CHAPTER 7

The Failure implies an Inaccessible

In this chapter we are finally going to prove the still remaining and promised second main theorem that –as we will see– will follow easily from known facts, stating the following:

Theorem 70. *Suppose there is a model of set theory* ZFC *such that*

$$(\gamma', \gamma) \not\longrightarrow (\kappa^+, \kappa)$$

holds for a given pair of cardinals $\gamma' > \gamma \geqslant \omega$ *and an uncountable regular cardinal* κ*. Then the following theory is consistent*

$$\mathsf{ZFC} \;\; + \;\; \text{“}\exists\tau\,(\,\tau \textit{ is inaccessible}\,)\text{”}.$$

For the remaining part of the chapter let $\gamma' > \gamma \geqslant \omega$ be arbitrary (but fixed) cardinals and, moreover, κ an uncountable and regular cardinal.

Because we will imitate a proof of Chang where he used model theoretical facts, let us remind the reader to call an infinite structure $\mathfrak{A}$ over a fixed language *κ-saturated* for a cardinal κ if for arbitrary $Y \subseteq \mathbb{A}$ such that $|Y| < \kappa$, a 1-type p over the structure $\langle\, \mathfrak{A}, y \,\rangle_{y \in Y}$ is always realized within the model $\langle\, \mathfrak{A}, y \,\rangle_{y \in Y}$. Here, with the structure $\langle\, \mathfrak{A}, y \,\rangle_{y \in Y}$ we mean the model $\mathfrak{A}$ extended on parameters of Y. Finally, call $\mathfrak{A}$ *saturated* if $\mathfrak{A}$ is $|\mathfrak{A}|$-saturated.

Roughly speaking, saturated models are large models where we cannot have a 1-type, that is a consistent set of formulae with just one fixed free variable, which is not a subset of an element type of a given suitable element of the model. We leave out the details on this terminology and strongly refer to standard introductional books on model theory like [ChaKei90] or [Hod93].

Let us now consider the following and very useful statement.

Theorem 71. *If κ^+ is a successor cardinal in* $\mathbf{L}$*, then for arbitrary cardinals $\gamma' > \gamma \geqslant \omega$ the transfer property $(\gamma', \gamma) \longrightarrow (\kappa^+, \kappa)$ holds.*

We are going to prove this statement along the idea of Chang, *cf.* [Cha63], re-written in [Sac72, §23], in his proof of his theorem mentioned already in the introduction in Chapter 1:

Lemma 72 (Chang, [Cha63], [Sac72, Theorem 23.4])**.** *If $\gamma' > \gamma \geqslant \omega$ and κ is a regular uncountable cardinal such that we have $2^{<\kappa} = \kappa$, then the transfer property $(\gamma', \gamma) \longrightarrow (\kappa^+, \kappa)$ holds.*

The key idea is the following: Taking a (γ', γ)-model $\mathfrak{B}$ within the language $\mathcal{L}$ we need to find saturated structures $\mathfrak{A}_0$ and $\mathfrak{A}_1$ of cardinality κ such that $\mathfrak{B} \equiv \mathfrak{A}_0 \precneqq \mathfrak{A}_1$ and $R^{\mathfrak{A}_0} = R^{\mathfrak{A}_1}$. Here, the predicate R is a new technical relation that Chang used for the proof which is not contained in the original language $\mathcal{L}$.

In fact, Chang used the following statement:

Lemma 73 (Chang, [Sac72, Proposition 23.1])**.** *Suppose for a given model $\mathfrak{B}$ we have that $|\mathfrak{B}| > |R^{\mathfrak{B}}| \geqslant \omega$. Let κ be a regular uncountable cardinal such that $2^{<\kappa} = \kappa$. Then there exist isomorphic saturated structures $\mathfrak{A}_0$ and $\mathfrak{A}_1$ such that $\mathfrak{B} \equiv \mathfrak{A}_0 \precneqq \mathfrak{A}_1$, $R^{\mathfrak{A}_0} = R^{\mathfrak{A}_1}$ and finally $|\mathfrak{A}_0| = |\mathfrak{A}_1| = |R^{\mathfrak{A}_1}| = \kappa$.*

Chang was using the property of a model $\mathfrak{A}$ being R-*saturated*, that is when every 1-type p of the following shape is already realized in $\mathfrak{A}$:

(a) p is a 1-type over the structure $\langle \mathfrak{A}, y \rangle_{y \in Y}$,
(b) $Y \subseteq \mathbb{A}$ such that $|Y| < |\mathbb{A}|$,
(c) $R(x) \in p$.

The interesting part around this property is now the following: The union of an elementary chain of R-saturated models is, in fact, again R-saturated. This does not hold for arbitrary chains of (just) saturated structures.

Having this, Chang was able to construct an elementary chain of models $\langle \mathfrak{A}_\nu \mid \nu < \kappa^+ \rangle$ such that for all $\nu > 1$, the structure $\mathfrak{A}_\nu$ is again R-saturated, $|\mathfrak{A}_\nu| = \kappa$ and $R^{\mathfrak{A}_{\nu'}} = R^{\mathfrak{A}_0}$ for all $\nu' < \nu$. Then, the model $\mathfrak{C} := \bigcup\{\alpha_\nu \mid \nu < \kappa^+\}$ bears the properties $|\mathfrak{C}| = \kappa^+$ and $R^{\mathfrak{C}} = R^{\mathfrak{A}_0}$ and is therefore as desired.

Now, knowing that for the fixed uncountable and regular cardinal κ, the successor κ^+ is even a successor within the constructible universe $\mathbf{L}$, choose a suitable ($\mathbf{L}$-)cardinal $\gamma < \kappa^+$ such that within the constructible universe we have $\gamma^+ = \kappa^{+\mathbf{V}}$.

Obviously, in the universe $\mathbf{V}$ there is a collapsing map σ from γ onto κ. Coding this map σ within an appropriate subset $D \subseteq \kappa$, we can finally arrange that the following holds:

$$\mathbf{L}[D] \models \kappa^+ = \kappa^{+\mathbf{V}}. \tag{5}$$

If there were already a (γ', γ)-model of the theory $\mathfrak{T}$ within the model $\mathbf{L}[D]$, then we could just apply Chang's Theorem within this set theoretical universe and the argumentation would be easier.

In any case, fix a (γ', γ)-model $\mathfrak{B}$ of the theory $\mathfrak{T}$ within the universe $\mathbf{V}$. Following Chang's idea, we can consider an appropriate extension $\mathfrak{S}$ of the complete theory of the structure $\mathfrak{B}$, that Chang suggested in his proof, *cf.* [Sac72, Proof of Proposition 23.1]. In $\mathbf{V}$, the structure $\mathfrak{B}$ will witness the consistency of this theory $\mathfrak{S}$.

However, even within the model $\mathbf{L}[D]$, this theory is consistent because otherwise we could find a (finite) proof sequence witnessing the inconsistency which would be absolute between $\mathbf{L}[D]$ and $\mathbf{V}$. To see this, we could assume by the theorem of Morley and Vaught, mentioned in the introductional Chapter 1, that $\gamma' = \aleph_1$ and $\gamma = \aleph_0$. Therefore, we can extend the predicate D, without loss of generality, such that D codes all ($\aleph_1$-many) finite sequences of formulae with parameters of the $(\aleph_1, \aleph_0)$-model $\mathfrak{B}$, providing enough set theory within $\mathbf{L}[D]$ to be able to speak about proof sequences. Note, because $\kappa \geqslant \aleph_1$ this is still possible such that $D \subseteq \kappa$.

And so we can apply Chang's proof idea within $\mathbf{L}[D]$. All he was further needing is $2^{<\kappa} = \kappa$ which we already have, given by Lemma 22 for the fixed uncountable and regular cardinal κ.

Therefore, having followed the proof idea of Chang, we finally have a $(\kappa^{+\mathbf{L}[D]}, \kappa)$-model $\mathfrak{C}$ within $\mathbf{L}[D]$. However, $\mathfrak{C}$ is still a model of $\mathfrak{T}$ within $\mathbf{V}$. Moreover, it is even a (κ^+, κ)-model in $\mathbf{V}$ because of the absoluteness of (κ and) κ^+, giving by (5).

Hence, the proof is finally done. $\boxtimes$(Lemma 71)

Clearly, if κ^+ is not a successor cardinal in $\mathbf{L}$, then is must be inaccessible because of having GCH and the fact, that regularity reflects downwards and so we can conclude the following

Corollary 74. *If for a regular and uncountable cardinal κ and a given pair of infinite cardinals $\gamma' > \gamma$ we have the failed transfer property $(\gamma', \gamma) \not\longrightarrow (\kappa^+, \kappa)$, then κ^+ is inaccessible within $\mathbf{L}$.*

And finally the second main statement, Theorem 70, is completely proved as well. $\boxtimes$(Theorem 70).