CHAPTER 6

A weaker Theory for the Counterexample

As promised in former chapters we are now giving another proof of the existence of the counterexample to the general Chang's Transfer Property, considering the question under what circumstances the following assertion for arbitrary infinite cardinals γ and any uncountable regular cardinal κ fails:

$$(\gamma^+, \gamma) \longrightarrow (\kappa^+, \kappa).$$

However, we now start from a weaker theory than we considered in Chapter 5. In fact, this will not change the claim of the main theorem, Theorem 59. Though, it might be interesting to know that the theory which is needed to get the desired failure of the above mentioned transfer property is indeed rather weak possible.

Moreover, we are even able to start from a ground model M that only satisfies GCH. This is indeed a much weaker assumption than we have used in Chapter 5, where we started (basically) from $\mathbf{L}$. Therefore we are going to prove the following

Theorem 68. *Let M be a model of set theory, satisfying GCH such that, in M, there is an inaccessible τ and $\kappa < \tau$ is an uncountable regular cardinal. Moreover, for Mitchell's notion of forcing $\mathbb{M}$ let G be an $\mathbb{M}$-generic filter over M. Then for arbitrary uncountable regular $\gamma \geqslant \tau$ or $\gamma = \omega$ we have*

$$M[G] \models (\gamma^+, \gamma) \not\longrightarrow (\kappa^+, \kappa).$$

This theorem will give us a lot of possibilities to get nice independent statements for the failure of Chang's Transfer Property with respect to large cardinals. Having a large cardinal, say a measurable one or even a larger cardinal –just providing there is an inaccessible cardinal below to work with– starting from a suitable model satisfying GCH, we then can apply the forcing of the last theorem and we are getting the desired failure of the transfer property in a universe where we still have the existence property of that large cardinal we have started from.

The reason for this is simply the fact that Mitchell's notion of forcing is in some sense a small one. That is, the forcing works very locally and it will not affect any really much larger cardinal properties beyond the considered inaccessible cardinal, as we already know.

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Now, to start with the proof, fix a model M of $\mathsf{ZFC} + \mathsf{GCH}$ such that τ is inaccessible and $\kappa < \tau$ is uncountable and regular.

Already in Chapter 4, we defined a theory $\mathfrak{T}$ which contains, *e.g.*, the axiom of constructibility. A model of this theory gives us very good control about constructing structures like the coarse morass.

There are two (and even more) important consequences we used within the fixed model of the theory $\mathfrak{T}$. At first, we had GCH and so we knew about the behavior of powers of cardinals. And secondly, we strongly used consequences of the very powerful condensation property of the constructible universe.

We will now start from a relatively weak theory such that its models satisfy the axiom $\mathbf{V} = \mathbf{L}[C]$ for a given $C \subseteq \mathrm{On}$. We will again have a symbol for the largest cardinal, $\dot{A}$, however, it might be that we loose GCH. Instead of this we assert that there are only a few bounded subsets of the interpretation of the symbol $\dot{A}$.

However, the price for this freedom will be a more complex structure theory during the proof. In fact, the levels of the morass, we are going to use here, will blow up. Each of them, the former intervals S_α, will now be a (wide-branching) tree. Therefore, we are going to argue with two trees within the new morass structure.

Moreover, because of the growing of the levels S_α, we now have to go over to use the models $\mathbf{L}^D_\beta$ themself as new indexes in the morass, not only their ordinal heights β (or even the old indexes ν as above), *cf.* Figure 2, p. 64.

Nevertheless, even now the main idea of the proof can be preserved.

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To fix the set of axioms let us define the new (weak version of the) theory $\mathfrak{T}'$ as follows:

$$\begin{aligned} \mathfrak{T}' \quad := \quad & \mathsf{ZFC}^- + \mathbf{V} = \mathbf{L}[C] \text{ for } C \subseteq \mathrm{On} + 2^{<\dot{A}} = \dot{A} \\ & + \dot{A} \text{ is the largest cardinal} + \dot{A} \text{ regular.} \end{aligned}$$

Trying to re-prove the Theorem 59, in a newer version given by Theorem 68, we will repeat the arguments we have stated in earlier chapters. For, fix a model $\mathfrak{A}$ of the new version $\mathfrak{T}'$ and let A be the largest cardinal. Let us work within this model $\mathfrak{A}$, doing all further constructions and definitions.

Then, obviously, we have $\mathbf{L}_A[C] = \mathbf{H}_A$, the set of all sets within $\mathfrak{A}$ that are hereditarily smaller than A. Furthermore, denote with $\mathbf{L}^D_\nu$ the model $\langle\, \mathbf{L}_\nu[D], \in, D \cap \nu \,\rangle$ and define as earlier, in Chapter 4,

$$\begin{aligned} S'_A := \{\, \nu \mid\ & \nu \text{ is a limit of p.r.-closed ordinals, } A < \nu, \\ & \mathbf{L}^B_\nu \models A \text{ is the largest cardinal} \,\}. \end{aligned}$$

However, this will not be the set where the tree is ranging on, as we will see very soon.

Now, for every $\nu \in S'_A$ let β_ν be again the smallest p.r.-closed β such that $\mathbf{L}^B_\beta \models |\nu| \leqslant A$. Moreover, define $S_A := \{\, \mathbf{L}^B_{\beta_\nu} \mid \nu \in S'_A \,\}$. As we can see now, we are going to use the whole models $\mathbf{L}^B_{\beta_\nu}$ as index in the morass structure. However, we cannot use *this* definition to get the missing levels S_α for $\alpha < A$. Note, we can obviously show now that

$$\begin{aligned} S_A = \{\, \mathbf{L}^B_\beta \mid\ & \text{there is } \nu < \beta \text{ such that } \nu \text{ is limit of p.r.-closed ordinals,} \\ & \beta \text{ is the smallest } \beta \text{ such that } \mathbf{L}^B_\beta \models |\nu| \leqslant A, \\ & \mathbf{L}^B_\nu \models A \text{ is the largest cardinal, and } A < \nu \,\}. \end{aligned}$$

With this in mind, we can define for all $\alpha < A$ the sets S_α as follows

$$\begin{aligned} S_\alpha := \{\, \mathbf{L}^D_\beta \mid\ & \text{there is } \nu < \beta \text{ such that } \nu \text{ is limit of p.r.-closed ordinals,} \\ & \beta \text{ is the smallest } \beta \text{ such that } \mathbf{L}^D_\beta \models |\nu| \leqslant \alpha, \\ & \mathbf{L}^D_\nu \models \alpha \text{ is the largest cardinal, } \alpha < \nu, \\ & D \subseteq \beta,\ \mathbf{L}^D_\nu \models (\mathbf{H}_\alpha = \mathbf{L}^D_\alpha) \,\}. \end{aligned}$$

Here, we have to use new predicates D because we will very often need condensation arguments and so we might loose the originally given predicate C.

Notice, that the models of S_A are linearly ordered by inclusion. However, all other collections of models in S_α for $\alpha < A$ are partially ordered by a relation $\sqsubset$, defined as follows: For elements $\mathbf{L}^{\bar{D}}_{\bar{\beta}}$ and $\mathbf{L}^D_\beta$ of S_α where $\alpha \leqslant A$ we set

$$\mathbf{L}^{\bar{D}}_{\bar{\beta}} \sqsubset \mathbf{L}^D_\beta \qquad \text{if and only if} \qquad \bar{\beta} < \beta \text{ and } \bar{D} = D \cap \bar{\beta}.$$

Then this relation obviously forms a tree on each level S_α. Moreover, we have expanded the former intervals and now we are going to check in the remaining part of this chapter that all of the earlier arguments are going through.

As above, define $S := \bigcup_{\alpha \leqslant A} S_\alpha$. Note, $\bigcup_{\alpha < A} S_\alpha$ is obviously a subset of $\mathbf{H}_A$. Therefore, because of the assumption given by the theory $\mathfrak{T}'$, namely $2^{<A} = A$, the cardinality of this set is at most (and obviously also at least) the cardinal A.

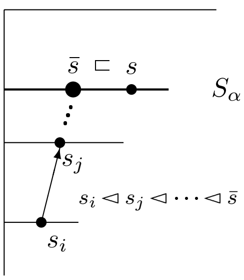
Moreover, repeating the argument of Lemma 54, we can prove that for stationary many $\alpha < A$ we have a non-empty set S_α.

Then for $\bar{s} := \mathbf{L}_{\bar{\beta}}^{\bar{D}}$ and $s := \mathbf{L}_\beta^D$ let $s(\alpha)$ be the α such that $s \in S_\alpha$. Furthermore, for $s = \mathbf{L}_\beta^D$ let $s(D)$ be the D and $s(\beta)$ be the β. And finally, let $s(\nu)$ be the smallest ν given by the definition of the set S_α.

Note, that with this notation we conclude that for elements $\bar{s}$ and s of S we have $\bar{s} \sqsubset s$ if and only if $\bar{s}(\alpha) = s(\alpha)$, $\bar{s}(\beta) < s(\beta)$ and $\bar{s}(D) = s(D) \cap \bar{\beta}$ and so, the relation $\sqsubset$ can be defined on the whole collection of models S than only seperately for each level S_α, still forming a tree and being linearly on S_A. In fact, we have for elements $\bar{s}$ and s of S_A obviously that $\bar{s} \sqsubset s$ if and only if $\bar{s} \subsetneq s$ if and only if $\bar{s}(\beta) < s(\beta)$.

Now, imitating the old definition, for $\bar{s} \in S_{\bar{\alpha}}$ and $s \in S_\alpha$ where $\bar{\alpha} < \alpha$ define $\bar{s} \lhd s$ if there is an elementary embedding $\pi : \bar{s} \longrightarrow s$ such that $\mathrm{crit}(\pi) = \bar{\alpha}$ and $\pi(\bar{\alpha}) = \alpha$. We call this map $\pi_{\bar{s}s}$.

And again, repeating the proof of Lemma 55 for structures $s = \mathbf{L}_\beta^D$ than the old $\mathbf{L}_{\beta_\nu}$'s we conclude that the maps $\pi_{\bar{s}s}$ are unique for fixed models $\bar{s}$ and s. Moreover, the same proof as of Lemma 56 shows that the relation $\lhd$ forms a tree on S.



Now, looking at the arguments used in the last part of Lemma 57 –where we constructed a useful elementary substructure X of a given tree element– we conclude that an $s \in S_A$ is a limit point in the tree relation and, moreover, if $\bar{s}$ and s are elements of S_α for $\alpha \leqslant A$ and $\bar{s} \sqsubset s$, then also $\bar{s}$ is a limit point in the tree relation $\lhd$.

And so we can give the general version of the coarse morass, let us simply call it *A-quasi-morass*, defined as follows

Definition 69. *Let the cardinal A, the sequence $\langle S_\alpha \mid \alpha \leqslant A \rangle$, the tree relations $\lhd$ and $\sqsubset$ with the sequence $\langle \pi_{\bar\nu\nu} \mid \bar\nu \lhd \nu \rangle$ of embeddings be defined as above. Then we call the structure*

$$\mathfrak{M} := \langle\ \mathfrak{B};\ A,\ \langle S_\alpha \mid \alpha \leqslant A \rangle,\ \lhd,\ \sqsubset,\ \langle \pi_{\bar\nu\nu} \mid \bar\nu \lhd \nu \rangle\ \rangle$$

the A-quasi-morass with the universe $\mathfrak{B}$.

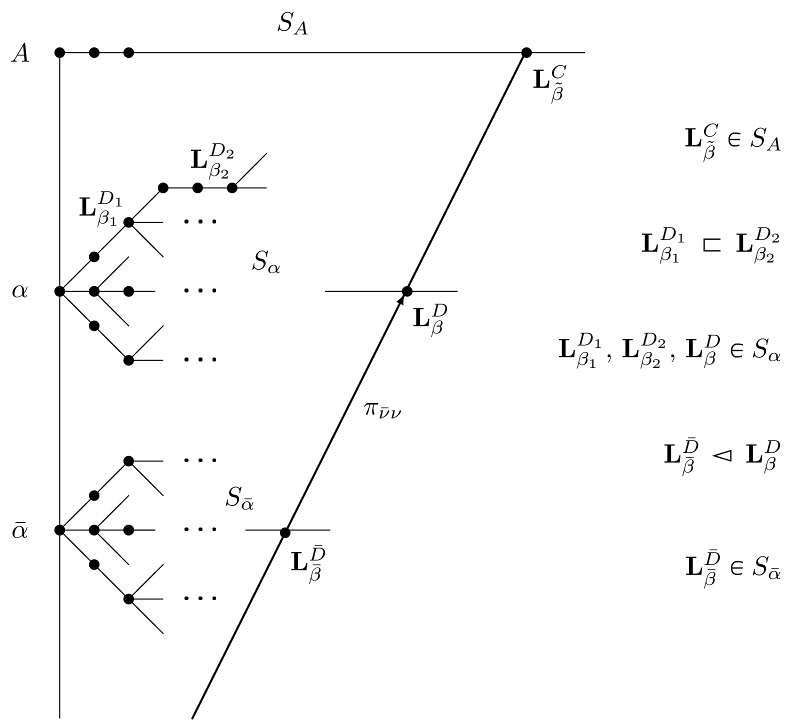


FIGURE 2. The A-quasi-morass

Here, the universe $\mathfrak{B}$ can be seen as collection of all models of the shape $\mathbf{L}^D_\beta$ where β is an ordinal and D a subset of β. Of course, using a suitable way of coding we can arrange $\mathfrak{B}$ again as collection of ordinals. However, not to fog the idea, we will work with the models instead of codes of ordinals.

In fact, with $\mathfrak{M} \restriction A$ we mean the initial segment of the given morass $\mathfrak{M}$ defined by

$$\mathfrak{M} \restriction A := \langle\, \mathfrak{B} \restriction A;\ A, \langle S_\alpha | \alpha < A \rangle, \lhd \restriction A, \sqsubset\restriction A, \langle \pi_{\bar s s} | \bar s \lhd s,\ s(\alpha) < A \rangle \rangle.$$

And again, $\mathfrak{B} \restriction A$ is the collection of all models $s \in \mathfrak{B}$ such that $s(\alpha) < A$. Note, in case we have coded the models as ordinals, this restriction could be seen simply as A itself—as in the earlier definition on page 45. Moreover, $\lhd \restriction A$ means the restriction of the relation $\lhd$ to models s such that $s(\alpha) < A$. The same holds for the relation $\sqsubset \restriction A$.

So, as in the beginning of Chapter 5, we can assume that we have again a ground model M satisfying (2), *cf.* p. 47, that is, M is a model of set theory satisfying $M \models$ "τ least inaccessible" $+ \mathbf{V} = \mathbf{L}[B]$ for a suitable subset $B \subseteq \kappa$ and also, without loss of generality, we assume (3), *cf.* p. 50, that is that the initial segment $\mathfrak{M} \restriction A$ is indeed an element of the ground model. Certainly, we again assume as in (4), *cf.* p. 50, that A is a subset of κ.

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Going on in the argumentation of the given proof for Theorem 59, being within the first case where we have a countable cofinal sequence $\langle \gamma_i \mid i < \omega \rangle$ within the ground model M, we defined the important sequences a_ν at page 51. For a fixed model $s \in S_A$, we now give the following definition of the desired sequence a_s as follows:

Let s_i be the unique tree element being the $\lhd$-smallest $\bar{s}$ such that $\bar{s} \lhd s$ and $\gamma_i \leqslant \bar{s}(\alpha)$. This tree element on the $\lhd$-branch below s is still well-defined. Then let a_s be the set of all s_i for arbitrary natural numbers i.

Moreover, as in the Remark 62, we know that for each $s \in S_A$, the initial segment $\mathfrak{M} \restriction s$ is uniquely definable from the parameters a_s and $\mathfrak{M} \restriction A$. Here, we define in the obvious way

$$\begin{aligned} \mathfrak{M} \restriction s \ := \ \langle\ \mathfrak{B} \restriction s;\ A,\ \langle S_\alpha \mid \alpha < s(\alpha) \rangle,\ \lhd \restriction s,\ \sqsubset \restriction s, \\ \langle \pi_{\bar{s}s'} \mid \bar{s} \lhd s',\ s'(\alpha) < s(\alpha) \rangle\ \rangle. \end{aligned}$$

And so, we again conclude for distinct s' and s, both elements of S_A, that $a_{s'} \neq a_s$. Note, the tree $\lhd$ within the initial segment $\mathfrak{M} \restriction A$ does not have unique limit points. However, we do not have to care about this fact because we only need this property for the models within S_A.

Furthermore, we have for arbitrary $\alpha \leqslant A$ and for each structure $s \in S_\alpha$ that the collection $S_\alpha \upharpoonright s := \{\bar{s} \in S_\alpha \mid \bar{s} \sqsubset s\}$ as the set of all elements $\mathbf{L}_{\bar{\beta}}^{s(D)\cap\bar{\beta}}$ in S_α where $\bar{\beta} < s(\beta)$, a subset of the model s, is even definable within the structure s.

Moreover, another property of the old morass structure we do not loose is the following: The cardinality of an $\sqsubset$-branch $\{\,\bar{s} \in S_\alpha \mid \bar{s} \sqsubset s\}$ for a fixed $s \in S_\alpha$ and, therefore, of an initial segment of the α-th level of the morass structure, $S_\alpha \upharpoonright s$, is strictly less than A because there are only less than the cardinal A many potentionally new ordinal heights $\bar{\beta} < s(\beta) < A$ for possible elements $\mathbf{L}_{\bar{\beta}}^{s(D)\cap\bar{\beta}}$.

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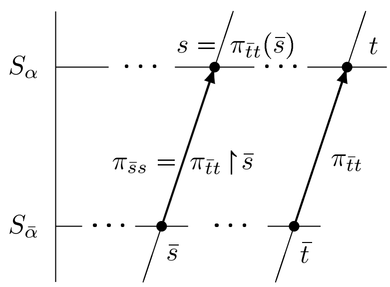
We now turn to the important tool we used in the proof of the main theorem, defining for $s \in S_A$ the following set of sequences:

$$\Theta(s) \;:=\; \{\, a_{\bar{s}} \mid \bar{s} \in S_A,\ \bar{s}(\beta) < s(\beta)\,\}.$$

Note, the elements s of the collection S_A of models are always of the shape $\mathbf{L}_{s(\beta)}^{C}$ and so we have a canonical (linear) order given by the ordinal height of these models.

We then can repeat the proof of Lemma 64 to get that for each $s \in S_A$, the sequence $\Theta(s)$ is uniformly definable from the parameters a_s, the morass segment $\mathfrak{M} \upharpoonright A$ and the (in A) cofinal sequence $\langle\, \gamma_i \mid i < \omega\,\rangle$ within the model $M[a_s]$.

For, we use the similar property as we had in the old proof, that is, that for all $\bar{s} \sqsubset \bar{t}$ where $\bar{s}$ and $\bar{t}$ are models of $S_{\bar{\alpha}}$, and moreover, $\bar{t} \lhd t$ for a model t of S_α and $\pi_{\bar{t}t}(\bar{s}) = s$, then we have that $s \sqsubset t$ such that $s \in S_\alpha$ and also $\bar{s} \lhd s$ and $\pi_{\bar{t}t} \upharpoonright \bar{s} = \pi_{\bar{s}s}$.



Now, consider an element $s_i = \mathbf{L}^{s_i(D)}_{s_i(\beta)}$ of the collection a_s for $s \in S_A$. Using an appropriate bijection between $\mathbf{H}_A$ and A we are able to code given elements s_i of $S_{s_i(\alpha)}$, a subset of $\mathbf{H}_A$, as ordinals below A. Note, here we use that by the choice of s_i we have $s_i(\alpha) < A$. Therefore, using such a coding we can consider the set a_s as an element of $[A]^\omega$ and together with (4), *cf.* p. 50, as an element of $[\kappa]^\omega$.

But then, the collection $\Theta(s)$ is a subset of $[\kappa]^\omega$. Hence, as above in Lemma 65, we conclude that in the model $M[a_s]$, the set $\Theta(s)$ has cardinality at most κ.

We are now able to finish the proof with the desired contradiction as above in the end of Chapter 5 as follows: Working within the forcing extension $M[G]$, let W be again the inner model $M[\bar{G}]$, where $\bar{G}$ is $\mathbb{P}(\tau)$-generic as above.

Following the old idea of Chapter 5, we consider $U := \{a_s \mid s \in S_A\}$, being the union of all $\Theta(s)$ for s ranging about all elements of S_A. Then $\langle U, <_U \rangle$ still forms a linear order, where the order relation is defined as: $a_{\bar{s}} <_U a_s$ if $\bar{s}(\beta) < s(\beta)$.

And again, considering a_s as a countable subset of κ using an appropriate coding, we know by Lemma 51 that a_s is indeed already an element of $M[\bar{G}]$. Hence, as above, $|U_x|^W \leqslant \kappa$.

After all, we use Lemma 9 again and conclude that the cardinality of U is strictly smaller than τ. However, we here have $|U| = |S_A| = \tau$ as well, and so the desired contradiction for the first case.

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In the second case, where the cofinality of A is uncountable, the contradiction follows exactly as on page 56: We construct the sequence $a_s := \{s_i \mid i < \bar{\kappa}\}$ where $\bar{\kappa} = \mathrm{cf}_M(A)$ as above using an uncountable and cofinal sequence in A and consider the set $X := \{a_s \mid s \in S_A\}$. We then again conclude that $|X| = |S_A| = \tau$.

On the other hand, again using Lemma 49 we know that each a_s is not only a subset but also an element of the ground model. Having $2^\kappa = \kappa^+$ within M, the cardinality of X is at most κ^{+M} which is strictly smaller than τ, a contradiction.

Furthermore, because of the forcing properties that we have already described in Chapter 3 and the fact that the ground model satisfies GCH, we know that within the generic extension $M[G]$, the assertion $2^\gamma = \gamma^+$ for $\gamma \geqslant \tau$ is preserved. Note, we also have $2^\omega = 2^\kappa = \tau$ and $\kappa^+ = \tau$. Hence, in $M[G]$ we trivially have $2^{<\gamma} = \gamma$ for $\gamma \geqslant \tau$ and so we always have a (γ^+, γ)-model of $\mathfrak{T}'$ for $\gamma \geqslant \tau$, considering the structure

$$\langle \mathbf{L}_{\gamma^+}[D], \gamma, \in, D \rangle$$

where $D \subseteq \gamma^+$ such that $\mathbf{L}_\gamma[D] = \mathbf{H}_\gamma$. Compare this to the upcoming Lemma 76.

Hence, all arguments of the old proof of Theorem 59 went through and so we have found the desired counterexample even for the weaker theory, defined on page 61, starting just from a model of GCH.

This finishes the survey through the proof. $\boxtimes$(Theorem 68)