpha < \beta$ that

$$(\beta, \alpha) \longrightarrow (\aleph_1, \aleph_0).$$

[1]This introduction gives just a quick overview what this paper is about. For more details the author refers to [ChaKei90, Dev84, Jec03, Kan94]. In fact, we will leave out many other interesting results next to the way we are walking on straight to the main theorem we are interested in.

In fact, using homogenous models they showed that for a consistent and countable theory, having a $(\beta, \alpha)$-model, they can construct an elementary chain of length $\aleph_1$ of models $\mathfrak{A}_\nu$ such that for arbitrary $\bar{\nu} < \nu < \aleph_1$ we always have $A^{\mathfrak{A}_{\bar{\nu}}} = A^{\mathfrak{A}_\nu}$ and $\mathfrak{A}_{\bar{\nu}}$ is an elementary submodel of $\mathfrak{A}_\nu$, being a proper subset. Moreover, $A^{\mathfrak{A}_\nu}$ and $\mathbb{A}^{\mathfrak{A}_\nu}$ are both countable for all $\nu < \aleph_1$. Then the union of this chain of models is obviously an $(\aleph_1, \aleph_0)$-model.

Using the Löwenheim-Skolem Theorem stated above we can find a generalized version of it, providing that for arbitrary infinite cardinals $\alpha < \beta$ we always have

$$(\beta, \alpha) \longrightarrow (\alpha^+, \alpha).$$

Furthermore, in [Cha63], Chang has proved for all infinite cardinals $\alpha < \beta$ and regular $\delta$ such that $2^{<\delta} = \delta$ that the following holds

$$(\beta, \alpha) \longrightarrow (\delta^+, \delta).$$

And so a natural question arises given by the so-called *gap-one conjecture* or *gap-one two cardinal problem* asserting that every theory $T$ of a countable language $\mathcal{L}$ which has an $(\alpha^+, \alpha)$-model, also has a $(\beta^+, \beta)$-model for infinite cardinals $\alpha$ and $\beta$.

Chang's result stated above shows the gap-one conjecture, where $\beta$ is a regular cardinal, follows from $\mathsf{GCH}$. Jensen, adding $\square_\kappa$ to the hypothesis, proved that $(\aleph_1, \aleph_0) \longrightarrow (\kappa^+, \kappa)$ when $\kappa$ is a singular cardinal. In fact, Jensen proved in [Jen72] that the full (and very strong) gap-one conjecture already follows from the axiom of constructibility.[2]

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Let us now look at a special version of the gap-one two cardinal problem — sometimes also called *Chang's Transfer Property*, in fact, on the following

[2]Moreover, we can formulate the so-called *gap-n* conjecture in an obvious way and Jensen then proved it even for the gap-*n* conjecture.

**Question 1.** *Under what circumstances can the following transfer property fail:*

$$(\aleph_1, \aleph_0) \longrightarrow (\aleph_2, \aleph_1)?$$

More precisely, we are going to answer the following questions:

**Question 2.** *What is the consistency strength of the failure of the above mentioned Chang's Transfer Property:* $(\aleph_1, \aleph_0) \longrightarrow (\aleph_2, \aleph_1)$*?*

**Question 3.** *What extensions of* ZFC *are consistent with the failure of Chang's or even more general transfer properties?*

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Chang's Transfer Property is closely related to a combinatorical problem of the existence of the following tree: For an infinite cardinal $\kappa$ we call a tree $T$ a $\kappa^+$-*Aronszajn tree* if $T$ has height $\kappa^+$ such that every branch and every level has cardinality at most $\kappa$. Let a *special Aronszajn tree* be an Aronszajn tree $T$ whose nodes are one-to-one functions from ordinals less than $\kappa^+$ into $\kappa$, ordered by inclusion. Or equivalently, there is a function $\sigma : T \longrightarrow \kappa$ such that $\sigma(x) \neq \sigma(y)$ for all tree elements $x <_T y$.

It is well-known that we can easily construct an $\aleph_1$-Aronszajn tree and, moreover, under GCH we can also construct a special $\kappa^+$-Aronszajn tree for every regular $\kappa$. We will, in fact, remind the reader of the proof in the appendix.

The connection now between special Aronszajn trees and the gap-one conjecture is given by the following statement:

**Theorem.** *There is a sentence $\varphi$ in a finite language such that for all infinite cardinals $\kappa$, $\varphi$ has a $(\kappa^+, \kappa)$-model if and only if there exists a special $\kappa^+$-Aronszajn tree.*

We will sketch the proof in Chapter 2. With this theorem in mind, a canonical counterexample to Chang's Transfer Property stated above involves the absence of a special $\aleph_2$-Aronszajn tree.

In [Mit72], Mitchell shows that it is consistent with ZFC that there is no special Aronszajn tree if and only if it is consistent that there exists a Mahlo cardinal. As a corollary, Mitchell shows that if it is consistent that there is a Mahlo cardinal, then it is consistent that Chang's Transfer Property, $(\aleph_1, \aleph_0) \not\longrightarrow (\aleph_2, \aleph_1)$, fails.

**Theorem** ([Mit72]). *The theory "ZFC and $\exists\tau(\tau$ is Mahlo$)$" is equiconsistent to the theory*

*"ZFC and there is no special $\aleph_2$-Aronszajn trees"*

*and implies the consistency of "ZFC and $(\aleph_1, \aleph_0) \not\longrightarrow (\aleph_2, \aleph_1)$".*

Mitchell's counterexample for the failure of the transfer property stated above, in fact, is given by the formula saying that there is a special $\aleph_2$-Aronszajn tree. This is sufficient for his theorem because there is always an Aronszajn tree of height $\aleph_1$.

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We will now improve the last statement, trying to get the failure of Chang's Transfer Property not only from a Mahlo but from an inaccessible cardinal, providing the existence of a special $\aleph_2$-Aronszajn tree. So, we have to take another suitable theory which will have enough $(\gamma^+, \gamma)$-models apart from the case $\gamma = \aleph_1$. In fact, in Chapter 6 we will look at a very weak theory to get the same statement. Moreover, this theory has other interesting properties as we will see in Chapter 8.

Furthermore, we know by the result of Chang we have mentioned above that we cannot expect to find the desired counterexample in an universe where GCH holds, in fact, where just $\aleph_1 = 2^{<\aleph_1} = 2^{\aleph_0}$ holds. However, we will find a model of set theory, proving the existence of the counterexample for the failure of Chang's Transfer Property such that $2^\kappa = \kappa^+$ holds for all uncountable $\kappa$ and $2^{\aleph_0} = \aleph_2$ and so, $\mathsf{GCH}$ only minimally fails.

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In fact, in Chapter 5 and Chapter 7 we are going to prove the following statement that is a corollary of the main theorems of this paper, Theorem 70 and Theorem 79:

**Theorem.** *The theory*

$$\mathsf{ZFC} \;+\; \text{“}\exists\tau\,(\,\tau \textit{ is inaccessible}\,)\text{”}$$

*is equi-consistent to the theory*

$$\mathsf{ZFC} \;+\; \text{“}(\aleph_1,\aleph_0) \not\longrightarrow (\aleph_2,\aleph_1)\text{”}.$$

This statement obviously improves Mitchell's theorem above and will follow from the next two theorems we are going to prove.

**Theorem.** *Suppose there is a model of* ZFC *with an inaccessible cardinal $\tau$. Moreover, let $\theta < \kappa$ be two regular cardinals below $\tau$. Then there is a forcing extension of* $\mathbf{L}$ *that is a model of the following:*

$$\begin{aligned}\mathsf{ZFC} \;&+\; 2^\theta = \kappa^+ \;+\; \text{“there is a special } \kappa^+\text{-Aronszajn tree”}\\ &+\; \text{“}2^\alpha = \alpha^+ \text{ for all infinite cardinals } \alpha < \theta \text{ or } \alpha \geqslant \kappa\text{”}\\ &+\; \text{“}(\gamma^+,\gamma) \not\longrightarrow (\kappa^+,\kappa) \text{ for all regular cardinals } \gamma \neq \kappa\text{”}.\end{aligned}$$

And moreover as we will see later, it follows easily from known facts the following:

**Theorem.** *Suppose there is a model of set theory* ZFC *such that*

$$(\gamma',\gamma) \not\longrightarrow (\kappa^+,\kappa)$$

*holds for a given pair of cardinals $\gamma' > \gamma \geqslant \omega$ and an uncountable regular cardinal $\kappa$. Then the following theory is consistent*

$$\mathsf{ZFC} \;+\; \text{“}\exists\tau\,(\,\tau \textit{ is inaccessible}\,)\text{”}.$$

It is enough to prove the last two theorems. In Chapter 5, considering the first theorem, starting from an appropiate ground model that has an inaccessible cardinal we will consider a suitable notion of forcing, due to Mitchell and to be defined in Chapter 3. Working then in the generic forcing extension, we will consider a theory $\boldsymbol{\mathfrak{T}}$ and show the failure of the above stated transfer property by constructing a counterexample. Moreover, in the forcing extension we will have a special $\kappa^+$-Aronszajn tree and –as desired– sufficiently small powers of cardinals. And so the proof will be done.

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Moreover, in Chapter 6 we are going to look at the proof of the first main theorem more closely. Although this theorem will be proved by Chapter 5, we are able to find the desired counterexample to the considered transfer property even with a much weaker theory and so we are going to improve the statement, proving Theorem 80.

On our way, considering the new theory, the main tool within the proof –the morass structure– is getting slightly more complex. Fortunately, the main idea of the old proof is preserved. In fact, we will consider the following theory:

$$\begin{aligned} \mathsf{ZFC}^- \;\; + \;\; \mathbf{V} = \mathbf{L}[C] \text{ for } C \subseteq \mathrm{On} \;\; + \;\; 2^{<\dot{A}} = \dot{A} \\ + \;\; \dot{A} \text{ is the largest cardinal } \;\; + \;\; \dot{A} \text{ regular,} \end{aligned}$$

and even this theory will have $(\gamma^+, \gamma)$-models for arbitrary regular cardinals $\gamma < \theta$ or $\gamma > \kappa$, working within the forcing extension we will have constructed by then.

More important, we will be able to construct the desired model of set theory –such that the transfer property above fails– as a forcing extension of a model of $\mathsf{GCH}$ as the following statement, in fact Theorem 80, promises:

**Theorem.** *Assuming* GCH*, let $\tau$ be inaccessible. Moreover, consider two more regular cardinals $\theta < \kappa$ below $\tau$. Then there is a forcing extension such that within this model of set theory we have $2^\theta = 2^\kappa = \kappa^+ = \tau$. Furthermore, we have for all regular cardinals $\gamma < \theta$ or $\gamma > \kappa$ the following failure of the transfer property:*

$$(\gamma^+, \gamma) \not\longrightarrow (\kappa^+, \kappa).$$

Of course, it is always possible to get a special $\kappa^+$-Aronszajn tree within the forcing extension by choosing $\tau$ appropiate as we will see. Moreover, the last theorem gives us many possibilities to get nice independent statements for the failure of Chang's Transfer Property with respect to large cardinals.

Having a large cardinal, say a measurable one or even a larger cardinal –just providing there is an inaccessible cardinal below to work with– starting from a suitable model satisfying GCH, we then can apply the forcing of the last theorem and we get the desired failure of the transfer property in a universe where we still have the existence property of that large cardinal we have started from.

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Finally, in Chapter 7, the proof of the second theorem will use the proof idea of Chang's statement that we mentioned above a few times. Working in a suitable $\mathbf{L}[D]$ by choosing the predicate $D$ carefully, we will be close enough to the universe $\mathbf{V}$ to have sufficient consistency preservation between $\mathbf{L}[D]$ and $\mathbf{V}$ and even close enough to the constructible universe to get sufficient fitting properties on powers of cardinals to be able to apply Chang's proof idea.

In the appendix we will remind the reader of a well-known and often used theorem of Jensen—he never has published but mentioned in [Jen72], giving a characterization of a weak version of the square principle with special Aronszajn trees.

As already mentioned in the preface, the author does not want to forget to stress that this paper would not even exist without the untiring support and the enriching ideas of Professor Jensen during the past years.

CHAPTER 2

# Fundamentals

The following statements are all provable in (sometimes just parts of) the Zermelo-Fraenkel set theory with the axiom of choice: ZFC.[1] The collection of these results should *not* be seen as a *complete* introduction to the theory we are using in the upcoming chapters. Most of the following statements will just be cited, anyway. For a detailed survey and proofs the author strongly refers to the standard books, *e.g.*, [ChaKei90, Dev84, Dra74, Jec03, Kan94]. For the conveniency of the reader we are (mostly) using standard notation.

With this in mind, the reader will find in this chapter some important standard facts and even some other (technical) basics we will need later.[2]

## Constructing Models

Let us start with some (very) basic set theory. A set $X$ is said to be *extensional* if for all distinct $u, v \in X$ there is an $x \in X$ such that $x \in u$ if and only if $x \notin v$.

**Lemma 4** ([Jec03, Theorem 6.15], Mostowski Collapse)**.** *For each extensional set $X$ there is a unique transitive set $M$ and an unique bijection $\pi : X \longleftrightarrow M$ such that $\pi : \langle X, \in \rangle \overset{\sim}{\longleftrightarrow} \langle M, \in \rangle$. Moreover, if $Y \subseteq X$ is transitive, then $\pi \restriction Y = \mathrm{id} \restriction Y$.*

[1]In fact, in most cases we will not need the presence of the Axiom of Choice here. However, in our applications we will have it.

[2]In most cases, the more famous the statement is, the less we are proving it here in this chapter.

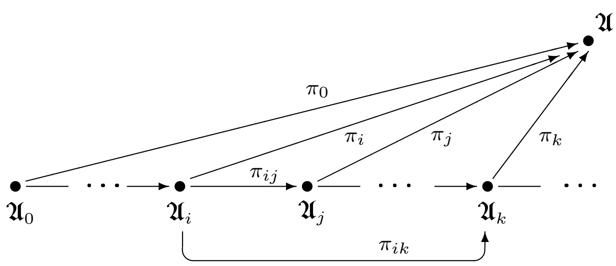
More generally, $X$ has not to be a set but it must be so-called set-like, meaning that for all $x \in X$ the collection of all $\in$-predecessors, $\in''\{x\}$, is a set.

Now let us look at models of, say, a countable language $\mathcal{L}$, and elementary embeddings between them.

**Definition 5.** *A directed system of models $\langle \mathfrak{A}_i \,|\, i < \omega \rangle$ has elementary embeddings $\pi_{ij} : \mathfrak{A}_i \longrightarrow \mathfrak{A}_j$ such that $\pi_{ik} = \pi_{jk} \circ \pi_{ij}$ for all natural numbers $i < j < k$.*

Then we can prove:

**Lemma 6** ([Jec03, Lemma 12.2], Direct Limit)**.** *If $\langle \mathfrak{A}_i, \pi_{ij} \,|\, i < j < \omega \rangle$ is a directed system of models, then there exists a model $\mathfrak{A}$, unique up to isomorphism, and elementary embeddings $\pi_i : \mathfrak{A}_i \longrightarrow \mathfrak{A}$ such that $\mathfrak{A} = \bigcup_{i<\omega} \mathrm{rng}(\pi_i)$ and $\pi_i = \pi_j \circ \pi_{ij}$ for all $i < j$.*




The model $\mathfrak{A}$ in the last lemma is called the *direct limit* of the given sequence $\langle \mathfrak{A}_i, \pi_{ij} \,|\, i < j < \omega \rangle$.

### Stationary Sets

Remember, for a regular cardinal $\kappa$ we call $X \subseteq \kappa$ a closed and unbounded set, *club* for short, if it is closed under limit points and unbounded in $\kappa$. We call a set $S \subseteq \kappa$ *stationary* if it meets all club sets.

There are nice properties, *e.g.*, the collection of all club sets is closed under intersections of strictly less than $\kappa$ many sets. Moreover, it is closed under diagonal intersection of length $\kappa$. Using this we can prove the following important and well-known fact:

**Lemma 7** ([Jec03, Theorem 8.7], Fodor)**.** *If $S \subseteq \kappa$ is stationary and $\pi : S \longrightarrow \kappa$ is a regressive ordinal function, that is $\pi(\xi) < \xi$ for all $\xi \in S \backslash \{\varnothing\}$, then there is a stationary subset $T \subseteq S$ and an ordinal $\gamma < \kappa$ such that $f(\alpha) = \gamma$ for all $\alpha \in T$.*

Sometimes useful –as we will see later– is also the following collection of well-known facts:

**Lemma 8.** *Let $\kappa > \omega$ be a regular cardinal.*

- *Let $\mathrm{cf}(\mu) > \kappa$. Then the set $\{\gamma < \mu \,|\, \mathrm{cf}(\gamma) = \kappa\}$ is a stationary subset of $\mu$.*
- *For each function $f : \kappa \longrightarrow \kappa$, the set $\{\alpha < \kappa \mid f''\alpha \subseteq \alpha\}$ is closed and unbounded in $\kappa$.*
- *The set of all limit points of a club set of $\kappa$ is club again.*

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Now, call a transitive class $W \subseteq \mathbf{V}$ an *inner model* if $W$ contains all ordinal numbers and satisfies ZFC. In fact, the constructible universe $\mathbf{L}$ is an inner model as we will discuss in one of the next sections. Then we can prove the following

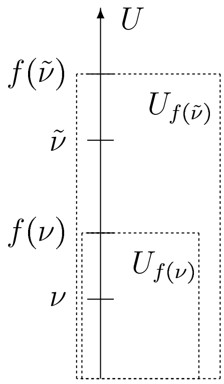
**Lemma 9.** *Let $W$ be an inner model and $\kappa < \tau$ be infinite cardinals such that $\kappa^{+W} < \tau$ and $\{ \lambda < \tau \mid W \models \mathrm{cf}(\lambda) = \kappa^+ \}$ is stationary in the universe $\mathbf{V}$. Moreover let $U := \langle U, <_U \rangle$ be a linear order such that $U \subseteq H$ for a suitable $H \in W$, $|H|^W = \tau$ and*

$$U_x := \{ z \mid z \leqslant_U x \} \in W, \ |U_x|^W \leqslant \kappa \text{ for any } x \in U.$$

*Then we have $|U| < \tau$.*

*Proof.* Suppose not. Without loss of generality, using a suitable bijection let $H$ be just $\tau$. So, let $\gamma$ be the cofinality of $\langle U, <_U \rangle$, that is the cardinality of a minimal subset of $U$ which lies cofinal. Hence, we have $\gamma \leqslant |U| = \tau$. However, letting $f : \gamma \longrightarrow U$ be a monotone and cofinal enumeration, we have $U = \bigcup_{\nu<\gamma} U_{f(\nu)}$ and so afterall also $\tau = |U| \leqslant \gamma \cdot \kappa \leqslant \tau$. This means we have $\gamma = \tau$.

Define $g(\nu) := \sup(U_{f(\nu)} \cap \nu)$ for $\nu < \tau$. Then $g$ is weakly monotone, that is $g(\nu) \leqslant \nu$, and $\sup_{\nu<\tau} g(\nu) = \sup U = \tau$. By our assumption, the set $S := \{ \nu < \tau \mid W \models \mathrm{cf}(\nu) = \kappa^+ \}$ is stationary in $\mathbf{V}$. For elements $\nu$ of $S$ is $g(\nu)$ strictly less than $\nu$ because in $W$ we have: $|U_{f(\nu)}| \leqslant \kappa < \kappa^+ = \mathrm{cf}(\nu) \leqslant \nu$, and so $U_{f(\nu)}$ is bounded in $\nu$.




Hence, $g \restriction S$ is a regressive function on a stationary set. Now, Theorem 7 of Fodor implies that there is a stationary subset $S'$ of $S$ such that $g(\nu) = \alpha$ for a suitable $\alpha < \tau$ and arbitrary $\nu \in S'$. But in this case we also have the following contradiction:

$$\tau = \sup_{\nu\in S} g(\nu) = \sup_{\nu\in S'} g(\nu) = \alpha < \tau.$$

Note, the first equality just holds because by definition we have for $x \leqslant_U y$ obviously $U_x \subseteq U_y$ and so

$$\bigcup\{ U_{f(\nu)} \mid \nu < \tau \} = \bigcup\{ U_{f(\nu)} \cap \nu \mid \nu < \tau \},$$

meaning the range of $f$ and $g$ are both cofinal in $\tau$.

Therefore, the lemma is proved. ⊠(Lemma 9)

## Forcing

Working in a so-called *ground model* $M$, we consider a partial order $\langle \mathbb{P}, <_{\mathbb{P}} \rangle$ and call it sometimes *notion of forcing* with the so-called *forcing conditions* as its elements. We also say that a condition $p$ *is stronger than* a condition $q$ if $p <_{\mathbb{P}} q$. We call a set $D \subseteq \mathbb{P}$ *dense* in $\mathbb{P}$ if for every $p \in \mathbb{P}$ there is a $q \in D$ such that $q \leqslant_{\mathbb{P}} p$.

Call a non-empty $G \subseteq \mathbb{P}$ a *filter* if firstly, whenever $p < q$ and $p \in G$, then also $q \in G$; and, secondly, if $p$, $q$ are elements of $G$, then there is an $r \in G$ such that $r$ is stronger than both, $p$ and $q$. Moreover, call a filter $G$ *generic over* $M$ (or just $M$*-generic*) if for every dense $D$ in $\mathbb{P}$ and $D \in M$, the filter $G$ always meets $D$.

Then we can construct the so-called *forcing extension* or *generic extension*, $M[G]$, of the ground model $M$ that satisfied ZFC, given a generic filter $G$, such that $M[G] \models$ ZFC, $M \subseteq M[G]$, $G \in M[G]$, $\mathrm{On}^{M[G]} = \mathrm{On}^M$ and it is minimal in the sense that if $N$ is a transitive model of ZF such that $M \subseteq N$ and $G \in N$, then $M[G] \subseteq N$.

The main idea now is that we are able to name elements of the generic extension within the ground model. Moreover, we can define the so-called *forcing relation* '$\Vdash$' within $M$ and so we are able to decide within the ground model what properties hold in the generic extension:

**Lemma 10** ([Jec03, Kun80])**.** *For every generic $G \subseteq \mathbb{P}$ over $M$ and every formula $\varphi$ of the forcing language we have*

$$M[G] \models \varphi \quad \textit{if and only if} \quad \exists p \in G\, p \Vdash \varphi.$$

For a collection of the properties of the forcing relation we refer to [Jec03, Theorem 14.7].

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In Chapter 3 we will use the connection to Boolean algebras. In fact, we can look at the universe $\mathbf{V}$ as collection of functions, ranging into the set $2 = \{0, 1\}$, that is, roughly speaking, the identification of sets with their characteristic functions. So we can identify the universe $\mathbf{V}$ with $\mathbf{V^2}$ where $\mathbf{2}$ is the simplest Boolean algebra. Then the formula $x \dot{\in} y$ has truth value 0 or 1.

Taking now a more complex Boolean algebra $\mathcal{B}$ we can look at atomic formulae $x \dot{\in} y$ and $x \dot{=} y$ where the truth value can be an element of $\mathcal{B}$ strictly between $0_{\mathcal{B}}$ and $1_{\mathcal{B}}$. Choosing $\mathcal{B}$ in an appropiate way we can try to decide formulae which we cannot in $\mathbf{V^2}$.

Furthermore, having a Boolean algebra $\mathcal{B}$ we can consider the related partial order $\langle \mathcal{B}', \leqslant_{\mathcal{B}'} \rangle$ defined as $\mathcal{B}' := \mathcal{B} \backslash \{0_{\mathcal{B}}\}$ and setting $b_0 \leqslant_{\mathcal{B}'} b_1$ if $p \cdot q = p$.

Moreover, we can start from a partial order to construct a related Boolean algebra as follows:

**Lemma 11** ([Jec03]). *For every partially ordered set $\langle \mathbb{P}, <_{\mathbb{P}} \rangle$ there is a complete Boolean algebra $\mathcal{B} = \mathcal{B}(\mathbb{P})$ and a mapping $\pi : \mathbb{P} \longrightarrow \mathcal{B}'$ where $\mathcal{B}'$ is as above such that*

- *if $p \leqslant_{\mathbb{P}} q$, then $\pi(p) \leqslant_{\mathcal{B}'} \pi(q)$,*
- *$p$ and $q$ are compatible if and only if $\pi(p) \cdot \pi(q) \neq 0_{\mathcal{B}}$,*
- *the set $\{ \pi(p) \mid p \in \mathbb{P} \}$ is dense in $B$.*

*Moreover, the Boolean algebra $\mathcal{B}$ is unique up to isomorphism.*

Having these complete Boolean algebras we can construct a Boolean-valued model (of the language of set theory) where the Boolean values of $\dot{=}$ and $\dot{\in}$ are given by two functions of two variables $[\![ x \dot{=} y ]\!]$ and $[\![ x \dot{\in} y ]\!]$, *cf.* [Jec03] for all details.

⋆ ⋆ ⋆

We are now interested in cardinal preserving properties for a given notion of forcing. For a cardinal $\kappa$ in the ground model, say a partial order yields the *$\kappa$-chain condition*, *$\kappa$-c.c.* for short, if every dense set has cardinality strictly less than $\kappa$.

**Lemma 12.** *If $\mathbb{P}$ yields the $\kappa$-c.c., then it preserves cofinalities above $\kappa$, that means, if $\lambda$ is a cardinal such that $\mathrm{cf}^M(\lambda) \geqslant \kappa$, then we have $\mathrm{cf}^M(\lambda) = \mathrm{cf}^{M[G]}(\lambda)$. Moreover, if $\kappa$ is regular, then cardinals are preserved above $\kappa$.*

A partial order $\mathbb{P}$ is *$\lambda$-closed* if whenever $\gamma < \lambda$ and $\{ p_\xi \mid \xi < \gamma \}$ is a decreasing sequence of elements of $\mathbb{P}$, that is $p_\xi \leqslant_{\mathbb{P}} p_\eta$ for $\eta < \xi$, then there is a $q \in \mathbb{P}$ such that for each $\xi < \gamma$ we have $q \leqslant_{\mathbb{P}} p_\xi$.

This property ensures that objects with suitable small cardinality within the forcing extension can already be found in the ground model.

**Lemma 13** ([Jec03, Kun80]). *If $\mathbb{P}$ is $\lambda$-closed for a cardinal $\lambda$, then there are no new sets of ordinals of cardinality strictly smaller than $\lambda$ in the forcing extension. Therefore, $\mathbb{P}$ preserves cofinalities below $\lambda$. Moreover, $\mathbb{P}$ preserves also cardinals below $\lambda$.*

We turn now to the problem of the iterated application of forcing, that is in the easiest case the following two-step product forcing.

**Lemma 14.** *Let $\mathbb{P}$ and $\mathbb{Q}$ be two notions of forcing in $M$. In order that $G \subseteq \mathbb{P} \times \mathbb{Q}$ is generic over $M$, it is necessary and sufficient that $G = G_1 \times G_2$ where $G_1 \subseteq \mathbb{P}$ is generic over $M$ and $G_2 \subseteq \mathbb{Q}$ is generic over $M[G_1]$. Moreover, in this case we have*

$$M[G] \;=\; M[G_1][G_2]\,(=\; M[G_2][G_1]\,).$$

Now, in general, in applications the second forcing might be not an element of the ground model $M$. The important fact here is that even then, in case of a two-step iteration, we can represent it by a single notion of forcing extension over the ground model.

Let $\mathbb{P}$ be a partial order in $M$ and $\dot{\mathbb{Q}}$ a name for a partial order, that is $\Vdash_{\mathbb{P}} (\dot{\mathbb{Q}}$ is partial order). Define then $\mathbb{P} \star \dot{\mathbb{Q}}$ as the set

$$\{\, (p, \dot{q}) \mid p \in \mathbb{P} \wedge \;\Vdash_{\mathbb{P}} \dot{q} \in \dot{\mathbb{Q}}\,\}$$

and, moreover,

$$(p_1, \dot{q}_1) \leqslant (p_2, \dot{q}_2) \quad \text{if and only if} \quad p_1 \leqslant p_2 \text{ and } p_1 \Vdash \dot{q}_1 \leqslant_{\dot{\mathbb{Q}}} \dot{q}_2.$$

For more details and facts we again refer to [Jec03, Kun80].

## Constructible Universe

For the whole section, we strongly refer to [Jen72, Dev84] for proofs and details. The idea of taking constructible sets is easy to describe: When we look at the von Neumann's view of $\mathbf{V}$, taking all subsets in the successor step $\mathbf{V}_{\alpha+1} = \mathcal{P}(\mathbf{V})$, then we realize that we have no idea what does this really mean. So an attempt could be just to take the subsets we really need, meaning all subsets that can be described or constructed in some sense.

Therefore, let us turn to the theory of constructible sets, looking at Gödel's constructible universe $\mathbf{L} := \bigcup_{\alpha \in \mathrm{On}} \mathbf{L}_\alpha$, where $\mathbf{L}_0 := \varnothing$, for limit ordinals $\lambda$ set $\mathbf{L}_\lambda := \bigcup_{\alpha<\lambda} \mathbf{L}_\lambda$ and, finally, take as $\mathbf{L}_{\alpha+1}$ the collection of all within $\mathbf{L}_\alpha$ from parameters taken from $\mathbf{L}_\alpha$ definable sets. Then $\mathbf{L}$ is an inner model of set theory, in fact, it is the smallest one.

**Lemma 15** ([Dev84]). *The following hold:*

(a) *Assume* $\mathbf{V} = \mathbf{L}$. *Then* $\mathsf{GCH}$ *and* $\mathsf{AC}$.

(b) $\mathbf{L} \models (\,\mathbf{V} = \mathbf{L} \,+\, \mathsf{ZFC} \,+\, \mathsf{GCH}\,)$.

Let $\mathsf{ZF}^-$ be all axioms of $\mathsf{ZF}$ without the power set axiom. Then sometimes it is useful to have the following

**Lemma 16** ([Dev84, Jen72]). *For a regular and uncountable cardinal* $\kappa$ *we have that* $\mathbf{L}_\kappa$ *is a model of* $\mathbf{V} = \mathbf{L}$ *and* $\mathsf{ZFC}^-$.

This is best possible situation, having $\mathbf{L}_{\kappa+} \models \mathcal{P}(\kappa) \notin \mathbf{V}$ because there are cofinal many ranks of subsets of $\kappa$. So, in case of a regular limit of cardinals, an ($\mathbf{L}$-)inaccessible, we would have found a model of full set theory — proving that we cannot expect to prove the existence of such a cardinal in the presence of Gödel's Incompleteness Theorem.

One of the most important results in this area is:

**Lemma 17** ([Dev84], Gödel, Condensation Lemma). *Let* $\alpha$ *be an arbitrary limit ordinal. If* $X \prec_1 \mathbf{L}_\alpha$, *that is preserving* $\exists$*-formulae, then there are unique* $\pi$ *and* $\beta$ *such that* $\beta \leqslant \alpha$ *and:*

(a) $\pi : \langle X, \in \rangle \overset{\sim}{\longleftrightarrow} \langle \mathbf{L}_\beta, \in \rangle$,

(b) *if* $Y \subseteq X$ *transitive, then* $\pi \upharpoonright Y = \mathrm{id} \upharpoonright Y$,

(c) $\pi(x) \leqslant_{\mathbf{L}} x$ *for all* $x \in X$.

Here, $\leqslant_{\mathbf{L}}$ is the canonical well-ordering of the constructible universe, *cf.* [Dev84] for details. And finally on the way to prove the Generalized Continuum Hypothesis we prove that bounded subsets will be caught by the next cardinal level of the constructible hierarchy.

**Lemma 18** ([Dev84]). *Assume* $\mathbf{V} = \mathbf{L}$ *and let* $\kappa$ *be a cardinal. If* $x$ *is a bounded subset of* $\kappa$, *or more generally, if* $x \subseteq \mathbf{L}_\alpha$ *for some* $\alpha < \kappa$, *then* $x \in \mathbf{L}_\kappa$.

Using the fine structure theory[3], Jensen was able to prove the important covering property for the constructible universe. For, let us say that $0^{\#}$ exists, if there is a non-trivial elementary embedding $\pi : \mathbf{L} \longrightarrow \mathbf{L}$.

**Lemma 19** ([Dev84], Jensen, Covering Lemma)**.** *Assume,* $0^{\#}$ *does not exist. If* $X$ *is an uncountable set of ordinals, then there is a constructible set,* $Y$*, of ordinals such that* $X \subseteq Y$ *and* $|X| = |Y|$*.*

⋆ ⋆ ⋆

Now, for many applications it is useful to consider a more general version of the constructible universe: for some set $A$, we can consider the class $\mathbf{L}[A]$, the universe of all sets constructible relative to $A$, in fact, let the levels $\mathbf{L}_\alpha[A]$ be similarly defined as for the usual hierarchy and let $\mathbf{L}_{\alpha+1}[A]$ be the set of all subsets of $\mathbf{L}_\alpha[A]$ that are definable over $\mathbf{L}_\alpha[A]$ using parameters from $\mathbf{L}_\alpha[A]$ and $A$ itself.

Then we have similiar properties as for the $\mathbf{L}_\alpha$-hierarchy, *cf.* [Dev84]. In particular we have for $\alpha \geqslant \omega$ that $|\mathbf{L}_\alpha[A]| = |\alpha|$ and for $B = A \cap \mathbf{L}[A]$,

$$\mathbf{L}[A] \;=\; \mathbf{L}[B] \;=\; (\,\mathbf{L}[B]\,)^{\mathbf{L}[A]}.$$

The price of having more freedom in the construction of subsets is loosing parts of GCH: One major tool for this assertion was the Condensation Lemma. But now, having an $X \prec_1 \mathbf{L}_\alpha[A]$ we just find $\pi$, $\beta$ and $B$ such that

$$\pi : X \stackrel{\sim}{\longrightarrow} \mathbf{L}_\beta[B],$$

where $B = \pi''(A \cap X)$. Thus, in general this does not lead to a structure of the $\mathbf{L}[A]$-hierarchy. However, we can then prove the following

**Lemma 20** ([Dev84])**.** *Let* $A \subseteq \kappa$*. Then* $\mathbf{L}[A]$ *is an inner model of* ZFC *and we have* $\mathbf{L}[A] \models 2^\lambda = \lambda^+$ *for* $\lambda \geqslant \kappa$*.*

Moreover, using a bit more technical tools we can finally prove

**Lemma 21** ([Dev84])**.** *Let* $\mathbf{V} = \mathbf{L}[A]$*, where* $A \subseteq \kappa^+$*. Then* $2^\kappa = \kappa^+$ *holds and so if* $\kappa = \aleph_0$*, we have the full* GCH*.*

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[3] *cf.* [Dev84, Jen72].

At the very last, let us prove the following statement we are going to use later.

**Lemma 22.** *For an uncountable and regular cardinal* $\kappa$ *and a subset* $D \subseteq \kappa$ *we have that* $\mathbf{L}[D] \models 2^{<\kappa} = \kappa$.

*Proof.* Let $b \subseteq \gamma < \kappa$ be a given small subset of $\kappa$. Then it is sufficient to prove the following

*Claim.* $b \in \mathbf{L}_\kappa[D]$.

With the claim in mind, we are obviously done, having small subsets of $\kappa$ already within the model $\mathbf{L}_\kappa[D]$ which has cardinality $\kappa$ as we already know. And so, it remains to prove the claim.

For, let $b \in \mathbf{L}_\xi[D] \models \mathsf{ZFC}^-$ for a suitable ordinal $\xi$. Heading a condensation argument, define simultanously two sequences $\langle X_i \mid i < \omega \rangle$ of elementary submodels of $\langle \mathbf{L}_\xi[D], \in, D \rangle$ and $\langle \kappa_i \mid i < \omega \rangle$ of ordinals as follows:

Let $X_0$ be the smallest elementary submodel of $\langle \mathbf{L}_\xi[D], \in, D \rangle$, containing $\gamma$ as a subset. Define $\kappa_i$ as the least upper bound of $X_i \cap \kappa$. Moreover, let $X_{i+1}$ be the smallest elementary submodel of $\langle \mathbf{L}_\xi[D], \in, D \rangle$, containing $\kappa_i$ as a subset. Finally set $X := \bigcup_{i<\omega} X_i$.

Then for $\bar\kappa := \sup_{i<\omega} \kappa_i$ we have $\bar\kappa < \kappa$. The model $X$ is an elementary substructure of $\langle \mathbf{L}_\xi[D], \in, D \rangle$. Moreover, we have that $\gamma \leqslant \bar\kappa \subseteq X$ and also $|X| = |\bar\kappa| < \kappa$.

Now, let $\sigma : \bar X \xrightarrow{\sim} X$ be the collapsing map. Then by the condensation property we have that $\bar X$ is isomorphic to $\langle \mathbf{L}_{\bar\xi}[\bar D], \in, \bar D \rangle$ for suitable $\bar\xi$ and $\bar D$ such that $\sigma \restriction \bar\kappa$ is the identity map. We also have that $\bar D \subseteq \bar\kappa$ and even more important, $\bar D = D \cap \bar\kappa$.

However, then we have that $\bar X = \langle \mathbf{L}_{\bar\xi}[D \cap \bar\kappa], \in, D \cap \bar\kappa \rangle$ which is clearly an element and especially a subset of $\mathbf{L}_\kappa[D]$ because $\bar\xi < \kappa$.

Therefore, $b = \sigma^{-1}(b) \in \bar X \subseteq \mathbf{L}_\kappa[D]$ as desired. ⊠(Lemma 22)

## Combinatorical Principles and Trees

We will not consider $\diamondsuit$-principles and related subjects like Souslin trees. Here, we are interested in constructing special Aronszajn trees. For, we consider coherent square-sequences, which Jensen introduced in [Jen72].

**Definition 23** ($\square_\kappa$-Sequence)**.** *For an infinite cardinal $\kappa$ call a sequence $\langle C_\alpha \mid \alpha \in \kappa^+ \cap \mathrm{Lim} \rangle$ a $\square_\kappa$-sequence if*

(a) $\forall \alpha \in \kappa^+ \cap \mathrm{Lim}\,(\, C_\alpha \subseteq \alpha$ *club* $)$,
(b) $\forall \alpha \in \kappa^+ \cap \mathrm{Lim}\,(\, \mathrm{cf}(\alpha) < \kappa \longrightarrow \mathrm{otp}(C_\alpha) < \kappa\,)$,
(c) *if $\beta < \alpha$ is a limit point of $C_\alpha$, then $C_\alpha = \beta \cap C_\alpha$.*

*We say, $\square_\kappa$ holds if there is a $\langle C_\alpha \mid \alpha \in \kappa^+ \cap \mathrm{Lim} \rangle$-sequence.*

For our purpose it will be interesting another weaker version of this combinatorical principle — the so-called *Weak Square*.

**Definition 24** ($\square^*_\kappa$-Sequence)**.** *For an infinite cardinal $\kappa$ call a sequence $\langle C_\alpha \mid \alpha \in \kappa^+ \cap \mathrm{Lim} \rangle$ a $\square^*_\kappa$-sequence if*

(a) $\forall \alpha \in \kappa^+ \cap \mathrm{Lim}\,(\, C_\alpha$ *is club in* $\alpha\,)$,
(b) $\forall \beta \in \kappa^+ \cap \mathrm{Lim}\,(\, |\{C_\alpha \cap \beta \,:\, \alpha \leqslant \beta\}| \leqslant \kappa\,)$,
(c) $\forall \alpha \in \kappa^+ \cap \mathrm{Lim}\,(\, \mathrm{otp}(C_\alpha) \leqslant \kappa\,)$.

*We say, $\square^*_\kappa$ holds if there is a $\square^*_\kappa$-sequence.*

Choose for all limit ordinals $\alpha \in \kappa^+$ club sets $C_\alpha \subseteq \alpha$ such that $\mathrm{otp}(C_\alpha) = \kappa$. If there are only $\kappa$ many bounded subsets of $\kappa$, then this forms trivially a $\square_\kappa$-sequence and finally we have

**Lemma 25** ([Dev84, Jen72])**.** *The following hold:*

(a) *If $\kappa^{<\kappa} = \kappa$, then $\square^*_\kappa$.*
(b) *If $\square_\kappa$, then $\square^*_\kappa$.*
(c) *Assume $\mathbf{V} = \mathbf{L}[A]$ for an $A \subseteq \kappa^+$ such that for all $\alpha < \kappa^+$,*

$$|\alpha|^{\mathbf{L}[A\cap\alpha]} \leqslant \kappa,$$

*then $\square_\kappa$ holds. In particular, if $\mathbf{V} = \mathbf{L}$, then $\square_\kappa$ holds for all infinite cardinals $\kappa$.*

Now, call a partial order $\langle T, <_T \rangle$ a *tree* if the set of all predecessors of an element of $T$ is well-ordered by $<_T$. Moreover, for a cardinal $\kappa$ we call a tree $T$ a $\kappa^+$-*Aronszajn tree* if $T$ has height $\kappa^+$ such that every branch and every level has cardinality at most $\kappa$. Let a *special* $\kappa^+$-*Aronszajn tree* be an Aronszajn tree $T$ whose nodes are one-to-one functions from ordinals less than $\kappa^+$ into $\kappa$, ordered by inclusion. Or equivalently, there is a function $\sigma : T \longrightarrow \kappa$ such that $\sigma(x) \neq \sigma(y)$ for all tree elements $x <_T y$.

Call a tree $T$ a $\kappa$-*Souslin tree* if $T$ has height $\kappa$ and every branch and every antichain has cardinality strictly less than $\kappa$. Obviously, a $\kappa$-Souslin tree is a $\kappa$-Aronszajn tree. However, a special $\kappa^+$-Aronszajn tree is not Souslin, because $A_\nu := \sigma^{-1}{}''\{\nu\}$ are antichains by the property of $\sigma$ defined above. But $\bigcup_{\nu<\kappa} A_\nu = T$ and $|T| = \kappa^+$.

**Lemma 26** ([Kan94]). *If $\kappa$ is regular and $2^{<\kappa} = \kappa$, then there is a $\kappa^+$-Aronszajn tree.*

There is an important connection to the theory of trees that we will use in Chapter 5 and give a proof in the appendix.

**Lemma 27** ([Dev84, Jen72]). *There is a special $\kappa^+$-Aronszajn tree if and only if $\Box^*_\kappa$ holds.*

The idea of the proof is easy to understand: Having a $\kappa^+$-Aronszajn tree we can consider suitable subsets of branches of the tree. The restrictions of the tree, having no cofinal branches and each tree level has cardinality at most $\kappa$, helps to prove to get a $\Box_\kappa$-sequence.

On the other hand, imitating the proof of an $\aleph_1$-Aronszajn tree, we now need the $\Box_\kappa$-sequence to survive the limit points during the construction without taking to many branches on such levels.

⋆ ⋆ ⋆

Remember, as in the first chapter, we call a model $\mathfrak{A} = \langle \mathbb{A}; A, \dots \rangle$ a $(\kappa, \lambda)$-model of a language $\mathcal{L} = \{\dot{A}, \dots\}$, if $|\mathbb{A}| = \kappa$ and $|A| = \lambda$. Then we have the following

**Lemma 28** ([ChaKei90, Theorem 7.2.11]). *There is a sentence $\varphi$ in a finite language such that for all infinite cardinals $\kappa$, $\varphi$ has a $(\kappa^+, \kappa)$-model if and only if there exists a special $\kappa^+$-Aronszajn tree.*

For, let $\mathcal{L} = \{U, T, <, r, f, g, h\}$ where $U$ is a unary relation, $T$ and $<$ are binary relations, $r$ and $f$ are unary functions and, finally, $g$ and $h$ are binary functions.

Then let $\varphi$ be the sentence of the language $\mathcal{L}$, saying that

(a) the relation $T$ acts like a tree, meaning that the partial order $\langle\, \mathrm{dom}(T) \cup \mathrm{rng}(T), T\, \rangle$ is a tree in the usual sense; the relation $<$ is a linear order; and the relation $U$ is an initial segment for $<$, that is, $\exists x\, \forall y\, (\, U(y) \longleftrightarrow y < x\, )$,

(b) the function $r$ acts like the tree order function or rank function, that is $x\, T\, y \longrightarrow r(x) < r(y)$, $\forall z\, \exists x\, (\, r(x) = z\, )$ and $z < r(y) \longrightarrow \exists x\, (xTy \wedge r(x) = z)$; and the function $f$ acts like the function for a special Aronszajn tree, that is $\forall x\, U(\, f(x)\, )$ and $xTy \longrightarrow f(x) \neq f(y)$,

(c) use the function $g$ to assert that for each $x$, the set of all predecessors $\{y | y < x\}$ has cardinality at most $|U|$; and finally, use the function $h$ to assert that for each $x$, the set of all elements of the same rank $\{y \mid r(y) = x\}$ has cardinality at most $|U|$.

It is an easy exercise to show that this sentence $\varphi$ will work to prove the lemma above.

To round up the theory we remind the reader of the following

**Lemma 29** ([Dev84]). *Assume* GCH*. Let $\kappa$ be an uncountable cardinal for which $\Box_\kappa$ holds. Then there is a $\kappa^+$-Souslin tree.*

Moreover, in [Jen72, p.286, Remark (3)], Jensen showed the following fact which we will use later

**Lemma 30** ([Jen72]). *If $\kappa^+$ is not Mahlo in $\mathbf{L}$, then $\Box_\kappa$ holds.*

In fact, we can prove the following generalization

**Lemma 31** ([Jen72]). *Suppose $\kappa^+$ is not a Mahlo cardinal in $\mathbf{L}[B]$ for a subset $B \subseteq \aleph_1$, then $\Box_\kappa$ holds.*

### Primitive Recursive Functions

The well-known primitive recursive functions on the natural numbers can be generalized to primitive recursive functions on ordinals. The easiest way here is to consider the canonical functions like successor function, addition, multiplication, taking powers and taking iterated powers.

Then we call an ordinal $\alpha$ *primitive recursive closed* if it is closed under these ordinal functions restricted to $\alpha$.

On the other hand we can generalize these functions to sets (not only ordinals) as are given by the next

**Definition 32** (Primitive Recursive Functions). *A (class) function $f : \mathbf{V}^n \longrightarrow \mathbf{V}$ is said to be primitive recursive (p.r.) if and only if it is generated by the following schemata:*

(a) $f(x_1, \ldots, x_n) = x_i$ *for* $1 \leqslant i \leqslant n$,

(b) $f(x_1, \ldots, x_n) = \{x_i, x_j\}$ *for* $1 \leqslant i, j \leqslant n$,

(c) $f(x_1, \ldots, x_n) = x_i \backslash x_j$ *for* $1 \leqslant i, j \leqslant n$,

(d) $f(x_1, \ldots, x_n) = h(\ g_1(x_1, \ldots, x_n), \ldots, g_k(x_1, \ldots, x_n)\ )$, *where* $h$, $g_1$,…,$g_k$ *are all p.r.*,

(e) $f(y, x_1, \ldots, x_n) = \bigcup_{z \in y} g(z, x_1, \ldots, x_n)$, *where* $g$ *is p.r.*,

(f) $f(x_1, \ldots, x_n) = \omega$,

(g) $f(y, x_1, \ldots, x_n) = g(y, x_1, \ldots, x_n, \langle f(z, x_1, \ldots, x_n) | z \in h(y) \rangle)$, *where* $g$ *and* $h$ *are p.r. and, moreover,*

$$z \in h(y) \ \longrightarrow\ \mathrm{rank}(z) < \mathrm{rank}(y).$$

In fact, in [JenKar71], it is shown that $\alpha$ is closed under ordinal primitive recursive functions if and only if $\mathbf{L}_\alpha$ is closed under the primitive recursive functions on general sets.

Even a rather complex function like $\langle \mathbf{L}_\nu | \nu \in \mathrm{On} \rangle$ is primitive recursive. This means that a level of the constructible universe with height a p.r.-closed ordinal has in most cases enough set theory within to work with. We are going to use such arguments in Chapter 5.

We will now look on cardinals and try to find many p.r.-closed ordinals.

**Lemma 33.** *Let $\kappa > \omega$ be a regular cardinal. Then $\kappa$ is p.r.-closed and there are cofinal many p.r.-closed ordinals below $\kappa$.*

*Proof.* Obviously, as a $\mathsf{ZF}^-$-model, $\mathbf{L}_\kappa$ is closed under functions of Definition 32. Moreover, let $\gamma_0 < \kappa$ be chosen. Then we can define $\gamma_{i+1}$ as the smallest $\gamma$ such that the union of all ranges of functions given by Definition 32 restricted to $\mathbf{L}_{\gamma_i}$ is a subset of $\mathbf{L}_\gamma$. Then $\sup_{i<\omega} \gamma_i < \kappa$ is p.r.-closed. ☒(Lemma 33)

Then the same argument proves that for a $\mathsf{ZF}^-$-model $\mathfrak{A}$ there are cofinal many p.r.-closed ordinals in $\mathrm{On}^{\mathfrak{A}}$ and even the following

**Corollary 34.** *Let $\mathfrak{A}$ be a model $\mathsf{ZF}^-$. Then there are cofinal many limits of p.r.-closed ordinals in $\mathrm{On}^{\mathfrak{A}}$.*

The next rather technical statement will allow us later to find p.r.-closed ordinals in elementary submodels. In fact, we are arguing to get finally Lemma 37 at the end of this section. For, let us define for $i < \omega$ and ordinals $\nu$:

$$\begin{aligned} g_0(\nu) &:= \nu + 1, \\ g_{i+1}(\nu) &:= g_i^{\nu+1}(\nu + 1). \end{aligned}$$

Here, the iterated power is defined in the usual way by induction on non-empty ordinals: let $g_i^1(\mu) := g_i(\mu)$, $g_i^{\nu'+1}(\mu) := g_i(\, g_i^{\nu'}(\mu)\, )$ and finally $g_i^\lambda(\mu) := \sup_{\nu'<\lambda} g_i^{\nu'}(\mu)$ for limit ordinals $\lambda$. Then these functions are obviously primitive recursive. Moreover, we have the following

**Lemma 35** ([JenKar71])**.** *Let $F$ be a p.r. set function. Then there is a $\Sigma_1$-formula $\varphi$ such that whenever $x_1, \ldots, x_n \in \mathbf{L}_\alpha[A]$ there is $i < \omega$ such that*

$$y = F(x_1, \ldots, x_n) \quad \Longleftrightarrow \quad \mathbf{L}_{g_i(\alpha)}[A] \models \varphi(y, x_1, \ldots, x_n).$$

This statement ensures that each primitive recursive function can be caught by the $g_i$-functions as rank in the **L**-hierarchy. But then we now have

**Corollary 36.** $\mathbf{L}_\alpha[A]$ *is p.r.-closed if and only if* $\alpha$ *is closed under the functions* $g_i$ *for* $i < \omega$.

Now, let $\varphi_i$ be the formula for the function $g_i$ given by Lemma 35. Then we have that $\alpha$ is p.r.-closed if and only if for all $i < \omega$ we have

$$\mathbf{L}_\alpha[A] \models \forall\nu\exists\xi\, \varphi_i(\xi, \nu).$$

Because even the sequence $\langle \varphi_i \mid i < \omega \rangle$ is p.r.-closed, we finally can code altogether in one formula, saying "On is p.r.-closed":

**Lemma 37** ([JenKar71])**.** *There is a formula* $\varphi$ *such that* $\alpha$ *is p.r.-closed if and only if* $\mathbf{L}_\alpha \models \varphi$.

## (Small) Large Cardinals

And finally we state an equivalence we will find useful in our construction later. Remember, we call a cardinal $\kappa$ *(strongly) inaccessible* if $\kappa$ is regular and for all $\lambda < \kappa$ we have $2^\lambda < \kappa$.

Call $\kappa$ *Mahlo* if the set $\{ \gamma < \kappa \mid \gamma \text{ is inaccessible} \}$ is stationary in $\kappa$, and finally, call a cardinal $\kappa$ *weakly compact* if the partition relation $\kappa \longrightarrow (\kappa)^2_2$ holds.

Here, $\kappa \longrightarrow (\kappa)^2_2$ means that every partition of $[\kappa]^2$, the set of all unordered pairs of $\kappa$, into two pieces has a homogeneous set of size $\kappa$. We refer to [Dra74, Jec03, Kan94] for more details.

⋆ ⋆ ⋆

In future facts we will use the following reflecting properties of the constructible universe:

**Lemma 38.** *The following hold:*

(a) *If $\kappa$ is a regular cardinal, then ($\kappa$ is a regular cardinal)$^{\mathbf{L}}$,*

(b) *If $\kappa$ is inaccessible, then ($\kappa$ is inaccessible)$^{\mathbf{L}}$,*

(c) *If $\kappa$ is Mahlo, then ($\kappa$ is Mahlo)$^{\mathbf{L}}$,*

(d) *If $\kappa$ is weakly compact, then ($\kappa$ is weakly compact)$^{\mathbf{L}}$.*

Finally, we state two well-known connections between large cardinals and the (non-)existence of trees:

**Lemma 39** ([Dev84, Jen72])**.** *If $\kappa$ is a regular uncountable cardinal which is not Mahlo in the constructible universe, then there is a constructible special Aronszajn tree of height $\kappa$.*

The proof uses arguments about combinatorical principles given by, *e.g.*, Lemma 27 and Lemma 30. Another well-known fact is the following:

**Lemma 40** ([Dev84, Dra74, Jec03, Kan94])**.** *Let $\kappa$ be an uncountable cardinal. The following are equivalent:*

(a) *$\kappa$ is weakly compact,*

(b) *$\kappa$ is inaccessible and there is no $\kappa$-Aronzsajn tree.*

CHAPTER 3

# The Forcing

We now define the notion of forcing we are going to use. In fact, it will be Mitchell's forcing that he introduced in [Mit72]. There he used it to prove the statement we are trying to improve with Theorem 59.

To start with, fix $\kappa$ an uncountable regular cardinal and $\tau > \kappa$ an inaccessible one in a (suitable) ground model for the remaining part of this chapter.

**Definition 41.** *Let* $\mathbb{P} := \mathbb{P}(\tau)$ *be* $\{\, p \,:\, \exists x\, (p \in {}^{x}2 \,\wedge\, |x| < \omega \,\wedge\, x \subseteq \tau\,\}$, *ordered by the usual reverse inclusion.*

Then the application of $\mathbb{P}$ adjoints in the usual way $\tau$-many (Cohen) reals. Now, the second forcing looks a bit more technical.

For $\alpha < \tau$, let $\mathbb{P}_\alpha := \{\, p \in \mathbb{P} \mid p \restriction \alpha = p \,\}$. If $s \subseteq \mathbb{P}$, then define

$$b_s := \{\, p \in \mathbb{P} \mid \forall q \leqslant_{\mathbb{P}} p\, \exists r \in s\, (\, r \text{ and } q \text{ are compatible}\,)\,\},$$

or equivalently, $b_s = \{\, p \in \mathbb{P} \mid p \Vdash_{\mathbb{P}} \bar{G} \cap s \neq \varnothing \,\}$ for a $\mathbb{P}$-generic $\bar{G}$ over the ground model $M$.

Let $\mathcal{B}$ be the boolean algebra associated with $\mathbb{P}$. Define then $\mathcal{B}_\alpha \subseteq \mathcal{B}$ by $\mathcal{B}_\alpha := \{\, b_s \mid s \subseteq \mathbb{P}_\alpha \,\}$. Then $\mathcal{B}_\alpha$ is canonically isomorphic to the complete boolean algebra associated with $\mathbb{P}_\alpha$.

Call a function $f \in M$ *acceptable*, if the following conditions hold:

(a) $\operatorname{dom}(f) \subseteq \tau$ and $\operatorname{rng}(f) \subseteq \mathcal{B}$,
(b) $|\operatorname{dom}(f)| < \kappa$,
(c) for all $\gamma < \tau$ we have $f(\gamma) \in \mathcal{B}_{\gamma+\omega}$.

The bounding property given by the last property will give us control of the time when we will have collapsed cardinals with the second forcing. This important fact was used by Mitchell in [Mit72] as we will see later.

Let $\mathcal{A}$ be the set of all acceptable functions in $M$. Moreover, let $\bar{G}$ be $\mathbb{P}$-generic and let $M'$ be $M[\bar{G}]$, the forcing extension by $\mathbb{P}$ and also ground model for second forcing $\dot{\mathbb{Q}}$. Then for $f \in \mathcal{A}$ define $\bar{f} : \mathrm{dom}(f) \longrightarrow 2$ in $M'$ by $\bar{f}(\gamma) = 1$ if and only if $f(\gamma) \cap \bar{G} \neq \varnothing$.

**Definition 42.** *With the notation above let* $\dot{\mathbb{Q}} := \dot{\mathbb{Q}}(\kappa, \tau)$ *be defined over the model* $M'$ *by letting the field be the set* $\mathcal{A}$ *of acceptable functions and letting* $f \leqslant_{\dot{\mathbb{Q}}} g$ *if and only if* $\bar{f} \supseteq \bar{g}$.

Note, although the field of $\dot{\mathbb{Q}}$ is a subset of the ground model $M$, the definition of the order $\leqslant_{\dot{\mathbb{Q}}}$ is using the $\mathbb{P}$-generic object $G$.

Finally, we can now put both forcings together. Note, $\dot{\mathbb{Q}}$ is defined in $M'$, a generic extension of $\mathbb{P}$. We, therefore, denote this partial order with a dot, to signify that we are using a name for it.

We then define the notion of forcing we are interested in:

**Definition 43** (Mitchell)**.** *Let* $\mathbb{M}(\kappa, \tau)$ *be the usual two-step product* $\mathbb{P} \star \dot{\mathbb{Q}}$ *of the forcings* $\mathbb{P}$ *and* $\dot{\mathbb{Q}}$ *defined above, that is*

$$\begin{aligned} \mathbb{M}(\kappa, \tau) \quad &:= \quad \mathbb{P} \times \mathcal{A}, \\ (p, f) \leqslant_{\mathbb{M}(\kappa,\tau)} (q, g) \quad &:\Longleftrightarrow \quad p \leqslant_{\mathbb{P}} q \ \text{ and } p \Vdash_{\mathbb{P}} f \leqslant_{\dot{\mathbb{Q}}} g. \end{aligned}$$

In the following we will cite a few lemmas proven in [Mit72]. Actually, Mitchell defines simultanously two such forcings depending which problem they should solve. We are using the second one, in fact, our $\mathbb{M}(\kappa, \tau)$ is his $R_2(\omega, \kappa, \tau)$ and we are stating the lemmas in our terminology.

Fix apart from $\bar{G}$ now also a $\dot{\mathbb{Q}}$-generic $\tilde{G}$ over $M'$ and set $N$ be the forcings extension $M[\bar{G}][\tilde{G}]$. We already know by elementary forcing arguments that then $\bar{G} \times \tilde{G}$ is $\mathbb{M}$-generic over the ground model $M$, *cf.* [Jec03, Kun80] for details.

**Lemma 44** ([Mit72]). *In the first step of the forcing using $\mathbb{P}$, we adjoin $\tau$-many reals to $M$, however, cardinals are preserved. In the second step we collapse $\tau$ to $\kappa^+$.*

One of the major observations for our desired preserving properties is the following:

**Lemma 45** ([Mit72, Lemma 3.3]). *The notion of forcing $\mathbb{M}(\kappa,\tau)$ has the $\tau$-chain condition.*

Therefore we get the following consequence:

**Corollary 46** ([Mit72, Lemma 3.4]). *For all ordinals $\delta$ such that $\delta \leqslant \kappa$ or $\tau \leqslant \delta$ we have $|\delta|^M = |\delta|^N$. Hence cardinals below $\kappa$ and above $\tau$ are preserved.*

And finally we can conclude:

**Corollary 47** ([Mit72, Corollary 3.5]). *In the forcing extension $N$ we have $2^\omega = 2^\kappa = \tau$.*

⋆ ⋆ ⋆

We will now try to look on the forcing construction in a rather different way. In fact, we will split it off in $\tau$-many parts $\mathbb{M}_\nu$ and $\mathbb{M}^\nu$ for $\nu < \tau$ where $\mathbb{M}_\nu$ will consists of conditions $p$ of the forcing $\mathbb{M}$ such that "$p$ works below $\nu$" and similarly for $\mathbb{M}^\nu$. In fact, $\mathbb{M}_\nu$ will add subsets of $\omega$ which can be described with conditions below $\nu$ and then we will collapse all ordinals below $\nu$ to $\kappa$. Of course this is only interesting in the case that $\nu$ is greater than $\kappa$.

Therefore we set

$$\mathbb{P}_\nu := \{\, p \in \mathbb{M} \;:\; p \restriction \nu = p \,\}, \qquad \mathbb{P}^\nu := \{\, p \in \mathbb{M} \;:\; p \restriction \nu = 0 \,\},$$
$$\mathcal{A}_\nu := \{\, f \in \mathcal{A} \;:\; f \restriction \nu = f \,\}, \qquad \mathcal{A}^\nu := \{\, f \in \mathcal{A} \;:\; f \restriction \nu = 0 \,\}.$$

And moreover

$$\mathbb{M}_\nu := \mathbb{P}_\nu \times \mathcal{A}_\nu, \qquad \mathbb{M}^\nu := \mathbb{P}^\nu \times \mathcal{A}^\nu,$$
$$G_\nu := G \cap \mathbb{M}_\nu, \qquad G^\nu := G \cap \mathbb{M}^\nu.$$

As noticed above we finally can prove the following

**Lemma 48** ([Mit72, Lemma 3.6])**.** *Let $\nu < \tau$ be a limit ordinal. Then $G_\nu$ is $\mathbb{M}_\nu$-generic over $M$, $K^\nu$ is $\mathbb{M}^\nu$-generic over $M[G_\nu]$, and $M[G_\nu][G^\nu] = M[G]$.*

The by far most important tool in analyzing Mitchell's notion of forcing is the following, providing that sequences of length with an uncountable cofinality such that their initial segments can be found in an initial segment of the forcing construction, in fact, are already an element of this segment:

**Lemma 49** ([Mit72, Lemma 3.8])**.** *Suppose that $\gamma$ has uncountable cofinality in the ground model $M$ and let $t : \gamma \longrightarrow M$ be such that $t \in M[G]$ and $t \restriction \alpha \in M[G_\nu]$ for every $\alpha < \nu$. Then $t \in M[G_\nu]$.*

⋆ ⋆ ⋆

Now we turn back to the question *when* exactly we add new reals. We already know that with $\mathbb{M}(\kappa, \tau)$ we add $\tau$-many Cohen reals because of the first forcing part. However, we can ask whether the second forcing is changing the powerset of $\omega$ again. In fact, it will not as we will prove with a tool that Mitchell has proved with the following

**Lemma 50** ([Mit72, Lemma 3.1])**.** *Suppose that $\dot{D}$ is a term such that*

$$\Vdash_{\mathbb{P}} (\ \dot{D} \text{ is strongly dense in } \dot{\mathbb{Q}}\ ),$$

*and $f \in \mathcal{A}$. Then there is $g \in \mathcal{A}$ such that $g \supseteq f$ and $p \Vdash_{\mathbb{P}} (\ g \in \dot{D}\ )$.*

And so, we are able to prove finally

**Lemma 51.** $\mathcal{P}(\omega) \cap M[\bar{G}] = \mathcal{P}(\omega) \cap M[G]$.

*Proof.* The first inclusion $\subseteq$ is obvious. Suppose now the other one does not hold. We are going to deduce a contradiction.

So suppose there is a subset $a$ of $\omega$ such that $a$ is an element of $M[G]$ but not in $M[\bar{G}]$, then using the theory of forcing there is a condition $f \in \dot{\mathbb{Q}}$ such that

$$M[\bar{G}] \models (\ \check{f} \Vdash_{\dot{\mathbb{Q}}} (\exists a \subseteq \omega)(a \notin \check{\mathbf{V}})\ ).$$

Then there is already a condition $p \in G \subseteq \mathbb{P}$ such that

$$p \Vdash_{\mathbb{P}} (\check{f} \Vdash_{\dot{\mathbb{Q}}} (\exists a \subseteq \omega)(a \notin \check{\mathbf{V}})).$$

So we can find a $\mathbb{P}$-name for a $\dot{\mathbb{Q}}$-name $\dot{a}$ such that

$$p \Vdash_{\mathbb{P}} (\check{f} \Vdash_{\dot{\mathbb{Q}}} (\dot{a} \subseteq \omega \ \wedge \ \dot{a} \notin \check{\mathbf{V}})).$$

Now consider the set

$$\{ g \in \dot{\mathbb{Q}} \mid (g \leqslant_{\dot{\mathbb{Q}}} \check{f} \ \wedge \ (g \Vdash \check{i} \in \dot{a} \vee g \Vdash \check{i} \notin \dot{a})) \ \vee \ g \perp \check{f} \}$$

and let $\dot{D}_i$ be a $\mathbb{P}$-name for it. Then we can conclude that

$$p \Vdash_{\mathbb{P}} (\dot{D}_i \text{ is dense in } \dot{\mathbb{Q}}).$$

Now, step by step using Lemma 50 we can find $f_i \in \mathcal{A}$ for $i < \omega$ such that $f_0 := f$, $f_{i+1} \supseteq f_i$ and $p \Vdash_{\mathbb{P}} \check{f}_{i+1} \in \dot{D}_i$. Finally define $f^* := \bigcup_{i<\omega} f_i$. Then we obviously have $p \Vdash \check{f}^* \leqslant_{\dot{\mathbb{Q}}} \check{f}_i$ for all $i < \omega$. Hence

$$M[G] \models (\check{f}^* \leqslant_{\dot{\mathbb{Q}}} \check{f} \ \wedge \ \check{f}^* \in \bigcap_{i<\omega} \dot{D}_i^G \ \wedge \ \check{f}^* \in \dot{\mathbb{Q}}).$$

Therefore, for each $i < \omega$ in $M[\bar{G}]$, already $\check{f}^*$ knows about whether $\check{i}$ is in $\dot{a}^G$ or not, that is

$$M[\bar{G}] \models (\check{f}^* \Vdash_{\dot{\mathbb{Q}}} \check{i} \in \dot{a}^G \ \vee \ \check{f}^* \Vdash_{\dot{\mathbb{Q}}} \check{i} \notin \dot{a}^G).$$

Define then in the ground model $M$,

$$b := \{ i < \omega \mid M[\bar{G}] \models (\check{f}^* \Vdash_{\dot{\mathbb{Q}}} \check{i} \in \dot{a}^G) \} \in M.$$

But then we have $\check{f}^* \Vdash \dot{a}^G = b \ \wedge \ \check{b} \in \check{\mathbf{V}}$. Hence $\check{f}^* \Vdash \dot{a}^G \in \check{\mathbf{V}}$.

However, we also have $\check{f}^* \Vdash \dot{a}^G \notin \check{\mathbf{V}}$ because of $\check{f}^* \leqslant_{\dot{\mathbb{Q}}} \check{f}$ and so we have deduced the desired contradiction. ⊠(Lemma 51)

Note, it is a similiar argument like using the property of a partial order being $\aleph_1$-closed. However, our forcing $\dot{\mathbb{Q}}$ does, in fact, not bear this property in $M[\bar{G}]$ where we need it. Although the conditions, functions within $\mathcal{A}$, live in the ground model, the order is defined using the generic object $\bar{G}$ and this causes the problems together with the fact that we added many countable subsets to $M$ when we got $M[\bar{G}]$.

Finally let us turn to a property of Mitchell's forcing that it does not kill stationary subsets of $\tau$.

**Lemma 52.** *Let $\mathbb{M} = \mathbb{M}(\kappa, \tau)$ be Mitchell's forcing and let $G$ be $\mathbb{M}$-generic over a ground model $M$. Then, in the extension $N = M[G]$, $S$ remains a stationary subset of $\tau$, where*

$$S := \{ \lambda < \tau \mid M \models \mathrm{cf}(\lambda) = \kappa^+ \}.$$

*Proof.* Let $\dot{C} \subseteq \check{\tau}$ be closed and unbounded. We are going to prove that

$$\dot{C}^G \cap \{ \lambda < \tau \mid M \models \mathrm{cf}(\lambda) = \kappa^+ \} \neq \varnothing.$$

For, let $\dot{\gamma}$ be a name of a monotone enumeration of $\dot{C}$, that is

$$\Vdash_{\mathbb{M}} \dot{C} = \langle \dot{\gamma}(\xi) \mid \xi < \check{\tau} \rangle \wedge \dot{\gamma} \text{ is monotone.}$$

Moreover, let $D_\nu$ be a maximal antichain in $\{ p \mid \exists \alpha\, p \Vdash_{\mathbb{M}} \dot{\gamma}(\check{\nu}) = \check{\alpha} \}$. Because of Lemma 45 we know that $|D_\nu| < \tau$. Now, for conditions $p \in D_\nu$ define $\gamma_{\nu,p}$ as the unique $\alpha$ such that $p \Vdash_{\mathbb{M}} \dot{\gamma}(\check{\nu}) = \check{\alpha}$ and, finally, $\Gamma_\nu := \{ \gamma_{\nu,p} \mid p \in D_\nu \}$. Then

$$\dot{\gamma}^G(\nu) \in \Gamma_\nu \in M. \tag{1}$$

In the ground model $M$, define a sequence $\langle \beta_\xi \mid \xi \leqslant \kappa^+ \rangle$ by setting $\beta := 0$, $\beta_\lambda := \bigcup_{\xi<\lambda} \beta_\xi$ and more interesting

$$\beta_{\xi+1} := \mathrm{lub}(\bigcup\{\Gamma_\nu \mid \nu < \beta_\xi\}).$$

Here, 'lub' means 'least upper bound'. Finally we have for $\beta := \beta_{\kappa^+}$ that $\bigcup_{\nu<\beta} \Gamma_\nu \subseteq \beta$ and $\mathrm{cf}(\beta) = \kappa^+$.

Then, in $M[G]$, the defined $\beta$ is an element of $\dot{C}^G$. For, let $\xi < \beta$. By our construction we have

$$\xi \leqslant \dot{\gamma}^G(\xi) < \Gamma_\xi \subseteq \bigcup\{\Gamma_\nu \mid \nu < \beta_{\xi+1}\} \leqslant \beta_{\xi+2} < \beta.$$

Hence, $\beta$ is a limit point of $\dot{C}^G$ because of (1) and so also an element of the closed set $\dot{C}^G$.

Therefore, $\Vdash_{\mathbb{M}} \check{\beta} \in \dot{C} \cap S$ and the lemma is proved. $\boxtimes$(Lemma 52)

CHAPTER 4

# The Coarse Morass

From now on consider a first order language $\mathcal{L}$ with a unary predicate $\dot{A}$ and a binary predicate $\dot{\in}$. Define the theory $\mathfrak{T}$ as follows:

$$\mathfrak{T} \quad := \quad \mathsf{ZFC}^- + \mathbf{V} = \mathbf{L} + \dot{A} \text{ regular} \\ + \dot{A} \text{ is the largest cardinal.}$$

Moreover, by a $(\tau,\kappa)$-model of $\mathcal{L}$ we understand a model of the shape $\mathfrak{A} = \langle \mathbb{A}; \dot{A}_{\mathfrak{A}}, \dot{\in}_{\mathfrak{A}}, \dots \rangle$ such that $|\mathbb{A}| = \tau$ and $|\dot{A}_{\mathfrak{A}}| = \kappa$. In the following we are working inside a model $\mathfrak{A}$ of our fixed theory $\mathfrak{T}$. Let $A = \dot{A}_{\mathfrak{A}}$ be the interpretation of $\dot{A}$ in $\mathfrak{A}$.

We will benefit from this theory in the next chapter, proving our main theory. For, we will use a tool, the so-called *coarse $A$-morass*. To be able to, we are going to define the structure theory in this chapter and prove facts we are going to apply later. This is not at all to understand as an introduction to the theory of morasses[1]. We will develop methods we are going to use in the next chapter.

Once and for all, in this chapter we are working within the fixed model $\mathfrak{A}$, otherwise we will state the opposite clearly. Note, because $\mathfrak{A}$ could be very different from our universe, possibly being ill-founded, and so forthcomming arguments will rarely be absolut.

[1] Only a few introductions to the theory of morasses can be found in the literatur — although they are sometimes used as a tool in the theory of inner models, proving statements around the cardinal transfer property. As a starting point we strongly refer to [Dev84].

At first, define sets $S_\alpha$ for ordinals $\alpha \leqslant A$ as follows:

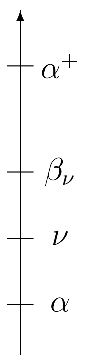
$$\begin{aligned} S_\alpha := \{ \nu \mid \ & \alpha < \nu < \alpha^+ \\ & \wedge \ \mathbf{L}_\nu \models (\alpha \text{ is the largest cardinal} \wedge \alpha \text{ is regular}) \\ & \wedge \ \forall \xi < \nu \, \exists \eta < \nu \, (\xi < \eta \ \wedge \ \eta \text{ is p.r.-closed}) \, \} \end{aligned}$$

Consider the set $S_A$. Because the fixed model $\mathfrak{A}$ of $\mathfrak{T}$ thinks that $A$ is regular and the largest cardinal and, by Corollary 34, there are indeed cofinal many limits of p.r.-closed ordinals within $\mathfrak{A}$, we obviously have

**Lemma 53.** $\sup S_A = \infty$.

Let us now look at these intervals $S_\alpha$ more closely and define for $\nu \in S_\alpha$

$$\begin{aligned} \beta_\nu \quad := \quad & \text{the smallest } \beta > \nu \text{ such that} \\ & \mathbf{L}_\beta \models |\nu| \leqslant \alpha \text{ and } \beta \text{ is p.r.-closed.} \end{aligned}$$




Then $\beta_\nu$ is well-defined for $\nu \in S_\nu$: Because $\nu$ lays between $\alpha$ and $\alpha^+$, it must be collapsed at some ordinal level $\xi < \alpha^+$. Of course, $\xi$ has not to be p.r.-closed. However, taking the closure of all (ordinal) primitive functions, possible by a countable limit construction, we get an ordinal $\beta$ (and therefore $\mathbf{L}_\beta$) satisfying the desired condition by Lemma 37. So we can choose a minimal one.

Although defined for all ordinals $\alpha$ below $A$ we can at least show that for a large set of indexes $\alpha$ the intervals $S_\alpha$ are non-empty, in fact, we will find a closed and unbounded set:

**Lemma 54.** *The set* $\{\alpha < A \mid S_\alpha \neq \varnothing \}$ *is stationary in* $A$.

*Proof.* By Lemma 53, $S_A$ is a non-empty set and so we can fix an arbitrary $\nu \in S_A$. We now define simultanously the following two sequences $\langle\, \alpha_\xi \mid \xi < A \,\rangle$ and $\langle\, X_\xi \mid \xi \leqslant A \,\rangle$, letting

$$\begin{aligned}
\alpha_\xi &:= X_\xi \cap A, \\
X_0 &:= \text{ the smallest } X \prec \mathbf{L}_{\beta_\nu} \text{ where } X \cap A \text{ transitive}, \\
X_{\xi+1} &:= \text{ the smallest } X \prec \mathbf{L}_{\beta_\nu} \text{ where } X_\xi \cap A \text{ transitive} \\
&\quad\; \text{ and } \alpha_\nu \in X, \\
X_\lambda &:= \bigcup\{X_\xi \mid \xi < \lambda\} \text{ for limit ordinals } \lambda.
\end{aligned}$$

Here, by 'smallest (elementary) submodel' we mean to take the submodel such that it is minimal for the inclusion relation. Then we obviously have $\alpha_\lambda = \sup_{\xi<\lambda} \alpha_\xi$.

Now, for each $\xi \leqslant A$ let $\pi : \mathbf{L}_{\beta(\xi)} \overset{\sim}{\longleftrightarrow} X_\xi$ be the Mostowski collapse. By construction we then have $\pi(\alpha_\xi) = A$. Moreover, $\pi$ is an elementary embedding of $\mathbf{L}_{\beta(\xi)}$ into $\mathbf{L}_{\beta_\nu}$. Let $\nu(\xi) \in \mathbf{L}_{\beta_\xi}$ such that $\pi(\ \nu(\xi)\ ) = \nu$. Then –because of the elementary property of $\pi$ and $\nu \in S_\alpha$– we also have that $\nu(\xi)$ lays in $S_{\alpha_\xi}$. In particular, $S_{\alpha_\xi}$ is not empty.

Finally, by our construction we have found a club set $\{\alpha_\xi \mid \xi < A\}$, witnessing the stationarity claimed. ⊠(Lemma 54)

On the set $S$, defined as the union $\bigcup_{\alpha\leqslant A} S_\alpha$ of the intervalls defined above, we will define a relation $\lhd$ and a suitable sequence of elementary embeddings $\langle\, \pi_{\bar\nu\nu} \mid \bar\nu \lhd \nu \,\rangle$ such that

$$\pi_{\bar\nu\nu} : \mathbf{L}_{\bar\nu} \longrightarrow \mathbf{L}_\nu \text{ and } \lhd \subseteq S \times S.$$

For, define $\alpha_\nu$ as the unique $\alpha$ such that $\nu \in S_\alpha$ and define

$$\begin{aligned}
\bar\nu \lhd \nu \quad :\Longleftrightarrow \qquad & (\ \alpha_{\bar\nu} < \alpha_\nu \ \wedge\ \pi(\bar\nu) = \nu \ \wedge\ \pi(\alpha_{\bar\nu}) = \alpha_\nu \ \wedge \\
& \quad \text{there is } \pi : \mathbf{L}_{\beta_{\bar\nu}} \prec \mathbf{L}_{\beta_\nu} \text{ such that } \mathrm{crit}(\pi) = \alpha_{\bar\nu}\ ).
\end{aligned}$$

And finally set $\pi_{\bar\nu\nu} := \pi \restriction \mathbf{L}_{\bar\nu}$.

Note, in the proof of Lemma 54 we actually showed that (in notation of the proof) we have $\pi_{\nu(\xi)\nu} = \pi \restriction \mathbf{L}_{\nu(\xi)}$ and so, we showed even more, namely $\nu(\xi) \lhd \nu$.

In fact, we can prove the following

**Lemma 55.** *The maps $\pi_{\bar{\nu}\nu}$ defined above are unique.*

*Proof.* Here we use the fact that we are working with **L**-like models and their definability properties. So let us define

$$X := \quad \text{the set of all } y \in \mathbf{L}_{\beta_\nu} \text{ such that } y \text{ is } \mathbf{L}_{\beta_\nu}\text{-definable using parameters from } \{\nu, \alpha_\nu\} \cup \alpha_\nu.$$

Then we know by the condensation property that $X$ is already a level of the $\mathbf{L}_\alpha$-hierarchy. Moreover, we can even describe the ordinal height of $X$ as follows

*Claim.* $X = \mathbf{L}_{\beta_\nu}$.

First note that, by definition, $X \prec \mathbf{L}_{\beta_\nu}$. Obviously, in $\mathbf{L}_{\beta_\nu}$ holds

$$(\exists f)\,(f : \alpha_\nu \xrightarrow{\text{onto}} \nu),$$

so it does in the substructure $X$. That means, there is an element $f$ of $X$ such that $\mathbf{L}_{\beta_\nu} \models f : \alpha_\nu \xrightarrow{\text{onto}} \nu$. So, by absolutness there is an $f$ in $X$ such that $f : \alpha_\nu \xrightarrow{\text{onto}} \nu$. Hence, for this $f$ we have $\text{dom}(f) = \alpha_\nu \subseteq X$ and so finally $\nu = \text{rng}(f) \subseteq X$.

Now, let $\sigma : \mathbf{L}_{\bar{\beta}} \overset{\sim}{\longleftrightarrow} X$ be the collapsing map. We just showed that $\nu$ is a subset of $X$ and therefore $\sigma$ restricted to $\nu + 1$ is the identity map. We also have $\bar{\beta} > \alpha_\nu$ is p.r.-closed and $\mathbf{L}_{\bar{\beta}} \models |\nu| \leqslant \alpha_\nu$ and so we have, because of the minimality property, that $\beta_\nu \leqslant \bar{\beta}$. However, trivially we have by our construction that $\bar{\beta} \leqslant \beta_\nu$ and the claim is proved.

With the claim in mind, there is everything within $\mathbf{L}_{\beta_\nu}$ definable with parameters taken of $\{\nu, \alpha_\nu\} \cup \alpha_\nu$. However, these parameters are fixed by $\pi : \mathbf{L}_{\beta_{\bar{\nu}}} \longrightarrow \mathbf{L}_{\beta_\nu}$. And so, $\pi$ is uniquely given by them. Therefore, $\pi_{\bar{\nu}\nu} = \pi \restriction \mathbf{L}_{\bar{\nu}}$ and the lemma is proved. ⊠(Lemma 55)

We now look at the above defined relation more closely, proving

**Lemma 56.** *The relation $\lhd$ forms a tree on $S$.*

*Proof.* It is an easy exercise to verify that the relation $\lhd$ is non-reflexive and transitive.

Moreover, consider a set $P$ of predecessors of an element of the tree. Then $\langle P, \lhd \rangle$ is obviously well-founded, because if $\bar{\nu} \lhd \nu$, then also $\alpha_{\bar{\nu}} < \alpha_\nu$.

It is left to prove that such a set $P$ is also linearly ordered by $\lhd$. If we showed this, we would even have the missing well-ordering property of a set of predecessors of an element of the tree.

*Claim.* *If* $\bar{\nu}, \nu \lhd \nu$, *then* $\bar{\nu} \trianglelefteq \nu'$ *or* $\nu' \trianglelefteq \bar{\nu}$.

For, let $\bar{\nu} \lhd \nu$ and $\nu' \lhd \nu$. Consider the two maps $\pi : \mathbf{L}_{\beta_{\bar{\nu}}} \longrightarrow \mathbf{L}_{\beta_\nu}$ and $\pi' : \mathbf{L}_{\beta_{\nu'}} \longrightarrow \mathbf{L}_{\beta_\nu}$ given by the definition of the tree where $\pi_{\bar{\nu}\nu} := \pi \restriction \mathbf{L}_{\bar{\nu}}$ and $\pi_{\nu'\nu} := \pi' \restriction \mathbf{L}_{\nu'}$. Then we have by construction (and well-known arguments, *e.g.*, condensation property) the following:

$$
\begin{aligned}
\operatorname{rng}(\pi) &= \text{the smallest } \bar{X} \prec \mathbf{L}_{\beta_\nu} \text{ such that} \\
& \quad A \cap \bar{X} \text{ is transitive and } \alpha_{\bar{\nu}} = A \cap \bar{X}, \\
\operatorname{rng}(\pi') &= \text{the smallest } X' \prec \mathbf{L}_{\beta_\nu} \text{ such that} \\
& \quad A \cap X' \text{ is transitive and } \alpha_{\nu'} = A \cap X'.
\end{aligned}
$$

Without loss of generality, let $\alpha_{\bar{\nu}} \leqslant \alpha_{\nu'}$. But then, $\bar{X}$ is a subset of $X'$. Therefore, $\pi'^{-1} \circ \pi : \mathbf{L}_{\beta_{\bar{\nu}}} \longrightarrow \mathbf{L}_{\beta_\nu}$ is an elementary embedding with the needed properties to conclude that $(\pi'^{-1} \circ \pi) \restriction \mathbf{L}_{\bar{\nu}} = \pi_{\bar{\nu}\nu}$ because of the uniqueness of the embeddings given by Lemma 55. And so, we finally have $\bar{\nu} \trianglelefteq \nu$.

With the claim also the lemma is proved. ⊠(Lemma 56)

There are two more properties we will find later useful. One of them says that there are, in fact, many limit points within the tree relation.

**Lemma 57.** *For* $\alpha \leqslant A$ *let* $\xi, \nu \in S_\alpha$ *where* $\xi < \nu$. *Then*

(a) $\beta_\xi < \nu$,

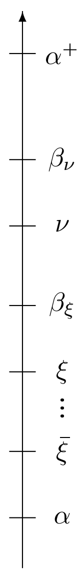
(b) $\sup \{ \bar{\xi} \mid \bar{\xi} \lhd \xi \} = \alpha$.

*Proof.* Let $\alpha$, $\xi$ and $\nu$ be as above. Then by definition, $\alpha$ is the largest cardinal in $\mathbf{L}_\nu$ and therefore we trivially have $\mathbf{L}_\nu \models |\xi| \leqslant \alpha$.

But this holds even below $\nu$, say at stage $\xi' < \nu$, and so $\mathbf{L}_{\xi'} \models |\xi| \leqslant \alpha$. We then are able to find an $\eta < \nu$ but above $\xi'$ such that $\eta$ is p.r.-closed. However, this is the condition $\beta_\xi$ should satisfy. Because of the minimality we finally have $\beta_\xi \leqslant \eta < \nu$. This proves the first fact.

For the second property, note that for $\bar{\xi} \lhd \xi$ we have that the model $\mathbf{L}_\xi$ thinks that $\alpha = \alpha_\xi$ is the largest cardinal and $\mathbf{L}_{\bar{\xi}}$ thinks the same about $\alpha_{\bar{\xi}} < \alpha$. Therefore, $\bar{\xi}$ cannot be greater than $\alpha$ because $\mathbf{L}_{\bar{\xi}}$ is a subset of $\mathbf{L}_\xi$.

On the other hand, let $\gamma < \alpha$. We will find a $\bar{\xi} < \alpha$ such that $\gamma \leqslant \bar{\xi}$ and $\bar{\xi} \lhd \xi$ as follows, working in $\mathbf{L}_\nu$: Starting from $X_0 := \gamma \cup \{\xi\}$ we set $X := \bigcup_{i<\omega} X_i$ where $X_{i+1}$ is the smallest $X'$ such that $X' \prec \mathbf{L}_{\beta_\xi}$ and $X_i \cap \alpha \in X'$. Then $X$ will be the smallest $X \prec \mathbf{L}_{\beta_\xi}$ such that $\gamma \cup \{\xi\} \subseteq X$ and $X \cap \alpha$ transitive.




Moreover, $\alpha$ looks like a regular cardinal in $\mathbf{L}_\nu$, so the cardinality of $X$ is strictly smaller than $\alpha$ even in this model. Consider then the collapse map $\sigma : \mathbf{L}_{\bar{\beta}} \overset{\sim}{\longleftrightarrow} X$ and we have $\sigma(\bar{\alpha}) = \alpha$ for the critical point $\bar{\alpha}$ of the embedding $\sigma$. Furthermore, even the map $\sigma$ is an element of $\mathbf{L}_\nu$. Now let $\bar{\xi}$ such that $\sigma(\bar{\xi}) = \xi$. Then we finally have

*Claim.* $\bar{\xi} \in S_{\bar{\alpha}}$.

To see this we have to look at the properties in the definition of $S_{\bar{\alpha}}$. Because $\alpha$ is strictly less than $\xi$ we trivially have $\bar{\alpha} < \bar{\xi}$. Also we know that $\xi$ is limit of p.r.-closed ordinals and so is $\bar{\xi}$ by the elementary preserving property of $\sigma$ and Lemma 37. The same reason shows the regularity and the property 'being the largest cardinal' of $\bar{\alpha}$ within $\mathbf{L}_{\bar{\xi}}$.

Moreover, because of $\bar{\xi} < \bar{\beta}$ we know that $\bar{\xi}$ is strictly smaller than $\bar{\alpha}$. This finishes the proof of the claim.

Trivially, we also have $\gamma \leqslant \bar{\xi} < \alpha$ by our construction and for the elementary embedding $\sigma : \mathbf{L}_{\bar{\beta}} \longrightarrow \mathbf{L}_{\beta_\xi}$ we know by definition and its properties that $\bar{\beta} = \beta_{\bar{\xi}}$ and $\bar{\alpha} = \alpha_{\bar{\xi}}$ holds and so by the uniqueness of the tree embeddings, given by Lemma 55, also $\sigma \upharpoonright \mathbf{L}_{\bar{\xi}} = \pi_{\bar{\xi}\xi}$.

And so, we finally have shown everything for $\bar{\xi} \lhd \xi$. ⊠(Lemma 57)

Note, the second part (b) of the last lemma claims that $\xi$ is a limit point within the tree relation $\lhd$. Moreover, together with Lemma 53 we have finally shown that each $\nu \in S_A$ is a limit point within the tree relation, that is

$$\sup\{\, \bar{\nu} \mid \bar{\nu} \lhd \nu \,\} = A.$$

Considering the figure that might help to understand the structure, we are now ready to define the complete structure we are aiming to:

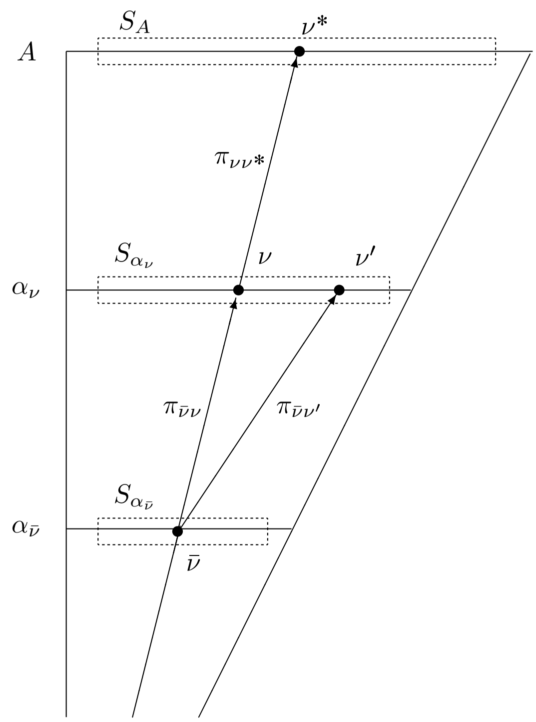



Figure 1. The coarse $A$-morass

**Definition 58** (The coarse $A$-Morass)**.** *Let the cardinal $A$, the sequence $\langle\, S_\alpha \mid \alpha \leqslant A \,\rangle$, the tree relation $\lhd$ with the sequence $\langle\, \pi_{\bar{\nu}\nu} \mid \bar{\nu} \lhd \nu \,\rangle$ of embeddings be defined as above. Then we call the structure*

$$\mathfrak{M} := \langle\, B, A, \langle\, S_\alpha \mid \alpha \leqslant A \,\rangle, \lhd, \langle\, \pi_{\bar{\nu}\nu} \mid \bar{\nu} \lhd \nu \,\rangle \rangle$$

*the* coarse $A$-morass *with the universe $B$ such that $A \subseteq B \subseteq \mathrm{On}$.*

CHAPTER 5

# An Inaccessible implies the Failure

We will now use the forcing we defined in the last chapter to prove the main theorem:

**Theorem 59.** *Suppose there is a model of* ZFC *with an inaccessible cardinal* $\tau$*. Moreover, let* $\kappa < \tau$ *be an uncountable regular cardinal. Then there is a forcing extension of* $\mathbf{L}$ *that is a model of the following:*

$$\begin{aligned} \mathsf{ZFC} \quad + \quad & 2^{\aleph_0} = \kappa^+ \quad + \quad \text{“}2^\alpha = \alpha^+ \textit{ for all cardinals } \alpha \geqslant \kappa\text{”} \\ + \quad & \text{“}(\gamma^+, \gamma) \not\longrightarrow (\kappa^+, \kappa) \textit{ for all regular cardinals } \gamma \neq \kappa\text{”} \\ + \quad & \text{“}\textit{there is a special } \kappa^+\textit{-Aronszajn tree}\text{”}. \end{aligned}$$

The proof of the theorem will last the remaining part of the chapter. Starting from a suitable ground model that has an inaccessible cardinal, we will work within the generic extension of the ground model, given by the forcing defined in the last chapter. There, we will consider the theory $\mathfrak{T}$ we have already defined and show the failure of the stated transfer property by constructing a counterexample. Moreover, in the forcing extension we will have a special $\kappa^+$-Aronszajn tree and –as desired– sufficient small powers of cardinals above $\kappa$. And so, the proof will be done.

Now, working in a set theoretical universe with an inaccessible cardinal, take an arbitrary (ground) model $M$, satisfying ZFC and $\mathbf{V} = \mathbf{L}[B]$ for any subset $B \subseteq \kappa$ such that $\tau$ is the least inaccessible cardinal above $\kappa$ in $M$:

$$\begin{aligned} M \models \mathsf{ZFC} \ + \ \mathbf{V} = \mathbf{L}[B] \text{ for } B \subseteq \kappa \\ + \ \tau \text{ least inaccessible above } \kappa. \end{aligned} \tag{2}$$

To start with, just take $M$ as Gödel's constructible universe $\mathbf{L}$, that is choosing $B = \varnothing$, and so $M$ obviously satisfies the condition (2) where $\tau$

is the least inaccessible cardinal above $\kappa$ in $M$, given by the assumption together with Lemma 38.

However, there will be a point during the up-coming construction where it might be convenient just to start with a model having the property given by (2), choosing the predicate $B$ in an appropiate way, than starting with $\mathbf{L}$.

We will now force with Mitchell's forcing $\mathbb{M}(\kappa, \tau)$ over $M$ having an $\mathbb{M}(\kappa, \tau)$-generic $G$ and finally getting the extension $M[G]$. Note, in $M[G]$, we have $2^{\aleph_0} = 2^\kappa = \kappa^+ = \tau$. Moreover, by construction of Mitchell's forcing defined in Definition 43 there is then a $\mathbb{P}(\tau)$-generic $\bar{G}$ and a $\dot{\mathbb{Q}}(\kappa, \tau)$-generic $\tilde{G}$ such that $M[G] = M[\bar{G}][\tilde{G}]$ where $\mathbb{P}(\tau)$ and $\dot{\mathbb{Q}}(\kappa, \tau)$ are defined as in Chapter 3, yielding the property of a two-step forcing

$$\mathbb{M}(\kappa, \tau) = \mathbb{P}(\tau) \star \dot{\mathbb{Q}}(\kappa, \tau).$$

Remember, we already defined the theory $\mathfrak{T}$ in Chapter 4 as follows:

$$\begin{array}{rcl} \mathfrak{T} & = & \mathsf{ZFC}^- + \ \mathbf{V} = \mathbf{L} \ + \ \dot{A} \text{ regular} \\ & & \qquad\quad + \ \ \dot{A} \text{ is the largest cardinal.} \end{array}$$

Aiming towards a contradiction, let us work with the theory $\mathfrak{T}$ and state the following

**Supposition 60.** *In $M[G]$, there is a $(\kappa^+, \kappa)$-model $\mathfrak{A}$ of $\mathfrak{T}$.*

⋆ ⋆ ⋆

In this chapter we are now going to deduce a contradiction to this assumption we just made. For, let $\mathfrak{M} := (\mathfrak{M})^{\mathfrak{A}}$ be the coarse $A$-morass defined in the last chapter within the fixed model $\mathfrak{A}$ and define

$$\mathfrak{M} \restriction A := \langle\ A,\ \langle S_\alpha \mid \alpha < A \rangle,\ \lhd \restriction A,\ \langle \pi_{\bar{\nu}\nu} \mid \bar{\nu} \lhd \nu < A \rangle\ \rangle.$$

Now, remember the second representation of the forcing extension $M[G]$, given with Lemma 48. In fact, let $\mathbb{M}_\nu$, $\mathbb{M}^\nu$, $G_\nu$ and $G^\nu$ for $\nu < \tau$ be defined as in the above mentioned lemma, then $M[G] = M[G_\nu][G^\nu]$.

Moreover, the dividing is in some sense cleverly choosen. In $M[G_\nu]$, we just have taken new subsets of $\omega$ that can be described by conditions "till $\nu$" and have then collapsed ordinals below $\nu$ to $\kappa$, which is of course only interesting for $\nu > \kappa$ anyway. This means, in $M[G_\nu]$, the forcing extension is already constructed up to $\nu$.

If we now consider an initial segment of the morass $\mathfrak{M}$, say $\mathfrak{M} \upharpoonright A$, then this small structure of cardinality $\kappa < \tau$ has to be already defined in an initial segment $M[G_\nu]$ of the forcing $M[G_\nu][G^\nu] = M[G]$ for suitable $\nu < \tau$. Therefore, we have the following

**Lemma 61.** *There is $\nu < \tau$ such that $\mathfrak{M} \upharpoonright A \in M[G_\nu]$.*

Now, choose $\nu' < \tau$ minimal such that $\mathfrak{M} \upharpoonright A \in M[G_{\nu'}]$ and define $\nu := \nu' + \kappa$. Then we can be sure that $\nu'$ and even $\nu$ is collapsed to $\kappa$ by our forcing at stage $\nu$, that is, in $M[G_\nu]$.

This property is important for us because it could have been that we consider just the case that $\nu$ is a cardinal, say $\nu = \kappa^+$, as the minimal one chosen and then we would have $\kappa^+$-many new reals but $\kappa^+$ would not be collapsed and so $2^{<\kappa} = \kappa$ would fail.

Moreover, $G_\nu$ as bounded subset of the generic filter $G$, living on forcing conditions up to $\nu$, is therefore a bounded subset of $M = \mathbf{L}[B]$ where $B \subseteq \kappa$. Remember, so far $B$ could be taken as the empty set.

Therefore, $G_\nu$ will be caught in an initial segment of the ground model, say $G_\nu \subseteq \mathbf{L}_{\bar{\nu}}[B]$ for a suitable $\bar{\nu}$. Choose $\bar{\nu}$ minimal with this property. Hence, in $\mathbf{L}[B][G_\nu] = M[G_\nu]$, the ordinal $\bar{\nu}$ is already collapsed to $\kappa$ because $\nu$ is, as we have seen above, and the minimal choice of $\bar{\nu}$.

Further, fix a bijection $f : \kappa \longleftrightarrow \bar{\nu}$ such that $f \in M[G_\nu]$ and consider the complete elementary theory of $\langle \mathbf{L}_{\bar{\nu}}[G_\nu], \in, G_\nu, \langle \dot{\xi} \mid \xi < \bar{\nu} \rangle \rangle$ where we use $f$ to code elements $\xi$ of $\bar{\nu}$ into elements $\dot{\xi}$ of $\kappa$. Then this theory is a subset of $\mathbf{L}_\kappa[B]$.

Therefore, we are able to code $G_\nu$ in a predicate $B' \subseteq \kappa$ such that $M[G_\nu] = \mathbf{L}[B][G_\nu] = \mathbf{L}[B']$ and the model $\mathbf{L}[B']$ still satisfies the property (2) for possible ground models. Note, the property of $\tau$, being inaccessible, we obviously did not change in $\mathbf{L}[B']$.

Now, $G^\nu$ is $\mathbb{M}^\nu$-generic over $M[G_\nu]$ where $\mathbb{M}^\nu$, defined in Chapter 3, is –roughly speaking– the forcing $\mathbb{M}(\kappa, \tau)$ but just taking conditions acting beyond $\nu$. So, the difference between both forcings is that the forcing $\mathbb{M}(\kappa, \tau)$ starts at level $\kappa$ whereas $\mathbb{M}^\nu$ begins later, at level $\nu$ such that $\kappa < \nu < \kappa^+$.

Because the forcing adds subsets of $\omega$ and collapses ordinals to $\kappa$, to start at stage $\nu < \kappa^+$ does not change anything in the arguments: For the indices $\nu$, the way towards the inaccessible $\tau$ is –roughly speaking– long enough to argue in the same way. Hence, for simplicity but without loss of generality, we may additionally assume

$$\mathfrak{M} \upharpoonright A \in M. \tag{3}$$

Furthermore, by the choice of the theory $\mathfrak{T}$ and the Supposition 60 we know that the interpretation of the predicat $\dot{A}$ is a set of cardinality $\kappa$ of ordinals. So, by renaming the elements using a suitable bijection, we can arrange $A$ as a subset of $\kappa$ and so we will also assume without loss of generality that

$$A \subseteq \kappa. \tag{4}$$

Apart from this, we do not know how $A$ looks like, in fact, with the model $\mathfrak{A}$ we could have a non-standard model of set theory and so $\langle A, \leqslant_A \rangle$ needs not to be well-founded. However, we can ask for the cofinality of the linear order $\langle A, <_A \rangle$ within $\mathfrak{A}$, knowing –as a subset of $\kappa$– this cardinal could be any regular cardinal below $\kappa$.

Hence, to go on with the proof we have to distinguish two cases.

**Case 1.** $\boxed{\mathrm{cf}_M(A) = \omega}$

Then under these circumstances, within the ground model we can find a countable sequence $\langle \gamma_i \mid i < \omega \rangle \in M$ being monotone and cofinal in $A$. Remember, we still work within the fixed model $\mathfrak{A}$.

Now, for $\nu \in S_A$ let $\nu_i$ be the unique tree element being the $\lhd$-smallest $\bar{\nu}$ such that $\bar{\nu} \lhd \nu$ and $\gamma_i \leqslant \alpha_{\bar{\nu}}$. Note, because for a fixed $\nu$ we have enough well-foundedness within the tree to define $\nu_i$ that way. We also know by construction of the coarse $A$-morass that $\nu_i < A < \nu$ for $\nu_i \lhd \nu \in S_A$ and therefore, by (4), we have $\nu_i \in \kappa$. Furthermore, define

$$a_\nu \;:=\; \{\nu_i \mid i < \omega\} \in [\kappa]^\omega.$$

Towards to the desired contradiction, we define within the fixed model $\mathfrak{A}$ for each $\nu \in S_A$ and $B' := \{\xi \in B \mid \xi < \nu\}$ the following initial segment of the morass structure

$$\begin{aligned} \mathfrak{M} \upharpoonright \nu \;:=\; & \langle\, B',\, \alpha_\nu,\, \langle S_\alpha \mid \alpha < \alpha_\nu \rangle,\, S_{\alpha_\nu} \cap B',\, \lhd \upharpoonright (S_{\alpha_\nu} \cap B'), \\ & \langle \pi_{\bar{\nu}\nu'} \mid \bar{\nu} \lhd \nu' < \nu \rangle \,\rangle. \end{aligned}$$

Now, working in the model $\mathfrak{A}$, consider the elementary embeddings $\pi_{\nu_i\nu_j} : \mathbf{L}_{\nu_i} \longrightarrow \mathbf{L}_{\nu_j}$ for every $i < j < \omega$ that we have by definition of the tree. Let us lift up these embeddings to maps of the shape

$$\tilde{\pi}_{\nu_i\nu_j} : \langle \mathbf{L}_{\nu_i}, \mathfrak{M} \upharpoonright \nu_i \rangle \longrightarrow \langle \mathbf{L}_{\nu_j}, \mathfrak{M} \upharpoonright \nu_j \rangle,$$

defined in the obvious way, that is

$$\begin{aligned} \tilde{\pi}_{\nu_i\nu_j} \upharpoonright \mathbf{L}_{\nu_i} \;&=\; \pi \upharpoonright \mathbf{L}_{\nu_i}, \\ \tilde{\pi}_{\nu_i\nu_j}(\langle S_\alpha \mid \alpha < \alpha_{\nu_i} \rangle) \;&=\; \langle S_\alpha \mid \alpha < \alpha_{\nu_j} \rangle, \\ \tilde{\pi}_{\nu_i\nu_j}(\, S_{\alpha_{\nu_i}} \cap B' \,) \;&=\; S_{\alpha_{\nu_j}} \cap B', \\ \tilde{\pi}_{\nu_i\nu_j}(\langle \pi_{\bar{\nu}\nu'} \mid \bar{\nu} \lhd \nu' < \nu_i \rangle) \;&=\; \langle \pi_{\bar{\nu}\nu'} \mid \bar{\nu} \lhd \nu' < \nu_j \rangle. \end{aligned}$$

But then we have $\tilde{\pi}_{\nu_i\nu_j} = \tilde{\pi}_{\nu_k\nu_j} \circ \tilde{\pi}_{\nu_i\nu_k}$ and therefore, using Lemma 6, the structure $\mathfrak{M} \restriction \nu$ is just the direct limit of the structure

$$\langle \langle \langle \mathbf{L}_{\nu_i}, \mathfrak{M} \restriction \nu_i \rangle \mid i < \omega \rangle, \langle \tilde{\pi}_{\nu_i\nu_j} \mid i \leqslant j < \omega \rangle \rangle$$

and so –up to isomorphism– this structure is unique. Therefore, we finally proved the following

**Remark 62.** *For each $\nu \in S_A$, up to isomorphism, $\mathfrak{M} \restriction \nu$ is uniquely definable from the parameters $\alpha_\nu$ and $\mathfrak{M} \restriction A$.*

Now, the set $a_\nu$ defines a countable path through the tree till the element $\nu$ (of the tree). And so, because of the uniqueness of limit points in this tree, we obviously have the following

**Lemma 63.** *For elements $\bar{\nu} \neq \nu$ of $S_\alpha$ we have $a_{\bar{\nu}} \neq a_\nu$.*

⋆ ⋆ ⋆

Let us define the technical but useful collection of all countable paths through the tree structure below $\nu$ for an element $\nu$ of $S_A$, letting within the model $\mathfrak{A}$,

$$\Theta(\nu) := \{ a_{\bar{\nu}} \mid \bar{\nu} \in S_A, \bar{\nu} < \nu \}.$$

Remembering that $M = \mathbf{L}[B]$, there is a first nice property as follows:

**Lemma 64.** *For each $\nu \in S_A$, the sequence $\Theta(\nu)$ is uniformly definable from parameters $a_\nu$, $\mathfrak{M} \restriction A$ and $\langle \gamma_i \mid i < \omega \rangle$ within the model $M[a_\nu]$.*

*Proof.* Note, by (3), the parameters $\mathfrak{M} \restriction A$ and $\langle \gamma_i \mid i < \omega \rangle$ are already elements of $M$, and hence even of $M[a_\nu]$. Still working in the model $\mathfrak{A}$, we will now define step by step the desired collection of sets as follows:

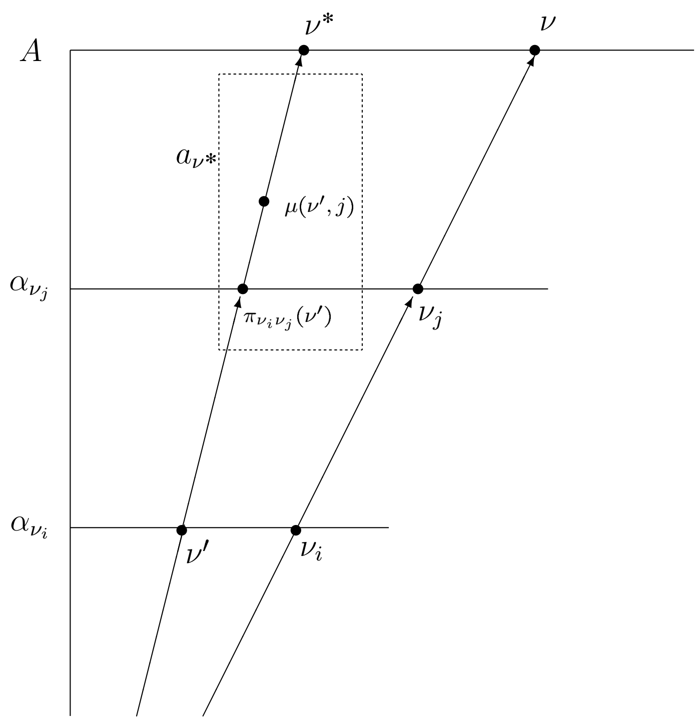
For each $\nu_i \in a_\nu$ and arbitrary $\nu' \in S_{\alpha_{\nu_i}}$ such that $\nu' < \nu_i$ define

$$P_0(\nu', i) := \{ \pi_{\nu_i\nu_j}(\nu') \mid i < j \}.$$

Then $P_0(\nu', i)$ is a cofinal set in the branch above $\nu'$ as a copy of the branch below $\nu$. Now set

$$P_1(\nu', i) := \{ \bar{\mu} \mid (\exists \mu \in P_0(\nu', i))(\bar{\mu} \lhd \mu) \}.$$

The set $P_1(\nu', i)$ collects all missing elements on the branch below an element of the set $P_0(\nu', i)$. Therefore, this set describes a branch of length $A$ and, by definition, it does not depend on the parameter $i$ and we have $P_1(\nu', i_0) = P_1(\nu', i_1)$ for all natural numbers $i_0$, $i_1$. Therefore, we define $P_1(\nu') := P_1(\nu', i)$ for an arbitrary natural number $i$.




Finally set

$$P_2(\nu') := \{ \mu(\nu', j) \mid j < \omega \},$$

where $\mu(\nu', j)$ denotes the well-defined $\lhd$-smallest $\mu$ of the set $P_1(\nu')$ of elements of the tree such that $\gamma_j \leqslant \alpha_\mu$.

Note, with the given parameters we can obviously define the above three sets within the model $M[a_\nu]$.

Consider now $\nu^* := \pi_{\nu_i \nu}(\nu')$ for an arbitrary $i < \omega$. Then $\nu^*$ again does not depend on the choice of $i$. Moreover, $\nu^*$ is the unique limit of the branch defined by $P_0(\nu', i)$ at tree level $A$. By definition we know then that $a_{\nu^*}$ is just the set $P_2(\nu')$ and so finally we have, defined within the model $M[a_\nu]$, the following

$$\begin{aligned} \Theta(\nu) &= \{ a_{\nu^*} \mid \nu^* \in S_A \ \wedge \ \nu^* < \nu \} \\ &= \{ P_2(\nu') \mid \exists \mu \in a_\nu \ ( \nu' < \mu \ \wedge \ \alpha_{\nu'} = \alpha_\mu )\}. \end{aligned}$$

Therefore, the proof is complete. ☒(Lemma 64)

Moreover, with Lemma 22, having $a_\nu$ as a subset of $\kappa$, we still have $2^{<\kappa} = \kappa$ within the ground model $M = \mathbf{L}[B]$ and also within the model $M[a_\nu]$. Hence, because $\Theta(\nu)$ is a subset of $[\kappa]^\omega$, we can sum up with the following

**Lemma 65.** *For $\nu \in S_A$, within model $M[a_\nu]$, the set $\Theta(\nu)$ has cardinality at most $\kappa$.*

Finally we are prepared to complete the desired contradiction using Lemma 9:

Let $W$ be the inner model $M[\bar{G}]$ and $\mathbf{V}$ be the final forcing extension $M[G] = M[\bar{G}][\tilde{G}]$. Moreover, let $\kappa$ be the given cardinal and $\tau$ be the inaccessible within the ground model $M$. Remember, by Lemma 44, we do not change cardinals forming the forcing extention $W$. So, we still have $\kappa^{+W} < \tau$. Lemma 45 then gives us immediately the desired stationarity of the set $\{ \lambda < \tau \mid W \models \mathrm{cf}(\lambda) = \kappa^+ \}$ that we need for the application of Lemma 9.

Now, let $H$ be $\mathcal{P}^W(\kappa)$ and so we have trivially $U \subseteq H \in W$ and, moreover, within $W$, also that $|H| = |\mathcal{P}(\kappa)| = \tau$. Remember, $W$ is the Cohen extension of $M$ by adding $\tau$ many reals.

Finally let $U$ be $\{ a_\nu | \nu \in S_A \}$. Then we have $U = \bigcup_{\nu \in S_A} \Theta(\nu)$. Further, $\langle U, <_U \rangle$ forms a linear order, where the order relation is defined by letting: $a_{\bar{\nu}} < a_\nu$ if and only if $\bar{\nu} < \nu$. For arbitrary $x \in U$, say $x = a_\nu$, we then have:

$$\begin{aligned} U_x &:= \{\, z \mid z <_U x \,\} = \{\, a_{\bar{\nu}} \mid a_{\bar{\nu}} <_U a_\nu \,\} \\ &= \{a_{\bar{\nu}} \mid \bar{\nu} < \nu;\ \bar{\nu}, \nu \in S_A \,\} \\ &= \Theta(\nu) \in M[\bar{G}] = W. \end{aligned}$$

Now, $a_\nu$ is obviously an element of $M[G]$ and as a countable subset of $\kappa$ we know by Lemma 51 that $a_\nu$ was not added by the second forcing step and so it is an element of $M[\bar{G}]$.

But then we know that $M[a_\nu] \subseteq M[\bar{G}]$ and so we can conclude finally

$$|U_x|^W = |\Theta(\nu)|^{M[\bar{G}]} \leqslant |\Theta(\nu)|^{M[a_\nu]} \leqslant \kappa.$$

Under these circumstances, Lemma 9 promised that the cardinality of $U$ is strictly smaller than $\tau$. However, the cardinality of $U$ is the same as the one of $S_A$ which is cofinal in the regular cardinal $\tau$. Therefore, the cardinality of $U$ is equal to $\tau$.

This desired contradiction finishes the first case.

We now turn to the remaining case in our proof and try to deduce a contradiction there as well.

**Case 2.** $\boxed{\mathrm{cf}_M(A) \neq \omega}$

We still can assume that the initial segment $\mathfrak{M} \restriction A$ of the morass is an element of the ground model $M$ as in the very beginning of the first case given by (3).

Now let $\bar{\kappa} := \mathrm{cf}_M(A) \neq \omega$ and $\langle \gamma_\nu \mid \nu < \bar{\kappa} \rangle \in M$ be an uncountable and cofinal sequence in the linear order $A$. For $\nu \in S_A$ define now as in the first case

$$\nu_i := \text{ the } \lhd\text{-smallest } \bar{\nu} \text{ such that } \bar{\nu} \lhd \nu \text{ and } \gamma_i \leqslant \alpha_{\bar{\nu}},$$

and finally let $a_\nu := \{\nu_i \mid i < \bar{\kappa}\}$, now an uncountable subset of $\kappa$.

Consider, within the forcing extension $M[G]$, the definable set

$$X := \{a_\nu \mid \nu \in S_A\}.$$

Note, $X$ is a subset of the ground model $M$. Moreover, the cardinality of $X$ is the same as the cardinality of $S_A$, by Lemma 63, and this is $\tau$ because of its regularity property and Lemma 53.

Because $\mathfrak{M} \restriction A$ lies already in the ground model $M$ and together with $\langle \gamma_i \mid i < \bar{\kappa} \rangle \in M$ we can define initial segments of $a_\nu \restriction i$ within the ground model. Hence, already $a_\nu \restriction i$ is an element of $M$ for arbitrary $i < \bar{\kappa}$ and so by Lemma 49 we also know that then the whole sequence $a_\nu$ is an element of the ground model.

However, by definition, $X$ is a subset of $\mathcal{P}^M(\kappa)$. Moreover, because of the inaccessibility of $\tau$ and $2^\kappa = \kappa^+$ within the ground model $M$, by Lemma 20 we finally conclude the following

$$\tau = |X|^M \leqslant \kappa^{+M} < \tau.$$

Hence, in both cases we were able to find a contradiction. This means our Supposition 60 was false and the main part of the proof of the Theorem 59 is already done.

To finish up with the proof let us look at the following two lemmas:

**Lemma 66.** *In $M[G]$, the theory $\mathfrak{T}$ has $(\gamma^+, \gamma)$-models for all regular $\gamma \neq \kappa$.*

*Proof.* Let us work in $M[G]$ and consider the models $\langle \mathbf{L}_{\gamma^+}, \gamma \rangle$ for a regular cardinal $\gamma < \kappa$. Then, by Lemma 16, this is a model of $\mathsf{ZFC}^-$ and $\mathbf{V} = \mathbf{L}$. And, moreover, $\gamma$ is indeed the largest cardinal in $\mathbf{L}_{\gamma^+}$ because of the preserving properties of the forcing by Corollary 46. And together with Lemma 33, we finally have found a $(\gamma^+, \gamma)$-model of the fixed theory $\mathfrak{T}$.

The same idea shows that $\langle \mathbf{L}_{\tau^{+(\alpha+1)}}, \tau^{+\alpha} \rangle$ is a $(\tau^{+(\alpha+1)}, \tau^{+\alpha})$-model of $\mathfrak{T}$ for arbitrary ordinals $\alpha$. And so, because $\tau = \kappa^{+M[G]}$, all cases are successfully discussed and therefore the lemma is proved.

⊠(Lemma 66)

In our first main theorem, we just proved that there cannot be a $(\kappa^+, \kappa)$-model. So, why does not work the model $\langle \mathbf{L}_{\kappa^+}, \kappa \rangle$? — The answer is easy when we remember that we collapsed $\tau$ to $\kappa^+$, and so, $\tau = \kappa^{+M[G]}$ –being inaccessible in the constructible universe– is not the cardinal successor of $\kappa$ in $\mathbf{L}$. Hence, in $(\mathbf{L}_{\kappa^+})^{M[G]} = \mathbf{L}_{\kappa^{+M[G]}}$, the cardinal $\kappa$ is not the largest one.

⋆ ⋆ ⋆

The last missing property in the statement of the main theorem we still have to show, uses the choice of $\tau$ being the minimal inaccessible cardinal above $\kappa$ within $M = \mathbf{L}[B]$ for a suitable subset $B$. In fact, analyzing our construction more deeply, we see that independent from the assumption (3), we did indeed start from the constructible universe — just using (2) and (3) to arguing in a more convenient way.

In this case, the cardinal $\tau$ is the least inaccessible above $\kappa$ even in the constructible universe. However, we could be able to argue within a general universe given by (2), just proving a similar statement for $\mathbf{L}[B]$ as Lemma 30 gives us for the constructible universe $\mathbf{L}$, *cf.* Lemma 31.

So in any case, we know then that $\tau = \kappa^{+M[G]}$ is not a Mahlo cardinal within the constructible universe, having started the forcing construction from **L**. But then, using Lemma 30, we know that in $M[G]$ we have a $\square_\tau$-sequence, and so together with the equivalence of Lemma 84 and Theorem 81, respectively, we finally proved the following

**Lemma 67.** *In $M[G]$, there is a special $\kappa^+$-Aronszajn tree.*

Using the facts of Chapter 3 that Mitchell proved in [Mit72], we conclude that within the forcing extension $M[G]$ we only somehow slightly damaged GCH –depending on the choice of $\kappa$–, that is, we have $2^\alpha = \alpha^+$ for all $\alpha \geqslant \kappa$. And even for the smallest infinite cardinal we have chosen a somehow minimal failure, $2^{\aleph_0} = \kappa^+$, again depending on the choice of $\kappa$.

Finally, our first main theorem is completely proved. $\boxtimes$(Theorem 59)

CHAPTER 6

# A weaker Theory for the Counterexample

As promised in former chapters we are now giving another proof of the existence of the counterexample to the general Chang's Transfer Property, considering the question under what circumstances the following assertion for arbitrary infinite cardinals $\gamma$ and any uncountable regular cardinal $\kappa$ fails:

$$(\gamma^+, \gamma) \longrightarrow (\kappa^+, \kappa).$$

However, we now start from a weaker theory than we considered in Chapter 5. In fact, this will not change the claim of the main theorem, Theorem 59. Though, it might be interesting to know that the theory which is needed to get the desired failure of the above mentioned transfer property is indeed rather weak possible.

Moreover, we are even able to start from a ground model $M$ that only satisfies $\mathsf{GCH}$. This is indeed a much weaker assumption than we have used in Chapter 5, where we started (basically) from $\mathbf{L}$. Therefore we are going to prove the following

**Theorem 68.** *Let $M$ be a model of set theory, satisfying* $\mathsf{GCH}$ *such that, in $M$, there is an inaccessible $\tau$ and $\kappa < \tau$ is an uncountable regular cardinal. Moreover, for Mitchell's notion of forcing $\mathbb{M}$ let $G$ be an $\mathbb{M}$-generic filter over $M$. Then for arbitrary uncountable regular $\gamma \geqslant \tau$ or $\gamma = \omega$ we have*

$$M[G] \models (\gamma^+, \gamma) \not\longrightarrow (\kappa^+, \kappa).$$

This theorem will give us a lot of possibilities to get nice independent statements for the failure of Chang's Transfer Property with respect to large cardinals. Having a large cardinal, say a measurable one or even a larger cardinal –just providing there is an inaccessible cardinal below to work with– starting from a suitable model satisfying $\mathsf{GCH}$, we then can apply the forcing of the last theorem and we are getting the desired failure of the transfer property in a universe where we still have the existence property of that large cardinal we have started from.

The reason for this is simply the fact that Mitchell's notion of forcing is in some sense a small one. That is, the forcing works very locally and it will not affect any really much larger cardinal properties beyond the considered inaccessible cardinal, as we already know.

⋆ ⋆ ⋆

Now, to start with the proof, fix a model $M$ of $\mathsf{ZFC} + \mathsf{GCH}$ such that $\tau$ is inaccessible and $\kappa < \tau$ is uncountable and regular.

Already in Chapter 4, we defined a theory $\mathfrak{T}$ which contains, *e.g.*, the axiom of constructibility. A model of this theory gives us very good control about constructing structures like the coarse morass.

There are two (and even more) important consequences we used within the fixed model of the theory $\mathfrak{T}$. At first, we had $\mathsf{GCH}$ and so we knew about the behavior of powers of cardinals. And secondly, we strongly used consequences of the very powerful condensation property of the constructible universe.

We will now start from a relatively weak theory such that its models satisfy the axiom $\mathbf{V} = \mathbf{L}[C]$ for a given $C \subseteq \mathrm{On}$. We will again have a symbol for the largest cardinal, $\dot{A}$, however, it might be that we loose $\mathsf{GCH}$. Instead of this we assert that there are only a few bounded subsets of the interpretation of the symbol $\dot{A}$.

However, the price for this freedom will be a more complex structure theory during the proof. In fact, the levels of the morass, we are going to use here, will blow up. Each of them, the former intervals $S_\alpha$, will now be a (wide-branching) tree. Therefore, we are going to argue with two trees within the new morass structure.

Moreover, because of the growing of the levels $S_\alpha$, we now have to go over to use the models $\mathbf{L}^D_\beta$ themself as new indexes in the morass, not only their ordinal heights $\beta$ (or even the old indexes $\nu$ as above), *cf.* Figure 2, p. 64.

Nevertheless, even now the main idea of the proof can be preserved.

⋆ ⋆ ⋆

To fix the set of axioms let us define the new (weak version of the) theory $\mathfrak{T}'$ as follows:

$$\mathfrak{T}' \quad := \quad \mathsf{ZFC}^- \;+\; \mathbf{V} = \mathbf{L}[C] \text{ for } C \subseteq \mathrm{On} \;+\; 2^{<\dot{A}} = \dot{A} \;+\; \dot{A} \text{ is the largest cardinal} \;+\; \dot{A} \text{ regular.}$$

Trying to re-prove the Theorem 59, in a newer version given by Theorem 68, we will repeat the arguments we have stated in earlier chapters. For, fix a model $\mathfrak{A}$ of the new version $\mathfrak{T}'$ and let $A$ be the largest cardinal. Let us work within this model $\mathfrak{A}$, doing all further constructions and definitions.

Then, obviously, we have $\mathbf{L}_A[C] = \mathbf{H}_A$, the set of all sets within $\mathfrak{A}$ that are hereditarily smaller than $A$. Furthermore, denote with $\mathbf{L}^D_\nu$ the model $\langle\, \mathbf{L}_\nu[D], \in, D \cap \nu \,\rangle$ and define as earlier, in Chapter 4,

$$S'_A := \{\, \nu \mid \nu \text{ is a limit of p.r.-closed ordinals, } A < \nu, \; \mathbf{L}^B_\nu \models A \text{ is the largest cardinal} \,\}.$$

However, this will not be the set where the tree is ranging on, as we will see very soon.

Now, for every $\nu \in S'_A$ let $\beta_\nu$ be again the smallest p.r.-closed $\beta$ such that $\mathbf{L}^B_\beta \models |\nu| \leqslant A$. Moreover, define $S_A := \{\, \mathbf{L}^B_{\beta_\nu} \mid \nu \in S'_A \,\}$. As we can see now, we are going to use the whole models $\mathbf{L}^B_{\beta_\nu}$ as index in the morass structure. However, we cannot use *this* definition to get the missing levels $S_\alpha$ for $\alpha < A$. Note, we can obviously show now that

$$\begin{aligned} S_A = \{\, \mathbf{L}^B_\beta \mid\ & \text{there is } \nu < \beta \text{ such that } \nu \text{ is limit of p.r.-closed ordinals,} \\ & \beta \text{ is the smallest } \beta \text{ such that } \mathbf{L}^B_\beta \models |\nu| \leqslant A, \\ & \mathbf{L}^B_\nu \models A \text{ is the largest cardinal, and } A < \nu \,\}. \end{aligned}$$

With this in mind, we can define for all $\alpha < A$ the sets $S_\alpha$ as follows

$$\begin{aligned} S_\alpha := \{\, \mathbf{L}^D_\beta \mid\ & \text{there is } \nu < \beta \text{ such that } \nu \text{ is limit of p.r.-closed ordinals,} \\ & \beta \text{ is the smallest } \beta \text{ such that } \mathbf{L}^D_\beta \models |\nu| \leqslant \alpha, \\ & \mathbf{L}^D_\nu \models \alpha \text{ is the largest cardinal, } \alpha < \nu, \\ & D \subseteq \beta,\ \mathbf{L}^D_\nu \models (\mathbf{H}_\alpha = \mathbf{L}^D_\alpha) \,\}. \end{aligned}$$

Here, we have to use new predicates $D$ because we will very often need condensation arguments and so we might loose the originally given predicate $C$.

Notice, that the models of $S_A$ are linearly ordered by inclusion. However, all other collections of models in $S_\alpha$ for $\alpha < A$ are partially ordered by a relation $\sqsubset$, defined as follows: For elements $\mathbf{L}^{\bar{D}}_{\bar{\beta}}$ and $\mathbf{L}^D_\beta$ of $S_\alpha$ where $\alpha \leqslant A$ we set

$$\mathbf{L}^{\bar{D}}_{\bar{\beta}} \sqsubset \mathbf{L}^D_\beta \qquad \text{if and only if} \qquad \bar{\beta} < \beta \text{ and } \bar{D} = D \cap \bar{\beta}.$$

Then this relation obviously forms a tree on each level $S_\alpha$. Moreover, we have expanded the former intervals and now we are going to check in the remaining part of this chapter that all of the earlier arguments are going through.

As above, define $S := \bigcup_{\alpha \leqslant A} S_\alpha$. Note, $\bigcup_{\alpha < A} S_\alpha$ is obviously a subset of $\mathbf{H}_A$. Therefore, because of the assumption given by the theory $\mathfrak{T}'$, namely $2^{<A} = A$, the cardinality of this set is at most (and obviously also at least) the cardinal $A$.

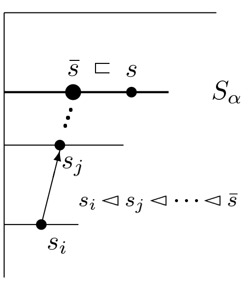
Moreover, repeating the argument of Lemma 54, we can prove that for stationary many $\alpha < A$ we have a non-empty set $S_\alpha$.

Then for $\bar{s} := \mathbf{L}_{\bar{\beta}}^{\bar{D}}$ and $s := \mathbf{L}_{\beta}^{D}$ let $s(\alpha)$ be the $\alpha$ such that $s \in S_\alpha$. Furthermore, for $s = \mathbf{L}_{\beta}^{D}$ let $s(D)$ be the $D$ and $s(\beta)$ be the $\beta$. And finally, let $s(\nu)$ be the smallest $\nu$ given by the definition of the set $S_\alpha$.

Note, that with this notation we conclude that for elements $\bar{s}$ and $s$ of $S$ we have $\bar{s} \sqsubset s$ if and only if $\bar{s}(\alpha) = s(\alpha)$, $\bar{s}(\beta) < s(\beta)$ and $\bar{s}(D) = s(D) \cap \bar{\beta}$ and so, the relation $\sqsubset$ can be defined on the whole collection of models $S$ than only seperately for each level $S_\alpha$, still forming a tree and being linearly on $S_A$. In fact, we have for elements $\bar{s}$ and $s$ of $S_A$ obviously that $\bar{s} \sqsubset s$ if and only if $\bar{s} \subsetneq s$ if and only if $\bar{s}(\beta) < s(\beta)$.

Now, imitating the old definition, for $\bar{s} \in S_{\bar{\alpha}}$ and $s \in S_\alpha$ where $\bar{\alpha} < \alpha$ define $\bar{s} \lhd s$ if there is an elementary embedding $\pi : \bar{s} \longrightarrow s$ such that $\mathrm{crit}(\pi) = \bar{\alpha}$ and $\pi(\bar{\alpha}) = \alpha$. We call this map $\pi_{\bar{s}s}$.

And again, repeating the proof of Lemma 55 for structures $s = \mathbf{L}_{\beta}^{D}$ than the old $\mathbf{L}_{\beta_\nu}$'s we conclude that the maps $\pi_{\bar{s}s}$ are unique for fixed models $\bar{s}$ and $s$. Moreover, the same proof as of Lemma 56 shows that the relation $\lhd$ forms a tree on $S$.




Now, looking at the arguments used in the last part of Lemma 57 –where we constructed a useful elementary substructure $X$ of a given tree element– we conclude that an $s \in S_A$ is a limit point in the tree relation and, moreover, if $\bar{s}$ and $s$ are elements of $S_\alpha$ for $\alpha \leqslant A$ and $\bar{s} \sqsubset s$, then also $\bar{s}$ is a limit point in the tree relation $\lhd$.

And so we can give the general version of the coarse morass, let us simply call it $A$-*quasi-morass*, defined as follows

**Definition 69.** *Let the cardinal $A$, the sequence $\langle S_\alpha \mid \alpha \leqslant A\rangle$, the tree relations $\lhd$ and $\sqsubset$ with the sequence $\langle \pi_{\bar\nu\nu} \mid \bar\nu \lhd \nu\rangle$ of embeddings be defined as above. Then we call the structure*

$$\mathfrak{M} := \langle\ \mathfrak{B};\ A,\ \langle S_\alpha \mid \alpha \leqslant A\rangle,\ \lhd,\ \sqsubset,\ \langle \pi_{\bar\nu\nu} \mid \bar\nu \lhd \nu\rangle\ \rangle$$

*the $A$-quasi-morass with the universe $\mathfrak{B}$.*

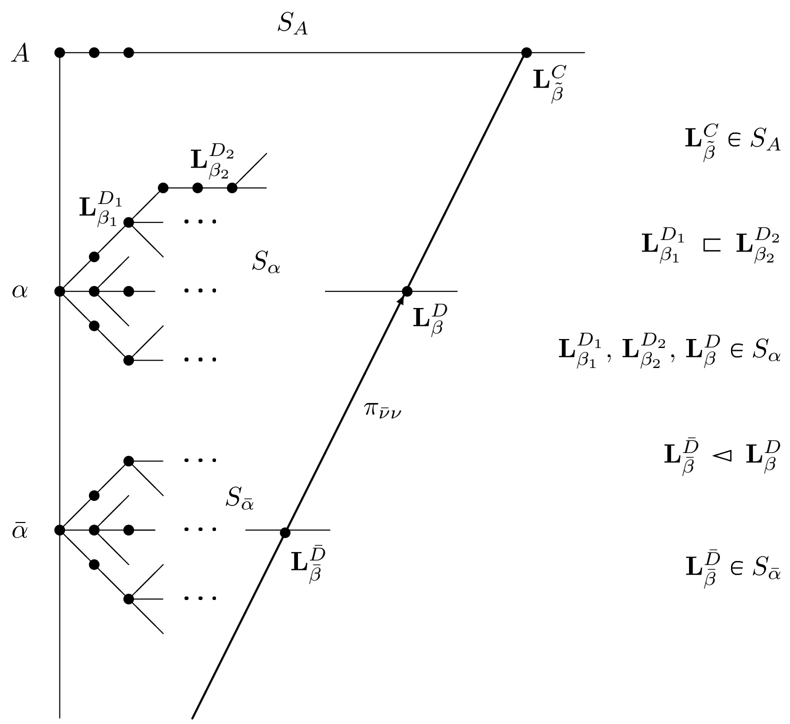



FIGURE 2. The $A$-quasi-morass

Here, the universe $\mathfrak{B}$ can be seen as collection of all models of the shape $\mathbf{L}^D_\beta$ where $\beta$ is an ordinal and $D$ a subset of $\beta$. Of course, using a suitable way of coding we can arrange $\mathfrak{B}$ again as collection of ordinals. However, not to fog the idea, we will work with the models instead of codes of ordinals.

In fact, with $\mathfrak{M}\restriction A$ we mean the initial segment of the given morass $\mathfrak{M}$ defined by

$$\mathfrak{M}\restriction A := \langle\, \mathfrak{B}\restriction A;\, A, \langle S_\alpha | \alpha < A\rangle, \lhd\restriction A, \sqsubset\restriction A, \langle \pi_{\bar s s} | \bar s \lhd s,\, s(\alpha) < A\rangle\,\rangle.$$

And again, $\mathfrak{B} \restriction A$ is the collection of all models $s \in \mathfrak{B}$ such that $s(\alpha) < A$. Note, in case we have coded the models as ordinals, this restriction could be seen simply as $A$ itself—as in the earlier definition on page 45. Moreover, $\lhd \restriction A$ means the restriction of the relation $\lhd$ to models $s$ such that $s(\alpha) < A$. The same holds for the relation $\sqsubset \restriction A$.

So, as in the beginning of Chapter 5, we can assume that we have again a ground model $M$ satisfying (2), *cf.* p. 47, that is, $M$ is a model of set theory satisfying $M \models$ "$\tau$ least inaccessible" $+\, \mathbf{V} = \mathbf{L}[B]$ for a suitable subset $B \subseteq \kappa$ and also, without loss of generality, we assume (3), *cf.* p. 50, that is that the initial segment $\mathfrak{M} \restriction A$ is indeed an element of the ground model. Certainly, we again assume as in (4), *cf.* p. 50, that $A$ is a subset of $\kappa$.

⋆ ⋆ ⋆

Going on in the argumentation of the given proof for Theorem 59, being within the first case where we have a countable cofinal sequence $\langle \gamma_i \mid i < \omega \rangle$ within the ground model $M$, we defined the important sequences $a_\nu$ at page 51. For a fixed model $s \in S_A$, we now give the following definition of the desired sequence $a_s$ as follows:

Let $s_i$ be the unique tree element being the $\lhd$-smallest $\bar{s}$ such that $\bar{s} \lhd s$ and $\gamma_i \leqslant \bar{s}(\alpha)$. This tree element on the $\lhd$-branch below $s$ is still well-defined. Then let $a_s$ be the set of all $s_i$ for arbitrary natural numbers $i$.

Moreover, as in the Remark 62, we know that for each $s \in S_A$, the initial segment $\mathfrak{M} \restriction s$ is uniquely definable from the parameters $a_s$ and $\mathfrak{M} \restriction A$. Here, we define in the obvious way

$$\mathfrak{M} \restriction s := \langle\, \mathfrak{B} \restriction s;\, A,\, \langle S_\alpha \mid \alpha < s(\alpha) \rangle,\, \lhd \restriction s,\, \sqsubset \restriction s,\\ \langle \pi_{\bar{s}s'} \mid \bar{s} \lhd s',\, s'(\alpha) < s(\alpha) \rangle \,\rangle.$$

And so, we again conclude for distinct $s'$ and $s$, both elements of $S_A$, that $a_{s'} \neq a_s$. Note, the tree $\lhd$ within the initial segment $\mathfrak{M} \restriction A$ does not have unique limit points. However, we do not have to care about this fact because we only need this property for the models within $S_A$.

Furthermore, we have for arbitrary $\alpha \leqslant A$ and for each structure $s \in S_\alpha$ that the collection $S_\alpha \upharpoonright s := \{\bar{s} \in S_\alpha \mid \bar{s} \sqsubset s\}$ as the set of all elements $\mathbf{L}_{\bar{\beta}}^{s(D)\cap\bar{\beta}}$ in $S_\alpha$ where $\bar{\beta} < s(\beta)$, a subset of the model $s$, is even definable within the structure $s$.

Moreover, another property of the old morass structure we do not loose is the following: The cardinality of an $\sqsubset$-branch $\{\,\bar{s} \in S_\alpha \mid \bar{s} \sqsubset s\}$ for a fixed $s \in S_\alpha$ and, therefore, of an initial segment of the $\alpha$-th level of the morass structure, $S_\alpha \upharpoonright s$, is strictly less than $A$ because there are only less than the cardinal $A$ many potentionally new ordinal heights $\bar{\beta} < s(\beta) < A$ for possible elements $\mathbf{L}_{\bar{\beta}}^{s(D)\cap\bar{\beta}}$.

⋆ ⋆ ⋆

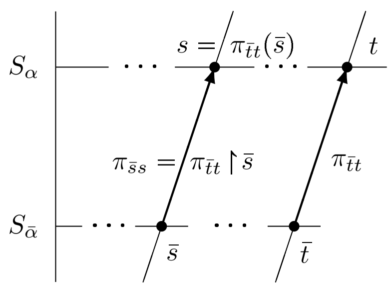
We now turn to the important tool we used in the proof of the main theorem, defining for $s \in S_A$ the following set of sequences:

$$\Theta(s) \;:=\; \{\, a_{\bar{s}} \mid \bar{s} \in S_A,\ \bar{s}(\beta) < s(\beta)\,\}.$$

Note, the elements $s$ of the collection $S_A$ of models are always of the shape $\mathbf{L}_{s(\beta)}^{C}$ and so we have a canonical (linear) order given by the ordinal height of these models.

We then can repeat the proof of Lemma 64 to get that for each $s \in S_A$, the sequence $\Theta(s)$ is uniformly definable from the parameters $a_s$, the morass segment $\mathfrak{M} \upharpoonright A$ and the (in $A$) cofinal sequence $\langle\, \gamma_i \mid i < \omega \,\rangle$ within the model $M[a_s]$.

For, we use the similar property as we had in the old proof, that is, that for all $\bar{s} \sqsubset \bar{t}$ where $\bar{s}$ and $\bar{t}$ are models of $S_{\bar{\alpha}}$, and moreover, $\bar{t} \lhd t$ for a model $t$ of $S_\alpha$ and $\pi_{\bar{t}t}(\bar{s}) = s$, then we have that $s \sqsubset t$ such that $s \in S_\alpha$ and also $\bar{s} \lhd s$ and $\pi_{\bar{t}t} \upharpoonright \bar{s} = \pi_{\bar{s}s}$.



Now, consider an element $s_i = \mathbf{L}_{s_i(\beta)}^{s_i(D)}$ of the collection $a_s$ for $s \in S_A$. Using an appropriate bijection between $\mathbf{H}_A$ and $A$ we are able to code given elements $s_i$ of $S_{s_i(\alpha)}$, a subset of $\mathbf{H}_A$, as ordinals below $A$. Note, here we use that by the choice of $s_i$ we have $s_i(\alpha) < A$. Therefore, using such a coding we can consider the set $a_s$ as an element of $[A]^\omega$ and together with (4), *cf.* p. 50, as an element of $[\kappa]^\omega$.

But then, the collection $\Theta(s)$ is a subset of $[\kappa]^\omega$. Hence, as above in Lemma 65, we conclude that in the model $M[a_s]$, the set $\Theta(s)$ has cardinality at most $\kappa$.

We are now able to finish the proof with the desired contradiction as above in the end of Chapter 5 as follows: Working within the forcing extension $M[G]$, let $W$ be again the inner model $M[\bar{G}]$, where $\bar{G}$ is $\mathbb{P}(\tau)$-generic as above.

Following the old idea of Chapter 5, we consider $U := \{a_s \mid s \in S_A\}$, being the union of all $\Theta(s)$ for $s$ ranging about all elements of $S_A$. Then $\langle U, <_U \rangle$ still forms a linear order, where the order relation is defined as: $a_{\bar{s}} <_U a_s$ if $\bar{s}(\beta) < s(\beta)$.

And again, considering $a_s$ as a countable subset of $\kappa$ using an appropriate coding, we know by Lemma 51 that $a_s$ is indeed already an element of $M[\bar{G}]$. Hence, as above, $|U_x|^W \leqslant \kappa$.

After all, we use Lemma 9 again and conclude that the cardinality of $U$ is strictly smaller than $\tau$. However, we here have $|U| = |S_A| = \tau$ as well, and so the desired contradiction for the first case.

⋆ ⋆ ⋆

In the second case, where the cofinality of $A$ is uncountable, the contradiction follows exactly as on page 56: We construct the sequence $a_s := \{s_i \,|\, i < \bar{\kappa}\}$ where $\bar{\kappa} = \mathrm{cf}_M(A)$ as above using an uncountable and cofinal sequence in $A$ and consider the set $X := \{a_s \,|\, s \in S_A\}$. We then again conclude that $|X| = |S_A| = \tau$.

On the other hand, again using Lemma 49 we know that each $a_s$ is not only a subset but also an element of the ground model. Having $2^\kappa = \kappa^+$ within $M$, the cardinality of $X$ is at most $\kappa^{+M}$ which is strictly smaller than $\tau$, a contradiction.

Furthermore, because of the forcing properties that we have already described in Chapter 3 and the fact that the ground model satisfies $\mathsf{GCH}$, we know that within the generic extension $M[G]$, the assertion $2^\gamma = \gamma^+$ for $\gamma \geqslant \tau$ is preserved. Note, we also have $2^\omega = 2^\kappa = \tau$ and $\kappa^+ = \tau$. Hence, in $M[G]$ we trivially have $2^{<\gamma} = \gamma$ for $\gamma \geqslant \tau$ and so we always have a $(\gamma^+, \gamma)$-model of $\mathfrak{T}'$ for $\gamma \geqslant \tau$, considering the structure

$$\langle \mathbf{L}_{\gamma^+}[D], \gamma, \in, D \rangle$$

where $D \subseteq \gamma^+$ such that $\mathbf{L}_\gamma[D] = \mathbf{H}_\gamma$. Compare this to the upcoming Lemma 76.

Hence, all arguments of the old proof of Theorem 59 went through and so we have found the desired counterexample even for the weaker theory, defined on page 61, starting just from a model of $\mathsf{GCH}$.

This finishes the survey through the proof. $\boxtimes$(Theorem 68)

CHAPTER 7

# The Failure implies an Inaccessible

In this chapter we are finally going to prove the still remaining and promised second main theorem that –as we will see– will follow easily from known facts, stating the following:

**Theorem 70.** *Suppose there is a model of set theory* ZFC *such that*

$$(\gamma', \gamma) \not\longrightarrow (\kappa^+, \kappa)$$

*holds for a given pair of cardinals* $\gamma' > \gamma \geqslant \omega$ *and an uncountable regular cardinal* $\kappa$*. Then the following theory is consistent*

$$\mathsf{ZFC} \;+\; \text{“}\exists\tau\,(\,\tau \textit{ is inaccessible}\,)\text{”}.$$

For the remaining part of the chapter let $\gamma' > \gamma \geqslant \omega$ be arbitrary (but fixed) cardinals and, moreover, $\kappa$ an uncountable and regular cardinal.

Because we will imitate a proof of Chang where he used model theoretical facts, let us remind the reader to call an infinite structure $\mathfrak{A}$ over a fixed language *$\kappa$-saturated* for a cardinal $\kappa$ if for arbitrary $Y \subseteq \mathbb{A}$ such that $|Y| < \kappa$, a 1-type $p$ over the structure $\langle\, \mathfrak{A}, y \,\rangle_{y \in Y}$ is always realized within the model $\langle\, \mathfrak{A}, y \,\rangle_{y \in Y}$. Here, with the structure $\langle\, \mathfrak{A}, y \,\rangle_{y \in Y}$ we mean the model $\mathfrak{A}$ extended on parameters of $Y$. Finally, call $\mathfrak{A}$ *saturated* if $\mathfrak{A}$ is $|\mathfrak{A}|$-saturated.

Roughly speaking, saturated models are large models where we cannot have a 1-type, that is a consistent set of formulae with just one fixed free variable, which is not a subset of an element type of a given suitable element of the model. We leave out the details on this terminology and strongly refer to standard introductional books on model theory like [ChaKei90] or [Hod93].

Let us now consider the following and very useful statement.

**Theorem 71.** *If $\kappa^+$ is a successor cardinal in* $\mathbf{L}$*, then for arbitrary cardinals $\gamma' > \gamma \geqslant \omega$ the transfer property $(\gamma', \gamma) \longrightarrow (\kappa^+, \kappa)$ holds.*

We are going to prove this statement along the idea of Chang, *cf.* [Cha63], re-written in [Sac72, §23], in his proof of his theorem mentioned already in the introduction in Chapter 1:

**Lemma 72** (Chang, [Cha63], [Sac72, Theorem 23.4])**.** *If $\gamma' > \gamma \geqslant \omega$ and $\kappa$ is a regular uncountable cardinal such that we have $2^{<\kappa} = \kappa$, then the transfer property $(\gamma', \gamma) \longrightarrow (\kappa^+, \kappa)$ holds.*

The key idea is the following: Taking a $(\gamma', \gamma)$-model $\mathfrak{B}$ within the language $\mathcal{L}$ we need to find saturated structures $\mathfrak{A}_0$ and $\mathfrak{A}_1$ of cardinality $\kappa$ such that $\mathfrak{B} \equiv \mathfrak{A}_0 \precneqq \mathfrak{A}_1$ and $R^{\mathfrak{A}_0} = R^{\mathfrak{A}_1}$. Here, the predicate $R$ is a new technical relation that Chang used for the proof which is not contained in the original language $\mathcal{L}$.

In fact, Chang used the following statement:

**Lemma 73** (Chang, [Sac72, Proposition 23.1])**.** *Suppose for a given model $\mathfrak{B}$ we have that $|\mathfrak{B}| > |R^{\mathfrak{B}}| \geqslant \omega$. Let $\kappa$ be a regular uncountable cardinal such that $2^{<\kappa} = \kappa$. Then there exist isomorphic saturated structures $\mathfrak{A}_0$ and $\mathfrak{A}_1$ such that $\mathfrak{B} \equiv \mathfrak{A}_0 \precneqq \mathfrak{A}_1$, $R^{\mathfrak{A}_0} = R^{\mathfrak{A}_1}$ and finally $|\mathfrak{A}_0| = |\mathfrak{A}_1| = |R^{\mathfrak{A}_1}| = \kappa$.*

Chang was using the property of a model $\mathfrak{A}$ being *$R$-saturated*, that is when every 1-type $p$ of the following shape is already realized in $\mathfrak{A}$:

(a) $p$ is a 1-type over the structure $\langle \mathfrak{A}, y \rangle_{y \in Y}$,
(b) $Y \subseteq \mathbb{A}$ such that $|Y| < |\mathbb{A}|$,
(c) $R(x) \in p$.

The interesting part around this property is now the following: The union of an elementary chain of $R$-saturated models is, in fact, again $R$-saturated. This does not hold for arbitrary chains of (just) saturated structures.

Having this, Chang was able to construct an elementary chain of models $\langle \mathfrak{A}_\nu \mid \nu < \kappa^+ \rangle$ such that for all $\nu > 1$, the structure $\mathfrak{A}_\nu$ is again $R$-saturated, $|\mathfrak{A}_\nu| = \kappa$ and $R^{\mathfrak{A}_{\nu'}} = R^{\mathfrak{A}_0}$ for all $\nu' < \nu$. Then, the model $\mathfrak{C} := \bigcup\{\alpha_\nu \mid \nu < \kappa^+\}$ bears the properties $|\mathfrak{C}| = \kappa^+$ and $R^{\mathfrak{C}} = R^{\mathfrak{A}_0}$ and is therefore as desired.

Now, knowing that for the fixed uncountable and regular cardinal $\kappa$, the successor $\kappa^+$ is even a successor within the constructible universe $\mathbf{L}$, choose a suitable ($\mathbf{L}$-)cardinal $\gamma < \kappa^+$ such that within the constructible universe we have $\gamma^+ = \kappa^{+\mathbf{V}}$.

Obviously, in the universe $\mathbf{V}$ there is a collapsing map $\sigma$ from $\gamma$ onto $\kappa$. Coding this map $\sigma$ within an appropriate subset $D \subseteq \kappa$, we can finally arrange that the following holds:

$$\mathbf{L}[D] \models \kappa^+ = \kappa^{+\mathbf{V}}. \tag{5}$$

If there were already a $(\gamma', \gamma)$-model of the theory $\mathfrak{T}$ within the model $\mathbf{L}[D]$, then we could just apply Chang's Theorem within this set theoretical universe and the argumentation would be easier.

In any case, fix a $(\gamma', \gamma)$-model $\mathfrak{B}$ of the theory $\mathfrak{T}$ within the universe $\mathbf{V}$. Following Chang's idea, we can consider an appropriate extension $\mathfrak{S}$ of the complete theory of the structure $\mathfrak{B}$, that Chang suggested in his proof, *cf.* [Sac72, Proof of Proposition 23.1]. In $\mathbf{V}$, the structure $\mathfrak{B}$ will witness the consistency of this theory $\mathfrak{S}$.

However, even within the model $\mathbf{L}[D]$, this theory is consistent because otherwise we could find a (finite) proof sequence witnessing the inconsistency which would be absolute between $\mathbf{L}[D]$ and $\mathbf{V}$. To see this, we could assume by the theorem of Morley and Vaught, mentioned in the introductional Chapter 1, that $\gamma' = \aleph_1$ and $\gamma = \aleph_0$. Therefore, we can extend the predicate $D$, without loss of generality, such that $D$ codes all ($\aleph_1$-many) finite sequences of formulae with parameters of the $(\aleph_1, \aleph_0)$-model $\mathfrak{B}$, providing enough set theory within $\mathbf{L}[D]$ to be able to speak about proof sequences. Note, because $\kappa \geqslant \aleph_1$ this is still possible such that $D \subseteq \kappa$.

And so we can apply Chang's proof idea within $\mathbf{L}[D]$. All he was further needing is $2^{<\kappa} = \kappa$ which we already have, given by Lemma 22 for the fixed uncountable and regular cardinal $\kappa$.

Therefore, having followed the proof idea of Chang, we finally have a $(\kappa^{+\mathbf{L}[D]}, \kappa)$-model $\mathfrak{C}$ within $\mathbf{L}[D]$. However, $\mathfrak{C}$ is still a model of $\mathfrak{T}$ within $\mathbf{V}$. Moreover, it is even a $(\kappa^+, \kappa)$-model in $\mathbf{V}$ because of the absoluteness of ($\kappa$ and) $\kappa^+$, giving by (5).

Hence, the proof is finally done. $\boxtimes$(Lemma 71)

Clearly, if $\kappa^+$ is not a successor cardinal in $\mathbf{L}$, then is must be inaccessible because of having GCH and the fact, that regularity reflects downwards and so we can conclude the following

**Corollary 74.** *If for a regular and uncountable cardinal $\kappa$ and a given pair of infinite cardinals $\gamma' > \gamma$ we have the failed transfer property $(\gamma', \gamma) \not\longrightarrow (\kappa^+, \kappa)$, then $\kappa^+$ is inaccessible within $\mathbf{L}$.*

And finally the second main statement, Theorem 70, is completely proved as well. $\boxtimes$(Theorem 70).

CHAPTER 8

# Further Remarks

Summarizing, we can put both theorems, Theorem 59 and Theorem 70, together and finally we get –as promised earlier– the following

**Theorem 75.** *The theory*

$$\mathsf{ZFC} \;+\; \text{“}\exists\tau\,(\,\tau \textit{ is inaccessible}\,)\text{”}$$

*is equi-consistent to the theory*

$$\mathsf{ZFC} \;+\; \text{“}(\aleph_1, \aleph_0) \not\longrightarrow (\aleph_2, \aleph_1)\text{”}.$$

To apply Theorem 59, we had to make sure that for the chosen $\kappa$ there is an inaccessible cardinal above it. Merging both theorems in a nice equi-consistent statement, we have chosen $\kappa$ minimal, taken $\kappa := \aleph_1$. However, there are obviously more possible choices of $\kappa$ ensuring that it always lies below an inaccessible cardinal.

⋆ ⋆ ⋆

Furthermore, the theory $\mathfrak{T}'$ that we used to find the counterexample in the past chapters to prove the Theorem 68 (or even Theorem 59), is –in fact– even more interesting as the following remarks will show.

Remember, we defined in Chapter 6 the theory as follows:

$$\begin{aligned}\mathfrak{T}' \quad = \quad & \mathsf{ZFC}^- + \mathbf{V} = \mathbf{L}[C] \text{ for } C \subseteq \mathrm{On} + 2^{<\dot{A}} = \dot{A} \\ & + \dot{A} \text{ is the largest cardinal } + \dot{A} \text{ regular.}\end{aligned}$$

First of all, note, that in case that we consider a cardinal $\kappa$ such that $2^{<\kappa} = \kappa$, we always find a canonical $(\kappa^+, \kappa)$-model, just considering

$\langle \mathbf{L}_{\kappa^+}[D], \kappa, \in, D \rangle$ where $D \subseteq \kappa^+$ such that $\mathbf{L}_\kappa[D] = \mathbf{H}_\kappa$. Therefore we have the following

**Lemma 76.** *If $2^{<\kappa} = \kappa$, then the theory $\mathfrak{T}'$ has a $(\kappa^+, \kappa)$-model.*

On the other hand, if there is an inaccessible cardinal $\tau$, then we are able to find –using Mitchell's idea– a forcing extension such that the continuum has cardinality $\kappa^+$, which will be $\tau$, and so the equality $2^{<\kappa} = \kappa$ fails badly. Furthermore, in this forcing extension, $\mathfrak{T}'$ does not have a $(\kappa^+, \kappa)$-model. This fact we have described in earlier chapters, proving Theorem 59 and Theorem 68. And so, we have finally shown the following

**Theorem 77.** *Assuming* GCH*, let $\tau$ be inaccessible. Moreover, consider a regular and uncountable $\kappa < \tau$. Then there is a forcing extension such that within this model of set theory we have $2^\omega = 2^\kappa = \kappa^+ = \tau$ and the theory $\mathfrak{T}'$ does not have a $(\kappa^+, \kappa)$-model. However, the theory $\mathfrak{T}'$ has $(\gamma^+, \gamma)$-models for all regular cardinals $\gamma > \kappa$ or $\gamma = \omega$.*

Now use the fact –mentioned among the fundamental material given in Chapter 2– that if $\kappa^+$ is not Mahlo in $\mathbf{L}$, then there is a special $\kappa^+$-Aronszajn tree. Therefore, choosing $\kappa$ in the first and $\tau$ in the second statement above appropriate, we can arrange the conclusions of both lemmas above either in the *presence* or *absence* of a special $\kappa^+$-Aronszajn tree.

⋆ ⋆ ⋆

Furthermore, analyzing the forcing $\mathbb{M}$ –that Mitchell was providing in [Mit72]– more precisely, we can prove the following fact, changing the focus to other (small) cardinals than just $\omega$. In fact, we are trying to determine the powerset of another small cardinal than $\omega$ as follows.

For, let $\kappa$ and $\tau$ be as usual (*i.e.*, as above in Chapter 3). Moreover, let $\theta < \kappa$ be any infinite regular cardinal. Then define $\mathbb{P}(\theta, \tau)$ as the set $\{\, p \,:\, \exists x\, (p \in {}^x 2 \;\wedge\; |x| < \theta \;\wedge\; x \subseteq \tau \,\}$, ordered by the usual reverse inclusion. Further, we define the set of acceptable functions as we did on page 33, changing the third condition (c) to the condition

(c') for all $\gamma < \tau$ we have $f(\gamma) \in \mathcal{B}_{\gamma+\theta}$.

All other definitions remain the same as in the Chapter 3, getting the second forcing $\dot{\mathbb{Q}}(\theta, \kappa, \tau)$. Finally we define analogously

$$\mathbb{M}(\theta) \ := \ \mathbb{M}(\theta, \kappa, \tau) \ := \ \mathbb{P}(\theta, \tau) \star \dot{\mathbb{Q}}(\theta, \kappa, \tau).$$

Then obviously, we have $\mathbb{M} = \mathbb{M}(\omega)$.

Now, following Mitchell's proven statements, namely [Mit72, Corollar 3.5], we finally obtain in a generic extension of a given ground model using the new partial order $\mathbb{M}(\theta)$ the equation

$$2^\theta = \tau.$$

Even more important, we need the following general version of Lemma 49, namely [Mit72, Lemma 3.8] now for $\mathbb{M}(\theta)$, saying the following:

**Lemma 78** ([Mit72, Lemma 3.8]). *Suppose that* $\mathrm{cf}_M(\gamma) > \theta$ *and let* $t : \gamma \longrightarrow M$ *be such that* $t \in M[G]$ *and* $t \restriction \alpha \in M[G_\nu]$ *for every* $\alpha < \nu$. *Then* $t \in M[G_\nu]$.

With this in mind, we have the following corollaries of the main theorems. Compare the first one to Theorem 59.

**Theorem 79.** *Suppose there is a model of* ZFC *with an inaccessible cardinal* $\tau$. *Moreover, let* $\theta < \kappa$ *be two regular cardinals below* $\tau$. *Then there is a forcing extension of* $\mathbf{L}$ *that is a model of the following:*

$$\begin{aligned} \mathsf{ZFC} \quad + \quad & 2^\theta = \kappa^+ \quad + \quad \text{“there is a special } \kappa^+\text{-Aronszajn tree”} \\ + \quad & \text{“} 2^\alpha = \alpha^+ \text{ for all infinite cardinals } \alpha < \theta \text{ or } \alpha \geqslant \kappa\text{”} \\ + \quad & \text{“} (\gamma^+, \gamma) \not\longrightarrow (\kappa^+, \kappa) \text{ for all regular cardinals } \gamma \neq \kappa\text{”.} \end{aligned}$$

The proof is very similar and almost literally the same as in the case of $\theta = \omega$. Analyzing the two cases within the old proof, on page 51 and 56, we see that the argument of the first case, Case 1, gives us

–even now– a contradiction as long as we have that $\mathrm{cf}_M(A) < \kappa$: This property was important for the argument when we applied the fact that $2^{<\kappa} = \kappa$ to get finally $|[\kappa]^\theta| \leqslant \kappa$.

However, in the second case, Case 2, we were able to deduce a contradiction whenever $\mathrm{cf}_M(A) > \theta$, just using an argument with the former version of Lemma 78 and the fact that we can easily generalize the Lemma 51 to the powerset of $\theta$. In fact, the proof of the last is again literally the same, because we only used the fact that the set of acceptable functions is closed under unions of increasing chains of such functions.

Note, here we use the fact that $\theta$ is strictly less than the regular $\kappa$ and that $\mathbb{P}(\theta, \tau)$ preserves $\mathsf{GCH}$ below $\theta$. Furthermore, $\dot{\mathbb{Q}}(\theta, \kappa, \tau)$ does not change powersets of cardinals below $\theta$. Hence, within the forcing extension we have indeed $2^\alpha = \alpha^+$ for all infinite cardinal $\alpha < \theta$.

In any case, we are getting a contradiction again and so the argument of the old proof goes through again. Hence, Theorem 79 is proved.

Furthermore, we also get the following statement. Compare this to Theorem 68 and Theorem 77.

**Theorem 80.** *Assuming* $\mathsf{GCH}$*, let* $\tau$ *be inaccessible. Moreover, consider two more regular and uncountable cardinals* $\theta < \kappa$ *below* $\tau$*. Then there is a forcing extension such that within this model of set theory we have* $2^\theta = 2^\kappa = \kappa^+ = \tau$ *and the theory* $\mathfrak{T}'$ *does not have a* $(\kappa^+, \kappa)$*-model. However, the theory* $\mathfrak{T}'$ *has* $(\gamma^+, \gamma)$*-models for all regular cardinals* $\gamma < \theta$ *or* $\gamma > \kappa$*. In particular, we have for all regular cardinals* $\gamma < \theta$ *or* $\gamma > \kappa$ *the following failure:*

$$(\gamma^+, \gamma) \not\longrightarrow (\kappa^+, \kappa).$$

Again, the proof is literally the same, because the old one was based on the former version of the theorem that we just proved. This is still possible even in the new situation, using the above considerations.

Note here again, because $\mathbb{P}(\theta, \tau)$ is the standard way of adding $\tau$-many subsets of $\theta$, we know that $\mathsf{GCH}$ is preserved below $\theta$. And as we just above mentioned, the second forcing $\dot{\mathbb{Q}}(\theta, \kappa, \tau)$ does not change powersets of cardinals below $\theta$. This means, in fact, within the forcing extension we still have the equation $2^{<\gamma} = \gamma$ for arbitrary $\gamma < \theta$ and so there are indeed as desired $(\gamma^+, \gamma)$-models of $\mathfrak{T}'$ for all $\gamma < \theta$.

⋆ ⋆ ⋆

This finishes our survey on properties of the theory $\mathfrak{T}'$ which is obviously independent from the existence of a special Aronszajn tree but strong enough to get the desired properties for the transfer property we have discussed in earlier chapters, proving the main statements: Theorem 59 and Theorem 70, or even the general versions of the first one with Theorem 79 and Theorem 80.

APPENDIX

# Weak Square and special Aronszajn Trees

## Introduction

In this chapter we will give a proof of a useful statement, Jensen has originally proved although he has never published it. In fact, he mentioned the equivalence we are going to prove now in his well-known paper [Jen72]. In Chapter 5 we were using this theorem to prove the main Theorem 59. The statement says the following

**Theorem 81** (Jensen). *There is a special $\kappa^+$-Aronszajn tree if and only if $\square^*_\kappa$ holds.*

To fix notation, given a tree $\langle T, <_T \rangle$, let $T_\alpha$ be the set of all elements of $T$ with tree level $\alpha$. Call $T \upharpoonright \alpha$ the (sub-)tree of all elements of $T$ with a tree height strictly less than $\alpha$. Moreover, denote with $\mathrm{rk}_T(x)$ the tree level of an element $x$ of the tree $T$.

We already defined in Chapter 2 what we mean with the combinatorical principle weak square, $\square^*_\kappa$, and with a special $\kappa^+$-Aronszajn tree. However, because we will extend the definitions, let us repeat the following

**Definition 82** (Aronszajn Tree). *We call $\langle T, <_T \rangle$ a $\kappa$-Aronszajn tree if the following hold:*

- *$T$ is a tree of height $\kappa$,*
- *for $\alpha < \kappa$, all levels $T_\alpha$ have cardinality strictly less than $\kappa$,*
- *all branches in $T$ have cardinality strictly less than $\kappa$,*
- *$T$ is normal, that is*
  - *$T$ has exactly one root,*
  - *for each $x \in T_\gamma$ and all level $\gamma' > \gamma$ there is $y \in T_{\gamma'}$ such that $x <_T y$,*

– *every element of $T$ has at least two distinct successors in the next tree level,*
– *above each branch in $T \upharpoonright \lambda$ there is exactly one element at limit height $\lambda$ in $T$.*

The property of being a normal tree is sometimes useful. However, the existence of a tree yielding just the first three conditions is equivalent to the existence of a tree given by the definition above.

We are interested in special Aronszajn trees given by the following

**Definition 83.** *A $\kappa^+$-Aronszajn tree is called special, if there is a function $\sigma : T \longrightarrow \kappa$ such that for all tree elements $x$ and $y$ we have: if $x <_T y$, then $\sigma(x) \neq \sigma(y)$.*

Trivially, a $\kappa$-Aronszajn tree has cardinality $\kappa$. And a $\kappa^+$-Aronszajn tree is special if and only if it is union of $\kappa$-many antichains: on the one hand take $A_\alpha := \sigma^{-1}{''}\{\alpha\}$ and on the other hand let $\sigma(x)$ be defined as the smallest $\alpha < \kappa$ such that $x \in A_\alpha$ for $T = \bigcup_{\alpha<\kappa} A_\alpha$.

Moreover, special $\kappa^+$-Aronszajn trees have antichains of cardinality $\kappa^+$. In particular, such trees are not Souslin trees.

## $\aleph_1$-Aronszajn Trees

Constructing an $\aleph_1$-Aronszajn tree is an easy exercise. To make clear where the problems occur when we try to construct Aronszajn trees for larger cardinals, we will remind the reader of the proof idea in case of the first uncountable cardinal.

By induction on countable ordinals we construct the tree levels $T_\alpha$ we are looking for such that its elements will be strictly monotone functions $f : \alpha \longrightarrow \mathbb{Q}$.

Here, with $\mathbb{Q}$ we denote the well-known linear order of the rational numbers. Moreover, we will have a function $\sup : T_\alpha \longrightarrow \mathbb{Q}$ such that $\sup(f) := \sup \operatorname{rng}(f)$ and whenever $x <_T y$, then $\sup(x) <_{\mathbb{Q}} \sup(y)$.

The induction is very easy. To start with, we take $T_0 := \{\varnothing\}$ and $\sup(\varnothing) := 0$. For the successor step set

$$T_{\beta+1} := \{\, f^\frown q \mid f \in T_\beta,\ \sup(f) < q,\ q \in \mathbb{Q}\,\},$$

and $\sup(f^\frown q) := (f^\frown q)(\beta) = q$ where $f^\frown q := f \cup \{\langle q, \beta \rangle\}$ for $f \in T_\beta$.

Now, in case where $\alpha$ is a countable limit ordinal and for $f \in T \restriction \alpha$ and $q \in \mathbb{Q}$ such that $\sup(f) < q$, we are going to define an extended function $g : \alpha \longrightarrow \mathbb{Q}$ such that $f \subsetneq g$ and $\sup(g) = q$ as follows:

Let $f \in T_\beta$ for a $\beta < \alpha$. Choose an unbounded $C \subseteq \alpha$ above $\beta$ of length $\omega$. Then $C$ is trivially a club subset of $\alpha$. Moreover, choose $\langle q_i \mid i \leqslant \omega \rangle$ such that $q_i < q_j$ for $i < j \leqslant \omega$, $q_0 := \sup(f)$ and $q_\omega = q$. Both can be done easily.

Now, along the (club) set $C = \langle c_i \mid i < \omega \rangle$ starting from $f$ let us look at a branch of the so far defined tree given by tree elements on levels indexed by elements of $C$: For each $c_i \in C$ choose an $f_i \in T_{c_i}$ such that $\sup(f_i) = q_i$. By the successor step, this is trivially possible because there are no limit stages within the enumeration of $C$. Then let $g_{f,q}$ be such that $f_i \subsetneq g_{f,q}$, $\mathrm{dom}(g_{f,q}) = \alpha$ and $\sup(g_{f,q}) = q$. Finally define

$$T_\alpha := \{\, g_{f,q} \mid f \in T \restriction \alpha,\ \sup(f) < q,\ q \in \mathbb{Q}\,\}.$$

It is an easy exercise to show that $T := \bigcup_{\alpha<\aleph_1} T_\alpha$ is an $\aleph_1$-Aronszajn tree. In fact, it is even a special one, considering the map $\sigma : T \longrightarrow \aleph_0$ defined by $\sigma := (f \circ \sup)$ where $f$ is an arbitrary but fixed bijection between $\mathbb{Q}$ and $\aleph_0$.

## The combinatorical Principle $\square^*_\kappa$

Combinatorical Principles, as small fragments of the constructible universe extracted in a useful assertion, have applications in many areas of mathematics.

We will in fact look at sequences with relatively weak properties of coherency. To start with, let us consider the next lemma speaking about equivalent statements, each asserting suitable weak square-sequences.

**Lemma 84.** *For an infinite cardinal and arbitrary closed and unbounded subsets* $\Gamma$ *and* $\Gamma'$ *of limit ordinals below* $\kappa^+$*, the following statements are equivalent:*

(a) *There is a sequence* $\langle C_\alpha \mid \alpha \in \kappa^+ \cap \mathrm{Lim} \rangle$ *such that*
- (i) $\forall \alpha \in \kappa^+ \cap \mathrm{Lim}\ ( C_\alpha \subseteq \alpha \text{ club} )$,
- (ii) $\forall \beta \in \kappa^+ \cap \mathrm{Lim}\ ( |\{C_\alpha \cap \beta \ : \ \alpha \leqslant \beta\}| \leqslant \kappa )$,
- (iii) $\forall \alpha \in \kappa^+ \cap \mathrm{Lim}\ ( \mathrm{otp}(C_\alpha) \leqslant \kappa )$.

(b) *There is a sequence* $\langle \mathcal{C}_\alpha \mid \alpha \in \kappa^+ \cap \mathrm{Lim} \rangle$ *such that*
- (i) $\forall \alpha \in \kappa^+ \cap \mathrm{Lim}\ ( \mathcal{C}_\alpha \subseteq \mathcal{P}(\alpha),\ |\mathcal{C}_\alpha| \leqslant \kappa )$,
- (ii) $\forall C \in \mathcal{C}_\alpha\ ( C \text{ is closed in } \alpha )$,
- (iii) $\exists C \in \mathcal{C}_\alpha\ ( C \text{ is unbounded in } \alpha )$,
- (iv) $\forall \alpha, \beta \in \mathrm{Lim}\ \forall C \in \mathcal{C}_\beta\ ( \alpha < \beta < \kappa^+ \longrightarrow \alpha \cap C \in \mathcal{C}_\alpha )$,
- (v) $\forall C \in \mathcal{C}_\alpha\ ( \mathrm{otp}(C) \leqslant \kappa )$.

(c) *There is a sequence* $\langle C_\alpha \mid \alpha \in \Gamma \rangle$ *such that*
- (i) $\forall \alpha \in \Gamma\ ( C_\alpha \subseteq \alpha \text{ club} )$,
- (ii) $\forall \beta \in \Gamma\ ( |\{C_\alpha \cap \beta \ : \ \alpha \leqslant \beta\}| \leqslant \kappa )$,
- (iii) $\forall \alpha \in \Gamma\ ( \mathrm{otp}(C_\alpha) \leqslant \kappa )$.

(d) *There is a sequence* $\langle \mathcal{C}_\alpha \mid \alpha \in \Gamma' \rangle$ *such that*
- (i) $\forall \alpha \in \Gamma'\ ( \mathcal{C}_\alpha \subseteq \mathcal{P}(\alpha),\ |\mathcal{C}_\alpha| \leqslant \kappa )$,
- (ii) $\forall C \in \mathcal{C}_\alpha\ ( C \text{ is closed in } \alpha )$,
- (iii) $\exists C \in \mathcal{C}_\alpha\ ( C \text{ is unbounded in } \alpha )$,
- (iv) $\forall \alpha, \beta \in \Gamma'\ \forall C \in \mathcal{C}_\beta\ ( \alpha < \beta < \kappa^+ \longrightarrow \alpha \cap C \in \mathcal{C}_\alpha )$,
- (v) $\forall C \in \mathcal{C}_\alpha\ ( \mathrm{otp}(C) \leqslant \kappa )$.

*Proof.* Considering $\mathcal{C}_\alpha := \{C_\beta \cap \alpha \mid \beta \geqslant \alpha\}$, we can conclude the implication from (a) to (b). Similarly, we show that (c) implies (d).

The other implication, to get (a) from (b) and (c) from (d), respectively, we simply choose as the desired $C_\alpha$ a closed and unbounded subset of the given $\mathcal{C}_\alpha$.

Take the restriction to $\Gamma$ of the sequence given by (a) and we get a sequence asserted by (c).

Now, let $\langle C^*_\alpha | \alpha \in \Gamma \rangle$ be a sequence in the sense of (c). Let $\langle \gamma_\beta | \beta < \kappa^+ \rangle$ be a monotone enumeration of $\Gamma$. Define $\Gamma^* := (\kappa^+ \cap \mathrm{Lim}) \setminus \Gamma$. If $\Gamma^*$ is empty, we would be done. Otherwise choose for $\alpha \in \Gamma^*$ a set $C_\alpha$ as follows:

Let $\beta < \kappa^+$ be such that $\gamma_\beta < \alpha < \gamma_{\beta+1}$. Then choose $C_\alpha$ as a closed and unbounded subset of $\alpha$ such that $\min C_\alpha > \gamma_\beta$ and $\mathrm{otp}(C_\alpha) \leqslant \kappa$. This can be done easily. For $\alpha \in \Gamma$ just take $C_\alpha := C^*_\alpha$ and we have constructed a sequence in the sense of (a).

Finally, the lemma is proved. $\boxtimes$(Lemma 84)

In [Jen72], Jensen called the equivalent assertions of the last lemma *weak square*, $\Box^*_\kappa$. The described sequence is called (in any of the four cases) a $\Box^*_\kappa$-sequence.

As we already have seen in Chapter 2, the following hold:

(a) If $\kappa^{<\kappa} = \kappa$, then $\Box^*_\kappa$.
(b) If $\Box_\kappa$, then $\Box^*_\kappa$.

## A special $\kappa^+$-Aronszajn Tree implies $\Box^*_\kappa$

Let $\langle T, <_T \rangle$ be a special $\kappa^+$-Aronszajn tree and $\sigma : T \longrightarrow \kappa$ such that whenever $x <_T y$, then $\sigma(x) \neq \sigma(y)$. We can assume, without loss of generality, that $T$ is just $\kappa^+$.

We are going to construct a weak square sequence $\langle C_\alpha \mid \alpha \in \Gamma \rangle$ in a sense of (c). For, consider the function $f : \kappa^+ \longrightarrow \kappa^+$, defined by $f(\alpha) := \max\{ \bigcup T_\alpha, \mathrm{rk}_T(\alpha) \}$. Then, using Lemma 8, we know that the set $\Delta := \{\alpha < \kappa^+ \mid f''\alpha \subseteq \alpha\}$ is a club subset of $\kappa^+$.

Now, take an element $\alpha$ of $\Delta$. Then we have $\alpha = T \restriction \alpha$, because, for the first inclusion, let $\beta < \alpha$ and $\gamma$ be the tree rank of $\beta$. Then $\beta \in T_\gamma$ and so $\gamma \leqslant f(\beta) \in f''\alpha \subseteq \alpha$. Hence, $\beta \in T_\gamma \subseteq T \restriction \alpha$.

For the second inclusion let $\beta$ be in $T_\gamma$ for its tree level $\gamma < \alpha$. Then we conclude $\beta \leqslant f(\gamma) \in f''\alpha \subseteq \alpha$.

Therefore, setting $\Gamma^* := \{\alpha < \kappa^+ \,:\, T \restriction \alpha = \alpha\}$, we know that the club set $\Delta$ is a subset of $\Gamma^*$ and because it is obviously closed, $\Gamma^*$ is a closed and unbounded subset of $\kappa^+$ as well.

Now, let $\Gamma$ be the set of all limit points of $\Gamma^*$. Then by Lemma 8, this set $\Gamma$ is still a closed and unbounded subset of $\kappa^+$.

Furthermore, choose for each $\alpha \in \Gamma$ an element $x_\alpha$ with tree level $\alpha$ and define branches $b_\alpha := \{z \in T \mid z <_T x_\alpha\}$ below each chosen $x_\alpha$. Then we can conclude that

$$b_\alpha \subseteq \alpha \text{ is an unbounded subset.} \tag{6}$$

This is easy to check: Obviously, $b_\alpha \subseteq T \restriction \alpha = \alpha$. So, let $\beta < \alpha$. We will show that $b_\alpha \nsubseteq \beta$. Because of $\alpha \in \Gamma$ we have an $\alpha^*$ such that $\beta < \alpha^* < \alpha$ and $\alpha^* \in \Gamma^*$. Therefore, there is a $y \in T_{\alpha^*} \cap b_\alpha$ and so $b_\alpha \nsubseteq T \restriction \alpha^* = \alpha^*$.

We now can fix an arbitrary $\alpha < \kappa^+$ and define $C_\alpha$ as a cofinal subset in $b_\alpha$ of order type at most $\kappa$ by choosing increasing elements $t^{(\alpha)}_\beta$ of the branch $b_\alpha$ minimal with respect to the given function $\sigma$ by induction as follows:

Let $t^{(\alpha)}_0 \in b_\alpha$ be such that $\sigma(t^{(\alpha)}_0) = \min \sigma'' b_\alpha$ and let $t^{(\alpha)}_\beta \in b_\alpha$ be such that $\sigma(t^{(\alpha)}_\beta) = \min\{\, \sigma(z) \mid (\forall \gamma < \beta)(t^{(\alpha)}_\gamma <_T z <_T x_\alpha) \,\}$. This is well-defined because of the one-to-one property of $\sigma$ on the branch $b_\alpha$.

We go on with the construction till it breaks down. Let $\gamma_\alpha$ be minimal such that $t_{\gamma_\alpha}$ does not exist. Then $\gamma_\alpha$ is a limit ordinal because otherwise, the set $\{z \mid t^{(\alpha)}_\beta <_T z <_T x_\alpha\}$ would be non-empty for $\gamma_\alpha = \beta + 1$ and so the definition would not break.

Moreover, the set $C'_\alpha := \{t^{(\alpha)}_\beta \mid \beta < \gamma_\alpha\}$ is cofinal in $b_\alpha$. Otherwise we would have that the set $\{z \mid (\forall \beta < \gamma)(t_\beta <_T z <_T x_\alpha)\}$ is non-empty and so $t_{\gamma_\alpha}$ would be defined again. Therefore, we also have by (6) that

$$C'_\alpha \text{ is cofinal in } \alpha. \tag{7}$$

Trivially, by definition, the sequence $\langle\, \sigma(t^{(\alpha)}_\beta) \,|\, \beta < \gamma_\alpha \,\rangle$ is strictly monotone in $\kappa$. And so, we conclude that the order type of $C'_\alpha$ is at most $\kappa$ and hence $\gamma_\alpha \leqslant \kappa$.

Finally, let $C_\alpha$ be the closure of $C'_\alpha$ by taking all limit points below $\alpha$ such that $C_\alpha$ is a closed and unbounded subset of $\alpha$ of order type at most $\kappa$.

The important property of a (weak) square sequence is its coherency we still have to prove with the next

**Lemma 85.** *For all $\alpha < \kappa^+$ we have $|\,\{\, C_\alpha \cap \beta \,:\, \alpha \geqslant \beta \,\}\,| \leqslant \kappa$.*

*Proof.* Define for every $x \in T_\xi$ where $\xi$ is a limit ordinal, the branch $b_x := \{z \,|\, z <_T x\}$ and the sequence $t^{(x)} := \{t^{(x)}_\beta \,|\, \beta < \gamma_x\}$ as we did for the $x_\alpha$'s above as follows:

Let $t^{(x)}_0$ be such that $\sigma(t^{(x)}_0) = \min \sigma'' b_x$ and again $t^{(x)}_\beta \in b_x$ satifying

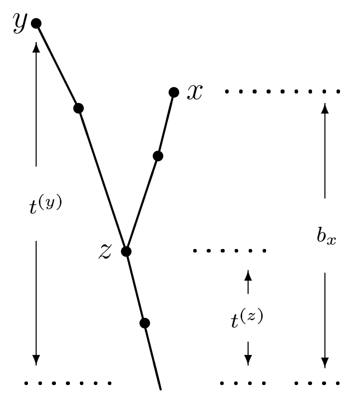
$$\sigma(t^{(x)}_\beta) = \min\{\, \sigma(z) \,|\, (\forall \gamma < \beta)(t^{(x)}_\gamma <_T z <_T x) \,\}. \tag{8}$$

Then for each limit ordinal $\alpha < \kappa^+$ we have $t^{(\alpha)} = t^{(x_\alpha)}$, and moreover, as above, that $t^{(x)}$ is cofinal in $b_x$. Further, $t^{(x)}$ is strictly monotone with respect to the tree relation $<_T$ and the order type of $t^{(x)}$ is $\gamma_x \leqslant \kappa$.

Consider now the sequence $t^{(y)} \cap b_x$ for a fixed tree element $y$ at limit height and let $z := \sup_T t^{(y)} \cap b_x$. Then we obviously have

$$t^{(y)} \cap b_x = t^{(z)} \tag{9}$$

by the definition given in (8) using the map $\sigma$ that is one-to-one on a fixed branch.



Therefore, for each limit ordinal $\beta < \kappa^+$ we then have that the interesting set $\{t^{(\alpha)} \cap \beta \mid \alpha \geqslant \beta\}$ is a subset of all small branches of the set $\{t^{(z)} \mid z \in T \restriction (\beta+1)\}$. However, the cardinality of the initial segment $T \restriction (\beta+1)$ of the tree is at most $\kappa$. And hence, the cardinality of the set $\{C_\alpha \cap \beta \,:\, \alpha \geqslant \beta\}$ is at most $\kappa$ for all $\beta < \kappa^+$ and so the lemma is proved. $\boxtimes$(Lemma 85)

Therefore, with the sequence $\langle C_\alpha \mid \alpha \in \Gamma \rangle$ we finally have found the desired $\square^*_\kappa$-sequence.

### Construction of the partial Order

We now try to imitate the desired behavior of the structure of the rational numbers with a new partial order of cardinality $\kappa$. Therefore let us prove the following

**Lemma 86.** *There is a partial order $\langle \mathbb{P}, <_\mathbb{P} \rangle$ and a subset $\mathcal{S}$ of the set of all sequences with elements of $\mathbb{P}$ such that:*

(a) *$\mathbb{P}$ is partial order, ${}^1\mathbb{P} \subseteq \mathcal{S}$, $|\mathcal{S}| \leqslant |\mathbb{P}| = \kappa$,*
(b) *$\mathbf{0}, \mathbf{1} \in \mathbb{P}$ such that for every $p \in \mathbb{P}$ we have $\mathbf{0} \leqslant_\mathbb{P} p \leqslant_\mathbb{P} \mathbf{1}$,*
(c) *for all $p, q \in \mathbb{P}$ there is an element $q' \in \mathbb{P}$ such that whenever $p <_\mathbb{P} q$, then $p <_\mathbb{P} q' <_\mathbb{P} q$,*
(d) *for every $s \in \mathcal{S}$ there is a limit ordinal $\alpha \in \kappa+1$ or $\alpha = 0$ such that $\mathrm{dom}(s) = \alpha + 1$ and $s$ is strictly monotone with respect to the relation $<_\mathbb{P}$,*
(e) *for every $s \in \mathcal{S}$ and for all limit ordinals $\alpha \in \mathrm{dom}(s)$ we have $s \restriction (\alpha+1) \in \mathcal{S}$,*
(f) *for every limit ordinal $\alpha \in \kappa + 1$ and all $p, q \in \mathbb{P}$, $p < q$, there is an $s \in \mathcal{S}$ such that $\mathrm{dom}(s) = \alpha + 1$, $s(0) = p$ and $s(\alpha) = q$.*

*Proof.* Consider an elementary submodel $H$ of $\mathbf{H}_{\kappa^+}$ of cardinality $\kappa$ such that $\kappa \subseteq H$. Then $H$ is transitive because for each element $x$ of $H$ there is a surjection from $\kappa$ onto $x$ within the elementary submodel $H$ of $\mathbf{H}_{\kappa^+}$ and therefore, with the domain also the whole range $x$ is a subset of $H$.

Let $\mathbb{P} := \{\, X \subseteq \kappa \mid X = \kappa \ \vee\ (X \in H \ \wedge\ |\kappa \backslash X| = \kappa\,)\,\}$ and define the following relation on $\mathbb{P}$ as follows

$$X \sqsubset Y \quad :\Longleftrightarrow \quad X \subseteq Y \subseteq \kappa \ \wedge\ |Y \backslash X| = \kappa.$$

We still have to work to define the set of sequences $\mathcal{S}$. Therefore, let $\Theta$ be a function such that for all $\alpha$ such that $0 < \alpha \leqslant \kappa$ and $X, Y \subseteq \kappa$ where $X \sqsubset Y$ we have $\Theta(\alpha, X, Y) := \langle\, Z_\gamma \mid \gamma \leqslant \alpha \,\rangle \in H$ such that

- $Z_0 = X$, $Z_\alpha = Y$;
- for every $\gamma < \gamma' \leqslant \alpha$ we have $Z_\gamma \sqsubset Z_{\gamma'}$;
- for every limit ordinal $\lambda \in \alpha + 1$ we have $Z_\lambda = \bigcup_{\gamma<\lambda} Z_\gamma$.

For, let $\alpha$ and $X$, $Y$ be given and fix a bijection $f : \alpha \times \kappa \longrightarrow Y \backslash X$ in $H$. Let $Z_\gamma$ be $X \cup f''(\gamma \times \kappa)$ for $\gamma \leqslant \alpha$. Then $Z_0$ and $Z_\alpha$ have the desired properties and because of $f''(\lambda \times \kappa) = \bigcup_{\gamma<\kappa} f''(\gamma \times \kappa)$ we have the property for $Z_\lambda$ where $\lambda$ is a limit ordinal. Moreover, the missing property is given by the following

$$|Z_{\gamma'} \backslash Z_\gamma| = |f''(\gamma' \times \kappa) \backslash f''(\gamma \times \kappa)| = |(\gamma' \backslash \gamma) \times \kappa| = \kappa.$$

Finally, let

$$\mathcal{S} \ := \ \{\, \Theta(\gamma, p, q) \restriction (\gamma' + 1) \ \mid \text{limit ordinals } \gamma' \leqslant \gamma \leqslant \kappa; \\ \text{or } \gamma' = 0;\ p,\ q \in \mathbb{P};\ p < q \,\}.$$

Moreover, for given $p$ and $q$, elements of $\mathbb{P}$, such that $p <_{\mathbb{P}} q$ we have for $q' := \Theta(2, p, q)(1)$ that $p <_{\mathbb{P}} q' <_{\mathbb{P}} q$.

Then the partial order $\langle\, \mathbb{P}, \sqsubset \,\rangle$ and the set $\mathcal{S}$ have the desired properties and so the lemma is proved. $\boxtimes$(Lemma 86)

## $\Box^*_\kappa$ implies a special $\kappa^+$-Aronszajn Tree

Having the partial order $\langle\, \mathbb{P}, <_{\mathbb{P}} \,\rangle$ we constructed in the last section, we will now build up a $\kappa^+$-Aronszajn tree using a suitable coherent sequence. For, let $\langle\, \mathcal{C}_\alpha \mid \alpha \in \kappa^+ \cap \mathrm{Lim} \,\rangle$ be a $\Box^*_\kappa$-sequence in the sense of (b) of the defining Lemma 84.

By induction on $\alpha < \kappa^+$, we are going to define the following objects

(a) the tree levels $T_\alpha$ consisting of suitable strictly monotone functions $f : \alpha \longrightarrow \mathbb{P}$,

(b) a function $\sup : T_\alpha \longrightarrow \mathbb{P}$ such that the following hold:
- $\sup(\varnothing) = 0$,
- if $\alpha$ is a limit ordinal, then $\sup(f)$ is a supremum of the range of $f$ in the partial order $\mathbb{P}$,
- if $\alpha = \beta + 1$, then $\sup(f)$ is just $f(\beta)$,

(c) a partial function $\mathsf{Ex}_\alpha : T_\alpha \times \mathbb{P} \longrightarrow T_\alpha$ such that $\mathsf{Ex}_\alpha(f, q)$ is defined if and only if $\sup(f) < q$, and $\mathsf{Ex}_\alpha(f, q) = g$ where $f \subseteq g$, $\sup(g) = q$ and $\mathrm{dom}(g) = \alpha$,

We simply can start the induction, letting $T_0 := \{\varnothing\}$. Even in the successor step $\alpha = \beta + 1$ we will extend the so far defined tree $T \restriction \alpha$ maximally possible in the following sense:

$$\begin{aligned} T_\alpha &:= \{ f^\frown q \mid f \in T_\beta,\ \sup(f) < q,\ q \in \mathbb{P} \}; \\ \sup(f^\frown q) &:= (f^\frown q)(\beta) = q; \\ \mathsf{Ex}_{\beta+1}(f, q) &:= \begin{cases} f^\frown q & : \text{if } f \in T_\beta,\ \sup(f) < q; \\ \mathsf{Ex}_\beta(f, q'_{f,q})^\frown q & : \text{if } f \in T \restriction \beta,\ \sup(f) < q. \end{cases} \end{aligned}$$

Here, we let $q'_{f,q}$ arbitrary be chosen such that $\sup(f) < q' < q$. The existence of such a $q'_{f,q}$ follows from the properties of $\mathbb{P}$.

And finally, for limit ordinals $\lambda < \kappa^+$, let a club set $C \in \mathcal{C}_\lambda$, a tree element $f \in T_\beta$ where $\beta < \lambda$, and a $q \in \mathbb{P}$ where $\sup(f) < q < \mathbf{1}_\mathbb{P}$ be given. Furthermore, define $X := \{\beta\} \cup C \backslash \beta \cup \{\lambda\}$. Then let $\delta$ be the order type of $X$ and $t : \delta \longrightarrow X$ be the monotone enumeration of $X$. Last but not least, choose an $s \in \mathcal{S}$ be such that $s(0) = \sup(f)$, $s(\delta - 1) = q$ and $\mathrm{dom}(s) = \delta$.

Define then a branch through the segment $T \restriction \lambda$ of the tree as follows: set $f_0 := f$, $f_{\gamma+1} := \mathsf{Ex}_{t(\gamma+1)}(f_\gamma, s(\gamma + 1))$, and $f_{\lambda'} := \bigcup_{\gamma<\lambda'} f_\gamma$. Here we use the coherency of the $\square^*_\kappa$-sequence that $C \cap t(\lambda')$ is an element of $\mathcal{C}_{t(\lambda')}$, and moreover, that $s \restriction (\lambda' + 1)$ is an element of $\mathcal{S}$. Hence, we indeed used $f_{\lambda'}$ for the definition of the tree level $T_{\lambda'}$.

Further, let $g(f, q, C, s)$ be $f_{\delta-1}$. Note, $t(\delta - 1) = \lambda$. Then define

$$T_\lambda := \{\, g(f, q, C, s) \;|\; f \in T \restriction \lambda,\ \sup(f) < q \in \mathbb{P},\ C \in \mathcal{C}_\lambda, \\ \delta \text{ defined as above}, \\ s \in \mathcal{S},\ s(0) = \sup(f),\ s(\delta - 1) = q \,\}$$

And finally, $\mathsf{Ex}_\lambda(f, q) := g(f, q, C, s)$ for arbitrary choosen $C \in \mathcal{C}_\lambda$ and $s \in \mathcal{S}$ such that $s(0) = \sup(f)$ and $s(\delta - 1) = q$ where $\delta$ is defined as above.

The exact choice of the club set $C$ and the sequence $s$ is not necessary to determine. What we need here is that we extend $f$ to the level $\lambda$ and the supremum $q$. Which way we exactly choose through the already defined tree $T \restriction \lambda$ is not important because of the coherency of the $\square^*_\kappa$-sequence and $\mathcal{S}$.

Last but not least, let $\sup_\lambda(\, g(f, q, C, s)\,) := q$ and the tree relation is then given by

$$f <_T g \quad \text{if and only if} \quad g \restriction \mathrm{dom}(f) = f \text{ and } f \neq g.$$

Then the set $T$ defined as the union of the defined levels $T_\alpha$ for $\alpha < \kappa^+$ is obviously a $\kappa^+$-Aronszajn tree. Moreover, the function $\sup : T \longrightarrow \mathbb{P}$ yields the property that for $f <_T g$ we have $\sup(f) <_\mathbb{P} \sup(g)$. That means, together with a fixed bijection between $\mathbb{P}$ and $\kappa$ we easily can find a map witnessing that $T$ is even special.

Finally, the theorem is completely proved. $\boxtimes$(Theorem 81)

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# Glossary

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