CHAPTER 8

# Further Remarks

Summarizing, we can put both theorems, Theorem 59 and Theorem 70, together and finally we get –as promised earlier– the following

**Theorem 75.** *The theory*

$$\mathsf{ZFC} \;+\; \text{“}\exists\tau\,(\,\tau \textit{ is inaccessible}\,)\text{”}$$

*is equi-consistent to the theory*

$$\mathsf{ZFC} \;+\; \text{“}(\aleph_1,\aleph_0) \not\longrightarrow (\aleph_2,\aleph_1)\text{”}.$$

To apply Theorem 59, we had to make sure that for the chosen $\kappa$ there is an inaccessible cardinal above it. Merging both theorems in a nice equi-consistent statement, we have chosen $\kappa$ minimal, taken $\kappa := \aleph_1$. However, there are obviously more possible choices of $\kappa$ ensuring that it always lies below an inaccessible cardinal.

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Furthermore, the theory $\mathfrak{T}'$ that we used to find the counterexample in the past chapters to prove the Theorem 68 (or even Theorem 59), is –in fact– even more interesting as the following remarks will show.

Remember, we defined in Chapter 6 the theory as follows:

$$\begin{aligned}\mathfrak{T}' \quad = \quad \mathsf{ZFC}^- \;+\; \mathbf{V} = \mathbf{L}[C] \text{ for } C \subseteq \mathrm{On} \;+\; 2^{<\dot{A}} = \dot{A}\\ +\; \dot{A} \text{ is the largest cardinal} \;+\; \dot{A} \text{ regular.}\end{aligned}$$

First of all, note, that in case that we consider a cardinal $\kappa$ such that $2^{<\kappa} = \kappa$, we always find a canonical $(\kappa^+,\kappa)$-model, just considering

$\langle \mathbf{L}_{\kappa^+}[D], \kappa, \in, D \rangle$ where $D \subseteq \kappa^+$ such that $\mathbf{L}_\kappa[D] = \mathbf{H}_\kappa$. Therefore we have the following

**Lemma 76.** *If* $2^{<\kappa} = \kappa$, *then the theory* $\mathfrak{T}'$ *has a* $(\kappa^+, \kappa)$*-model.*

On the other hand, if there is an inaccessible cardinal $\tau$, then we are able to find –using Mitchell's idea– a forcing extension such that the continuum has cardinality $\kappa^+$, which will be $\tau$, and so the equality $2^{<\kappa} = \kappa$ fails badly. Furthermore, in this forcing extension, $\mathfrak{T}'$ does not have a $(\kappa^+, \kappa)$-model. This fact we have described in earlier chapters, proving Theorem 59 and Theorem 68. And so, we have finally shown the following

**Theorem 77.** *Assuming* GCH, *let* $\tau$ *be inaccessible. Moreover, consider a regular and uncountable* $\kappa < \tau$. *Then there is a forcing extension such that within this model of set theory we have* $2^\omega = 2^\kappa = \kappa^+ = \tau$ *and the theory* $\mathfrak{T}'$ *does not have a* $(\kappa^+, \kappa)$*-model. However, the theory* $\mathfrak{T}'$ *has* $(\gamma^+, \gamma)$*-models for all regular cardinals* $\gamma > \kappa$ *or* $\gamma = \omega$.

Now use the fact –mentioned among the fundamental material given in Chapter 2– that if $\kappa^+$ is not Mahlo in $\mathbf{L}$, then there is a special $\kappa^+$-Aronszajn tree. Therefore, choosing $\kappa$ in the first and $\tau$ in the second statement above appropriate, we can arrange the conclusions of both lemmas above either in the *presence* or *absence* of a special $\kappa^+$-Aronszajn tree.

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Furthermore, analyzing the forcing $\mathbb{M}$ –that Mitchell was providing in [Mit72]– more precisely, we can prove the following fact, changing the focus to other (small) cardinals than just $\omega$. In fact, we are trying to determine the powerset of another small cardinal than $\omega$ as follows.

For, let $\kappa$ and $\tau$ be as usual (*i.e.*, as above in Chapter 3). Moreover, let $\theta < \kappa$ be any infinite regular cardinal. Then define $\mathbb{P}(\theta, \tau)$ as the set $\{ p : \exists x\, (p \in {}^x2 \wedge |x| < \theta \wedge x \subseteq \tau \}$, ordered by the usual reverse inclusion. Further, we define the set of acceptable functions as we did on page 33, changing the third condition (c) to the condition

(c') for all $\gamma < \tau$ we have $f(\gamma) \in \mathcal{B}_{\gamma+\theta}$.

All other definitions remain the same as in the Chapter 3, getting the second forcing $\dot{\mathbb{Q}}(\theta, \kappa, \tau)$. Finally we define analogously

$$\mathbb{M}(\theta) \ := \ \mathbb{M}(\theta, \kappa, \tau) \ := \ \mathbb{P}(\theta, \tau) \star \dot{\mathbb{Q}}(\theta, \kappa, \tau).$$

Then obviously, we have $\mathbb{M} = \mathbb{M}(\omega)$.

Now, following Mitchell's proven statements, namely [Mit72, Corollar 3.5], we finally obtain in a generic extension of a given ground model using the new partial order $\mathbb{M}(\theta)$ the equation

$$2^\theta = \tau.$$

Even more important, we need the following general version of Lemma 49, namely [Mit72, Lemma 3.8] now for $\mathbb{M}(\theta)$, saying the following:

**Lemma 78** ([Mit72, Lemma 3.8]). *Suppose that* $\mathrm{cf}_M(\gamma) > \theta$ *and let* $t : \gamma \longrightarrow M$ *be such that* $t \in M[G]$ *and* $t \restriction \alpha \in M[G_\nu]$ *for every* $\alpha < \nu$. *Then* $t \in M[G_\nu]$.

With this in mind, we have the following corollaries of the main theorems. Compare the first one to Theorem 59.

**Theorem 79.** *Suppose there is a model of* ZFC *with an inaccessible cardinal* $\tau$. *Moreover, let* $\theta < \kappa$ *be two regular cardinals below* $\tau$. *Then there is a forcing extension of* $\mathbf{L}$ *that is a model of the following:*

$$\begin{aligned} \mathsf{ZFC} \quad + \quad & 2^\theta = \kappa^+ \quad + \quad \text{"there is a special } \kappa^+\text{-Aronszajn tree"} \\ + \quad & \text{"}2^\alpha = \alpha^+ \text{ for all infinite cardinals } \alpha < \theta \text{ or } \alpha \geqslant \kappa\text{"} \\ + \quad & \text{"}(\gamma^+, \gamma) \not\longrightarrow (\kappa^+, \kappa) \text{ for all regular cardinals } \gamma \neq \kappa\text{".} \end{aligned}$$

The proof is very similar and almost literally the same as in the case of $\theta = \omega$. Analyzing the two cases within the old proof, on page 51 and 56, we see that the argument of the first case, Case 1, gives us

–even now– a contradiction as long as we have that $\mathrm{cf}_M(A) < \kappa$: This property was important for the argument when we applied the fact that $2^{<\kappa} = \kappa$ to get finally $|[\kappa]^\theta| \leqslant \kappa$.

However, in the second case, Case 2, we were able to deduce a contradiction whenever $\mathrm{cf}_M(A) > \theta$, just using an argument with the former version of Lemma 78 and the fact that we can easily generalize the Lemma 51 to the powerset of $\theta$. In fact, the proof of the last is again literally the same, because we only used the fact that the set of acceptable functions is closed under unions of increasing chains of such functions.

Note, here we use the fact that $\theta$ is strictly less than the regular $\kappa$ and that $\mathbb{P}(\theta, \tau)$ preserves GCH below $\theta$. Furthermore, $\dot{\mathbb{Q}}(\theta, \kappa, \tau)$ does not change powersets of cardinals below $\theta$. Hence, within the forcing extension we have indeed $2^\alpha = \alpha^+$ for all infinite cardinal $\alpha < \theta$.

In any case, we are getting a contradiction again and so the argument of the old proof goes through again. Hence, Theorem 79 is proved.

Furthermore, we also get the following statement. Compare this to Theorem 68 and Theorem 77.

**Theorem 80.** *Assuming* GCH*, let $\tau$ be inaccessible. Moreover, consider two more regular and uncountable cardinals $\theta < \kappa$ below $\tau$. Then there is a forcing extension such that within this model of set theory we have $2^\theta = 2^\kappa = \kappa^+ = \tau$ and the theory $\mathfrak{T}'$ does not have a $(\kappa^+, \kappa)$-model. However, the theory $\mathfrak{T}'$ has $(\gamma^+, \gamma)$-models for all regular cardinals $\gamma < \theta$ or $\gamma > \kappa$. In particular, we have for all regular cardinals $\gamma < \theta$ or $\gamma > \kappa$ the following failure:*

$$(\gamma^+, \gamma) \not\longrightarrow (\kappa^+, \kappa).$$

Again, the proof is literally the same, because the old one was based on the former version of the theorem that we just proved. This is still possible even in the new situation, using the above considerations.

Note here again, because $\mathbb{P}(\theta, \tau)$ is the standard way of adding $\tau$-many subsets of $\theta$, we know that $\mathsf{GCH}$ is preserved below $\theta$. And as we just above mentioned, the second forcing $\dot{\mathbb{Q}}(\theta, \kappa, \tau)$ does not change powersets of cardinals below $\theta$. This means, in fact, within the forcing extension we still have the equation $2^{<\gamma} = \gamma$ for arbitrary $\gamma < \theta$ and so there are indeed as desired $(\gamma^+, \gamma)$-models of $\mathfrak{T}'$ for all $\gamma < \theta$.

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This finishes our survey on properties of the theory $\mathfrak{T}'$ which is obviously independent from the existence of a special Aronszajn tree but strong enough to get the desired properties for the transfer property we have discussed in earlier chapters, proving the main statements: Theorem 59 and Theorem 70, or even the general versions of the first one with Theorem 79 and Theorem 80.