CHAPTER 5

# An Inaccessible implies the Failure

We will now use the forcing we defined in the last chapter to prove the main theorem:

**Theorem 59.** *Suppose there is a model of* $\mathsf{ZFC}$ *with an inaccessible cardinal* $\tau$*. Moreover, let* $\kappa < \tau$ *be an uncountable regular cardinal. Then there is a forcing extension of* $\mathbf{L}$ *that is a model of the following:*

$$\begin{aligned} \mathsf{ZFC} \quad + \quad 2^{\aleph_0} = \kappa^+ \quad + \quad & \text{“}2^\alpha = \alpha^+ \textit{ for all cardinals } \alpha \geqslant \kappa\text{”} \\ + \quad & \text{“}(\gamma^+, \gamma) \not\longrightarrow (\kappa^+, \kappa) \textit{ for all regular cardinals } \gamma \neq \kappa\text{”} \\ + \quad & \text{“}\textit{there is a special } \kappa^+\textit{-Aronszajn tree}\text{”.} \end{aligned}$$

The proof of the theorem will last the remaining part of the chapter. Starting from a suitable ground model that has an inaccessible cardinal, we will work within the generic extension of the ground model, given by the forcing defined in the last chapter. There, we will consider the theory $\mathfrak{T}$ we have already defined and show the failure of the stated transfer property by constructing a counterexample. Moreover, in the forcing extension we will have a special $\kappa^+$-Aronszajn tree and –as desired– sufficient small powers of cardinals above $\kappa$. And so, the proof will be done.

Now, working in a set theoretical universe with an inaccessible cardinal, take an arbitrary (ground) model $M$, satisfying $\mathsf{ZFC}$ and $\mathbf{V} = \mathbf{L}[B]$ for any subset $B \subseteq \kappa$ such that $\tau$ is the least inaccessible cardinal above $\kappa$ in $M$:

$$\begin{aligned} M \models \mathsf{ZFC} \ + \ \mathbf{V} = \mathbf{L}[B] \text{ for } B \subseteq \kappa \\ + \ \tau \text{ least inaccessible above } \kappa. \end{aligned} \tag{2}$$

To start with, just take $M$ as Gödel's constructible universe $\mathbf{L}$, that is choosing $B = \varnothing$, and so $M$ obviously satisfies the condition (2) where $\tau$

is the least inaccessible cardinal above $\kappa$ in $M$, given by the assumption together with Lemma 38.

However, there will be a point during the up-coming construction where it might be convenient just to start with a model having the property given by (2), choosing the predicate $B$ in an appropiate way, than starting with $\mathbf{L}$.

We will now force with Mitchell's forcing $\mathbb{M}(\kappa, \tau)$ over $M$ having an $\mathbb{M}(\kappa, \tau)$-generic $G$ and finally getting the extension $M[G]$. Note, in $M[G]$, we have $2^{\aleph_0} = 2^\kappa = \kappa^+ = \tau$. Moreover, by construction of Mitchell's forcing defined in Definition 43 there is then a $\mathbb{P}(\tau)$-generic $\bar{G}$ and a $\dot{\mathbb{Q}}(\kappa, \tau)$-generic $\tilde{G}$ such that $M[G] = M[\bar{G}][\tilde{G}]$ where $\mathbb{P}(\tau)$ and $\dot{\mathbb{Q}}(\kappa, \tau)$ are defined as in Chapter 3, yielding the property of a two-step forcing

$$\mathbb{M}(\kappa, \tau) = \mathbb{P}(\tau) \star \dot{\mathbb{Q}}(\kappa, \tau).$$

Remember, we already defined the theory $\mathfrak{T}$ in Chapter 4 as follows:

$$\boxed{\begin{array}{rcl} \mathfrak{T} & = & \mathsf{ZFC}^- + \mathbf{V} = \mathbf{L} + \dot{A} \text{ regular} \\ & & \qquad + \ \dot{A} \text{ is the largest cardinal.} \end{array}}$$

Aiming towards a contradiction, let us work with the theory $\mathfrak{T}$ and state the following

**Supposition 60.** *In $M[G]$, there is a $(\kappa^+, \kappa)$-model $\mathfrak{A}$ of $\mathfrak{T}$.*

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In this chapter we are now going to deduce a contradiction to this assumption we just made. For, let $\mathfrak{M} := (\mathfrak{M})^{\mathfrak{A}}$ be the coarse $A$-morass defined in the last chapter within the fixed model $\mathfrak{A}$ and define

$$\mathfrak{M} \restriction A := \langle\ A,\ \langle S_\alpha \mid \alpha < A \rangle,\ \lhd \restriction A,\ \langle \pi_{\bar{\nu}\nu} \mid \bar{\nu} \lhd \nu < A \rangle\ \rangle.$$

Now, remember the second representation of the forcing extension $M[G]$, given with Lemma 48. In fact, let $\mathbb{M}_\nu$, $\mathbb{M}^\nu$, $G_\nu$ and $G^\nu$ for $\nu < \tau$ be defined as in the above mentioned lemma, then $M[G] = M[G_\nu][G^\nu]$.

Moreover, the dividing is in some sense cleverly choosen. In $M[G_\nu]$, we just have taken new subsets of $\omega$ that can be described by conditions "till $\nu$" and have then collapsed ordinals below $\nu$ to $\kappa$, which is of course only interesting for $\nu > \kappa$ anyway. This means, in $M[G_\nu]$, the forcing extension is already constructed up to $\nu$.

If we now consider an initial segment of the morass $\mathfrak{M}$, say $\mathfrak{M} \upharpoonright A$, then this small structure of cardinality $\kappa < \tau$ has to be already defined in an initial segment $M[G_\nu]$ of the forcing $M[G_\nu][G^\nu] = M[G]$ for suitable $\nu < \tau$. Therefore, we have the following

**Lemma 61.** *There is $\nu < \tau$ such that $\mathfrak{M} \upharpoonright A \in M[G_\nu]$.*

Now, choose $\nu' < \tau$ minimal such that $\mathfrak{M} \upharpoonright A \in M[G_{\nu'}]$ and define $\nu := \nu' + \kappa$. Then we can be sure that $\nu'$ and even $\nu$ is collapsed to $\kappa$ by our forcing at stage $\nu$, that is, in $M[G_\nu]$.

This property is important for us because it could have been that we consider just the case that $\nu$ is a cardinal, say $\nu = \kappa^+$, as the minimal one chosen and then we would have $\kappa^+$-many new reals but $\kappa^+$ would not be collapsed and so $2^{<\kappa} = \kappa$ would fail.

Moreover, $G_\nu$ as bounded subset of the generic filter $G$, living on forcing conditions up to $\nu$, is therefore a bounded subset of $M = \mathbf{L}[B]$ where $B \subseteq \kappa$. Remember, so far $B$ could be taken as the empty set.

Therefore, $G_\nu$ will be caught in an initial segment of the ground model, say $G_\nu \subseteq \mathbf{L}_{\bar{\nu}}[B]$ for a suitable $\bar{\nu}$. Choose $\bar{\nu}$ minimal with this property. Hence, in $\mathbf{L}[B][G_\nu] = M[G_\nu]$, the ordinal $\bar{\nu}$ is already collapsed to $\kappa$ because $\nu$ is, as we have seen above, and the minimal choice of $\bar{\nu}$.

Further, fix a bijection $f : \kappa \longleftrightarrow \bar{\nu}$ such that $f \in M[G_\nu]$ and consider the complete elementary theory of $\langle \mathbf{L}_{\bar{\nu}}[G_\nu], \in, G_\nu, \langle \dot{\xi} \mid \xi < \bar{\nu} \rangle \rangle$ where we use $f$ to code elements $\xi$ of $\bar{\nu}$ into elements $\dot{\xi}$ of $\kappa$. Then this theory is a subset of $\mathbf{L}_\kappa[B]$.

Therefore, we are able to code $G_\nu$ in a predicate $B' \subseteq \kappa$ such that $M[G_\nu] = \mathbf{L}[B][G_\nu] = \mathbf{L}[B']$ and the model $\mathbf{L}[B']$ still satisfies the property (2) for possible ground models. Note, the property of $\tau$, being inaccessible, we obviously did not change in $\mathbf{L}[B']$.

Now, $G^\nu$ is $\mathbb{M}^\nu$-generic over $M[G_\nu]$ where $\mathbb{M}^\nu$, defined in Chapter 3, is –roughly speaking– the forcing $\mathbb{M}(\kappa, \tau)$ but just taking conditions acting beyond $\nu$. So, the difference between both forcings is that the forcing $\mathbb{M}(\kappa, \tau)$ starts at level $\kappa$ whereas $\mathbb{M}^\nu$ begins later, at level $\nu$ such that $\kappa < \nu < \kappa^+$.

Because the forcing adds subsets of $\omega$ and collapses ordinals to $\kappa$, to start at stage $\nu < \kappa^+$ does not change anything in the arguments: For the indices $\nu$, the way towards the inaccessible $\tau$ is –roughly speaking– long enough to argue in the same way. Hence, for simplicity but without loss of generality, we may additionally assume

$$\mathfrak{M} \upharpoonright A \in M. \tag{3}$$

Furthermore, by the choice of the theory $\mathfrak{T}$ and the Supposition 60 we know that the interpretation of the predicat $\dot{A}$ is a set of cardinality $\kappa$ of ordinals. So, by renaming the elements using a suitable bijection, we can arrange $A$ as a subset of $\kappa$ and so we will also assume without loss of generality that

$$A \subseteq \kappa. \tag{4}$$

Apart from this, we do not know how $A$ looks like, in fact, with the model $\mathfrak{A}$ we could have a non-standard model of set theory and so $\langle A, \leqslant_A \rangle$ needs not to be well-founded. However, we can ask for the cofinality of the linear order $\langle A, <_A \rangle$ within $\mathfrak{A}$, knowing –as a subset of $\kappa$– this cardinal could be any regular cardinal below $\kappa$.

Hence, to go on with the proof we have to distinguish two cases.

**Case 1.** $\boxed{\mathrm{cf}_M(A) = \omega}$

Then under these circumstances, within the ground model we can find a countable sequence $\langle \gamma_i \mid i < \omega \rangle \in M$ being monotone and cofinal in $A$. Remember, we still work within the fixed model $\mathfrak{A}$.

Now, for $\nu \in S_A$ let $\nu_i$ be the unique tree element being the $\lhd$-smallest $\bar{\nu}$ such that $\bar{\nu} \lhd \nu$ and $\gamma_i \leqslant \alpha_{\bar{\nu}}$. Note, because for a fixed $\nu$ we have enough well-foundedness within the tree to define $\nu_i$ that way. We also know by construction of the coarse $A$-morass that $\nu_i < A < \nu$ for $\nu_i \lhd \nu \in S_A$ and therefore, by (4), we have $\nu_i \in \kappa$. Furthermore, define

$$a_\nu \ := \ \{\nu_i \mid i < \omega\} \in [\kappa]^\omega.$$

Towards to the desired contradiction, we define within the fixed model $\mathfrak{A}$ for each $\nu \in S_A$ and $B' := \{\xi \in B \mid \xi < \nu\}$ the following initial segment of the morass structure

$$\begin{aligned} \mathfrak{M} \restriction \nu \ \ := \ \ \langle\, & B',\, \alpha_\nu,\, \langle S_\alpha \mid \alpha < \alpha_\nu \rangle,\, S_{\alpha_\nu} \cap B',\, \lhd \restriction (S_{\alpha_\nu} \cap B'), \\ & \langle \pi_{\bar{\nu}\nu'} \mid \bar{\nu} \lhd \nu' < \nu \rangle \,\rangle. \end{aligned}$$

Now, working in the model $\mathfrak{A}$, consider the elementary embeddings $\pi_{\nu_i\nu_j} : \mathbf{L}_{\nu_i} \longrightarrow \mathbf{L}_{\nu_j}$ for every $i < j < \omega$ that we have by definition of the tree. Let us lift up these embeddings to maps of the shape

$$\tilde{\pi}_{\nu_i\nu_j} : \langle \mathbf{L}_{\nu_i}, \mathfrak{M} \restriction \nu_i \rangle \longrightarrow \langle \mathbf{L}_{\nu_j}, \mathfrak{M} \restriction \nu_j \rangle,$$

defined in the obvious way, that is

$$\begin{aligned} \tilde{\pi}_{\nu_i\nu_j} \restriction \mathbf{L}_{\nu_i} \ &= \ \pi \restriction \mathbf{L}_{\nu_i}, \\ \tilde{\pi}_{\nu_i\nu_j}(\langle S_\alpha \mid \alpha < \alpha_{\nu_i} \rangle) \ &= \ \langle S_\alpha \mid \alpha < \alpha_{\nu_j} \rangle, \\ \tilde{\pi}_{\nu_i\nu_j}(\, S_{\alpha_{\nu_i}} \cap B'\,) \ &= \ S_{\alpha_{\nu_j}} \cap B', \\ \tilde{\pi}_{\nu_i\nu_j}(\langle \pi_{\bar{\nu}\nu'} \mid \bar{\nu} \lhd \nu' < \nu_i \rangle) \ &= \ \langle \pi_{\bar{\nu}\nu'} \mid \bar{\nu} \lhd \nu' < \nu_j \rangle. \end{aligned}$$

But then we have $\tilde{\pi}_{\nu_i\nu_j} = \tilde{\pi}_{\nu_k\nu_j} \circ \tilde{\pi}_{\nu_i\nu_k}$ and therefore, using Lemma 6, the structure $\mathfrak{M}\restriction\nu$ is just the direct limit of the structure

$$\langle\langle\, \langle \mathbf{L}_{\nu_i}, \mathfrak{M}\restriction\nu_i \rangle \mid i < \omega \,\rangle, \langle\, \tilde{\pi}_{\nu_i\nu_j} \mid i \leqslant j < \omega \,\rangle\rangle$$

and so –up to isomorphism– this structure is unique. Therefore, we finally proved the following

**Remark 62.** *For each $\nu \in S_A$, up to isomorphism, $\mathfrak{M}\restriction\nu$ is uniquely definable from the parameters $\alpha_\nu$ and $\mathfrak{M}\restriction A$.*

Now, the set $a_\nu$ defines a countable path through the tree till the element $\nu$ (of the tree). And so, because of the uniqueness of limit points in this tree, we obviously have the following

**Lemma 63.** *For elements $\bar{\nu} \neq \nu$ of $S_\alpha$ we have $a_{\bar{\nu}} \neq a_\nu$.*

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Let us define the technical but useful collection of all countable paths through the tree structure below $\nu$ for an element $\nu$ of $S_A$, letting within the model $\mathfrak{A}$,

$$\Theta(\nu) := \{\, a_{\bar{\nu}} \mid \bar{\nu} \in S_A, \bar{\nu} < \nu \,\}.$$

Remembering that $M = \mathbf{L}[B]$, there is a first nice property as follows:

**Lemma 64.** *For each $\nu \in S_A$, the sequence $\Theta(\nu)$ is uniformly definable from parameters $a_\nu$, $\mathfrak{M}\restriction A$ and $\langle\, \gamma_i \mid i < \omega \,\rangle$ within the model $M[a_\nu]$.*

*Proof.* Note, by (3), the parameters $\mathfrak{M}\restriction A$ and $\langle\, \gamma_i \mid i < \omega \,\rangle$ are already elements of $M$, and hence even of $M[a_\nu]$. Still working in the model $\mathfrak{A}$, we will now define step by step the desired collection of sets as follows:

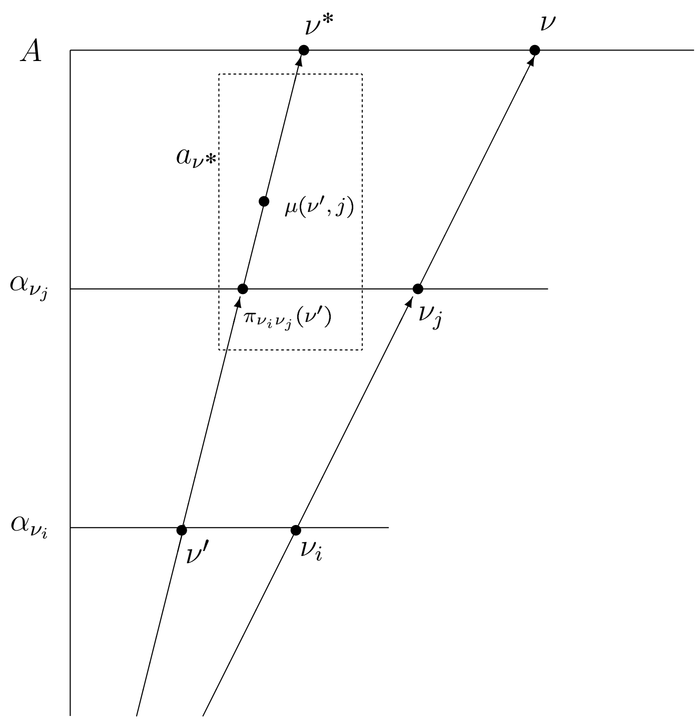
For each $\nu_i \in a_\nu$ and arbitrary $\nu' \in S_{\alpha_{\nu_i}}$ such that $\nu' < \nu_i$ define

$$P_0(\nu', i) := \{\, \pi_{\nu_i\nu_j}(\nu') \mid i < j \,\}.$$

Then $P_0(\nu', i)$ is a cofinal set in the branch above $\nu'$ as a copy of the branch below $\nu$. Now set

$$P_1(\nu', i) := \{ \bar{\mu} \mid (\exists \mu \in P_0(\nu', i))( \bar{\mu} \lhd \mu ) \}.$$

The set $P_1(\nu', i)$ collects all missing elements on the branch below an element of the set $P_0(\nu', i)$. Therefore, this set describes a branch of length $A$ and, by definition, it does not depend on the parameter $i$ and we have $P_1(\nu', i_0) = P_1(\nu', i_1)$ for all natural numbers $i_0$, $i_1$. Therefore, we define $P_1(\nu') := P_1(\nu', i)$ for an arbitrary natural number $i$.




Finally set

$$P_2(\nu') := \{ \mu(\nu', j) \mid j < \omega \},$$

where $\mu(\nu', j)$ denotes the well-defined $\lhd$-smallest $\mu$ of the set $P_1(\nu')$ of elements of the tree such that $\gamma_j \leqslant \alpha_\mu$.

Note, with the given parameters we can obviously define the above three sets within the model $M[a_\nu]$.

Consider now $\nu^* := \pi_{\nu_i \nu}(\nu')$ for an arbitrary $i < \omega$. Then $\nu^*$ again does not depend on the choice of $i$. Moreover, $\nu^*$ is the unique limit of the branch defined by $P_0(\nu', i)$ at tree level $A$. By definition we know then that $a_{\nu^*}$ is just the set $P_2(\nu')$ and so finally we have, defined within the model $M[a_\nu]$, the following

$$\begin{aligned}\Theta(\nu) &= \{\, a_{\nu^*} \mid \nu^* \in S_A \ \wedge\ \nu^* < \nu \,\} \\ &= \{\, P_2(\nu') \mid \exists \mu \in a_\nu \ (\, \nu' < \mu \ \wedge\ \alpha_{\nu'} = \alpha_\mu \,)\}.\end{aligned}$$

Therefore, the proof is complete. ☒(Lemma 64)

Moreover, with Lemma 22, having $a_\nu$ as a subset of $\kappa$, we still have $2^{<\kappa} = \kappa$ within the ground model $M = \mathbf{L}[B]$ and also within the model $M[a_\nu]$. Hence, because $\Theta(\nu)$ is a subset of $[\kappa]^\omega$, we can sum up with the following

**Lemma 65.** *For $\nu \in S_A$, within model $M[a_\nu]$, the set $\Theta(\nu)$ has cardinality at most $\kappa$.*

Finally we are prepared to complete the desired contradiction using Lemma 9:

Let $W$ be the inner model $M[\bar{G}]$ and $\mathbf{V}$ be the final forcing extension $M[G] = M[\bar{G}][\tilde{G}]$. Moreover, let $\kappa$ be the given cardinal and $\tau$ be the inaccessible within the ground model $M$. Remember, by Lemma 44, we do not change cardinals forming the forcing extention $W$. So, we still have $\kappa^{+W} < \tau$. Lemma 45 then gives us immediately the desired stationarity of the set $\{\, \lambda < \tau \mid W \models \mathrm{cf}(\lambda) = \kappa^+ \,\}$ that we need for the application of Lemma 9.

Now, let $H$ be $\mathcal{P}^W(\kappa)$ and so we have trivially $U \subseteq H \in W$ and, moreover, within $W$, also that $|H| = |\mathcal{P}(\kappa)| = \tau$. Remember, $W$ is the Cohen extension of $M$ by adding $\tau$ many reals.

Finally let $U$ be $\{a_\nu \mid \nu \in S_A\}$. Then we have $U = \bigcup_{\nu \in S_A} \Theta(\nu)$. Further, $\langle U, <_U \rangle$ forms a linear order, where the order relation is defined by letting: $a_{\bar{\nu}} < a_\nu$ if and only if $\bar{\nu} < \nu$. For arbitrary $x \in U$, say $x = a_\nu$, we then have:

$$\begin{aligned} U_x &:= \{ z \mid z <_U x \} = \{ a_{\bar{\nu}} \mid a_{\bar{\nu}} <_U a_\nu \} \\ &= \{ a_{\bar{\nu}} \mid \bar{\nu} < \nu;\ \bar{\nu}, \nu \in S_A \} \\ &= \Theta(\nu) \in M[\bar{G}] = W. \end{aligned}$$

Now, $a_\nu$ is obviously an element of $M[G]$ and as a countable subset of $\kappa$ we know by Lemma 51 that $a_\nu$ was not added by the second forcing step and so it is an element of $M[\bar{G}]$.

But then we know that $M[a_\nu] \subseteq M[\bar{G}]$ and so we can conclude finally

$$|U_x|^W = |\Theta(\nu)|^{M[\bar{G}]} \leqslant |\Theta(\nu)|^{M[a_\nu]} \leqslant \kappa.$$

Under these circumstances, Lemma 9 promised that the cardinality of $U$ is strictly smaller than $\tau$. However, the cardinality of $U$ is the same as the one of $S_A$ which is cofinal in the regular cardinal $\tau$. Therefore, the cardinality of $U$ is equal to $\tau$.

This desired contradiction finishes the first case.

We now turn to the remaining case in our proof and try to deduce a contradiction there as well.

**Case 2.** $\boxed{\mathrm{cf}_M(A) \neq \omega}$

We still can assume that the initial segment $\mathfrak{M} \restriction A$ of the morass is an element of the ground model $M$ as in the very beginning of the first case given by (3).

Now let $\bar{\kappa} := \mathrm{cf}_M(A) \neq \omega$ and $\langle \gamma_\nu \mid \nu < \bar{\kappa} \rangle \in M$ be an uncountable and cofinal sequence in the linear order $A$. For $\nu \in S_A$ define now as in the first case

$$\nu_i := \text{ the } \lhd\text{-smallest } \bar{\nu} \text{ such that } \bar{\nu} \lhd \nu \text{ and } \gamma_i \leqslant \alpha_{\bar{\nu}},$$

and finally let $a_\nu := \{\nu_i \mid i < \bar{\kappa}\}$, now an uncountable subset of $\kappa$.

Consider, within the forcing extension $M[G]$, the definable set

$$X := \{a_\nu \mid \nu \in S_A\}.$$

Note, $X$ is a subset of the ground model $M$. Moreover, the cardinality of $X$ is the same as the cardinality of $S_A$, by Lemma 63, and this is $\tau$ because of its regularity property and Lemma 53.

Because $\mathfrak{M} \restriction A$ lies already in the ground model $M$ and together with $\langle \gamma_i \mid i < \bar{\kappa} \rangle \in M$ we can define initial segments of $a_\nu \restriction i$ within the ground model. Hence, already $a_\nu \restriction i$ is an element of $M$ for arbitrary $i < \bar{\kappa}$ and so by Lemma 49 we also know that then the whole sequence $a_\nu$ is an element of the ground model.

However, by definition, $X$ is a subset of $\mathcal{P}^M(\kappa)$. Moreover, because of the inaccessibility of $\tau$ and $2^\kappa = \kappa^+$ within the ground model $M$, by Lemma 20 we finally conclude the following

$$\tau = |X|^M \leqslant \kappa^{+M} < \tau.$$

Hence, in both cases we were able to find a contradiction. This means our Supposition 60 was false and the main part of the proof of the Theorem 59 is already done.

To finish up with the proof let us look at the following two lemmas:

**Lemma 66.** *In $M[G]$, the theory $\mathfrak{T}$ has $(\gamma^+, \gamma)$-models for all regular $\gamma \neq \kappa$.*

*Proof.* Let us work in $M[G]$ and consider the models $\langle\, \mathbf{L}_{\gamma^+}, \gamma\,\rangle$ for a regular cardinal $\gamma < \kappa$. Then, by Lemma 16, this is a model of $\mathsf{ZFC}^-$ and $\mathbf{V} = \mathbf{L}$. And, moreover, $\gamma$ is indeed the largest cardinal in $\mathbf{L}_{\gamma^+}$ because of the preserving properties of the forcing by Corollary 46. And together with Lemma 33, we finally have found a $(\gamma^+, \gamma)$-model of the fixed theory $\mathfrak{T}$.

The same idea shows that $\langle\, \mathbf{L}_{\tau^{+(\alpha+1)}}, \tau^{+\alpha}\,\rangle$ is a $(\tau^{+(\alpha+1)}, \tau^{+\alpha})$-model of $\mathfrak{T}$ for arbitrary ordinals $\alpha$. And so, because $\tau = \kappa^{+M[G]}$, all cases are successfully discussed and therefore the lemma is proved.

$\boxtimes$(Lemma 66)

In our first main theorem, we just proved that there cannot be a $(\kappa^+, \kappa)$-model. So, why does not work the model $\langle\, \mathbf{L}_{\kappa^+}, \kappa\,\rangle$? — The answer is easy when we remember that we collapsed $\tau$ to $\kappa^+$, and so, $\tau = \kappa^{+M[G]}$ –being inaccessible in the constructible universe– is not the cardinal successor of $\kappa$ in $\mathbf{L}$. Hence, in $(\,\mathbf{L}_{\kappa^+}\,)^{M[G]} = \mathbf{L}_{\kappa^{+M[G]}}$, the cardinal $\kappa$ is not the largest one.

⋆ ⋆ ⋆

The last missing property in the statement of the main theorem we still have to show, uses the choice of $\tau$ being the minimal inaccessible cardinal above $\kappa$ within $M = \mathbf{L}[B]$ for a suitable subset $B$. In fact, analyzing our construction more deeply, we see that independent from the assumption (3), we did indeed start from the constructible universe — just using (2) and (3) to arguing in a more convenient way.

In this case, the cardinal $\tau$ is the least inaccessible above $\kappa$ even in the constructible universe. However, we could be able to argue within a general universe given by (2), just proving a similar statement for $\mathbf{L}[B]$ as Lemma 30 gives us for the constructible universe $\mathbf{L}$, *cf.* Lemma 31.

So in any case, we know then that $\tau = \kappa^{+M[G]}$ is not a Mahlo cardinal within the constructible universe, having started the forcing construction from **L**. But then, using Lemma 30, we know that in $M[G]$ we have a $\square_\tau$-sequence, and so together with the equivalence of Lemma 84 and Theorem 81, respectively, we finally proved the following

**Lemma 67.** *In $M[G]$, there is a special $\kappa^+$-Aronszajn tree.*

Using the facts of Chapter 3 that Mitchell proved in [Mit72], we conclude that within the forcing extension $M[G]$ we only somehow slightly damaged GCH –depending on the choice of $\kappa$–, that is, we have $2^\alpha = \alpha^+$ for all $\alpha \geqslant \kappa$. And even for the smallest infinite cardinal we have chosen a somehow minimal failure, $2^{\aleph_0} = \kappa^+$, again depending on the choice of $\kappa$.

Finally, our first main theorem is completely proved. $\boxtimes$(Theorem 59)