CHAPTER 4

The Coarse Morass

From now on consider a first order language $\mathcal{L}$ with a unary predicate $\dot{A}$ and a binary predicate $\dot{\in}$. Define the theory $\mathfrak{T}$ as follows:

$$\mathfrak{T} \quad := \quad \mathsf{ZFC}^- + \ \mathbf{V} = \mathbf{L} \ + \ \dot{A} \text{ regular} \\ + \ \dot{A} \text{ is the largest cardinal.}$$

Moreover, by a (τ,κ)-model of $\mathcal{L}$ we understand a model of the shape $\mathfrak{A} = \langle\ \mathbb{A}; \dot{A}_{\mathfrak{A}}, \dot{\in}_{\mathfrak{A}}, \dots\ \rangle$ such that $|\mathbb{A}| = \tau$ and $|\dot{A}_{\mathfrak{A}}| = \kappa$. In the following we are working inside a model $\mathfrak{A}$ of our fixed theory $\mathfrak{T}$. Let $A = \dot{A}_{\mathfrak{A}}$ be the interpretation of $\dot{A}$ in $\mathfrak{A}$.

We will benefit from this theory in the next chapter, proving our main theory. For, we will use a tool, the so-called *coarse A-morass*. To be able to, we are going to define the structure theory in this chapter and prove facts we are going to apply later. This is not at all to understand as an introduction to the theory of morasses[1]. We will develop methods we are going to use in the next chapter.

Once and for all, in this chapter we are working within the fixed model $\mathfrak{A}$, otherwise we will state the opposite clearly. Note, because $\mathfrak{A}$ could be very different from our universe, possibly being ill-founded, and so forthcomming arguments will rarely be absolut.

[1]Only a few introductions to the theory of morasses can be found in the literatur — although they are sometimes used as a tool in the theory of inner models, proving statements around the cardinal transfer property. As a starting point we strongly refer to [Dev84].

At first, define sets S_α for ordinals $\alpha \leqslant A$ as follows:

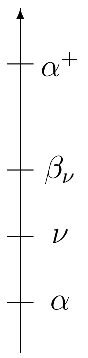
$$\begin{aligned} S_\alpha := \{\, \nu \mid\ & \alpha < \nu < \alpha^+ \\ & \wedge\ \mathbf{L}_\nu \models (\alpha \text{ is the largest cardinal} \wedge \alpha \text{ is regular}) \\ & \wedge\ \forall \xi < \nu\, \exists \eta < \nu\, (\xi < \eta\ \wedge\ \eta \text{ is p.r.-closed})\,\} \end{aligned}$$

Consider the set S_A. Because the fixed model $\mathfrak{A}$ of $\mathfrak{T}$ thinks that A is regular and the largest cardinal and, by Corollary 34, there are indeed cofinal many limits of p.r.-closed ordinals within $\mathfrak{A}$, we obviously have

Lemma 53. $\sup S_A = \infty$.

Let us now look at these intervals S_α more closely and define for $\nu \in S_\alpha$

$$\begin{aligned} \beta_\nu \quad := \quad & \text{the smallest } \beta > \nu \text{ such that} \\ & \mathbf{L}_\beta \models |\nu| \leqslant \alpha \text{ and } \beta \text{ is p.r.-closed.} \end{aligned}$$



Then β_ν is well-defined for $\nu \in S_\nu$: Because ν lays between α and α^+, it must be collapsed at some ordinal level $\xi < \alpha^+$. Of course, ξ has not to be p.r.-closed. However, taking the closure of all (ordinal) primitive functions, possible by a countable limit construction, we get an ordinal β (and therefore $\mathbf{L}_\beta$) satisfying the desired condition by Lemma 37. So we can choose a minimal one.

Although defined for all ordinals α below A we can at least show that for a large set of indexes α the intervals S_α are non-empty, in fact, we will find a closed and unbounded set:

Lemma 54. *The set* $\{\alpha < A \mid S_\alpha \neq \varnothing\,\}$ *is stationary in* A.

Proof. By Lemma 53, S_A is a non-empty set and so we can fix an arbitrary $\nu \in S_A$. We now define simultanously the following two sequences $\langle \alpha_\xi \mid \xi < A \rangle$ and $\langle X_\xi \mid \xi \leqslant A \rangle$, letting

$$\begin{aligned} \alpha_\xi &:= X_\xi \cap A, \\ X_0 &:= \text{ the smallest } X \prec \mathbf{L}_{\beta_\nu} \text{ where } X \cap A \text{ transitive}, \\ X_{\xi+1} &:= \text{ the smallest } X \prec \mathbf{L}_{\beta_\nu} \text{ where } X_\xi \cap A \text{ transitive} \\ &\quad\ \text{ and } \alpha_\nu \in X, \\ X_\lambda &:= \bigcup \{ X_\xi \mid \xi < \lambda \} \text{ for limit ordinals } \lambda. \end{aligned}$$

Here, by ‘smallest (elementary) submodel’ we mean to take the submodel such that it is minimal for the inclusion relation. Then we obviously have $\alpha_\lambda = \sup_{\xi<\lambda} \alpha_\xi$.

Now, for each $\xi \leqslant A$ let $\pi : \mathbf{L}_{\beta(\xi)} \overset{\sim}{\longleftrightarrow} X_\xi$ be the Mostowski collapse. By construction we then have $\pi(\alpha_\xi) = A$. Moreover, π is an elementary embedding of $\mathbf{L}_{\beta(\xi)}$ into $\mathbf{L}_{\beta_\nu}$. Let $\nu(\xi) \in \mathbf{L}_{\beta_\xi}$ such that $\pi(\ \nu(\xi)\ ) = \nu$. Then –because of the elementary property of π and $\nu \in S_\alpha$– we also have that $\nu(\xi)$ lays in S_{α_ξ}. In particular, S_{α_ξ} is not empty.

Finally, by our construction we have found a club set $\{\alpha_\xi \mid \xi < A\}$, witnessing the stationarity claimed. ☒(Lemma 54)

On the set S, defined as the union $\bigcup_{\alpha \leqslant A} S_\alpha$ of the intervalls defined above, we will define a relation $\lhd$ and a suitable sequence of elementary embeddings $\langle \pi_{\bar{\nu}\nu} \mid \bar{\nu} \lhd \nu \rangle$ such that

$$\pi_{\bar{\nu}\nu} : \mathbf{L}_{\bar{\nu}} \longrightarrow \mathbf{L}_\nu \text{ and } \lhd \subseteq S \times S.$$

For, define α_ν as the unique α such that $\nu \in S_\alpha$ and define

$$\begin{aligned} \bar{\nu} \lhd \nu \quad :\Longleftrightarrow \qquad & (\ \alpha_{\bar{\nu}} < \alpha_\nu \ \wedge\ \pi(\bar{\nu}) = \nu \ \wedge\ \pi(\alpha_{\bar{\nu}}) = \alpha_\nu \ \wedge \\ & \quad \text{there is } \pi : \mathbf{L}_{\beta_{\bar{\nu}}} \prec \mathbf{L}_{\beta_\nu} \text{ such that } \mathrm{crit}(\pi) = \alpha_{\bar{\nu}}\ ). \end{aligned}$$

And finally set $\pi_{\bar{\nu}\nu} := \pi \restriction \mathbf{L}_{\bar{\nu}}$.

Note, in the proof of Lemma 54 we actually showed that (in notation of the proof) we have $\pi_{\nu(\xi)\nu} = \pi \restriction \mathbf{L}_{\nu(\xi)}$ and so, we showed even more, namely $\nu(\xi) \lhd \nu$.

In fact, we can prove the following

Lemma 55. *The maps $\pi_{\bar{\nu}\nu}$ defined above are unique.*

Proof. Here we use the fact that we are working with **L**-like models and their definability properties. So let us define

$$X := \quad \text{the set of all } y \in \mathbf{L}_{\beta_\nu} \text{ such that } y \text{ is } \mathbf{L}_{\beta_\nu}\text{-definable using parameters from } \{\nu, \alpha_\nu\} \cup \alpha_\nu.$$

Then we know by the condensation property that X is already a level of the $\mathbf{L}_\alpha$-hierarchy. Moreover, we can even describe the ordinal height of X as follows

Claim. $X = \mathbf{L}_{\beta_\nu}$.

First note that, by definition, $X \prec \mathbf{L}_{\beta_\nu}$. Obviously, in $\mathbf{L}_{\beta_\nu}$ holds

$$(\exists f)\,(f : \alpha_\nu \overset{\text{onto}}{\longrightarrow} \nu),$$

so it does in the substructure X. That means, there is an element f of X such that $\mathbf{L}_{\beta_\nu} \models f : \alpha_\nu \overset{\text{onto}}{\longrightarrow} \nu$. So, by absolutness there is an f in X such that $f : \alpha_\nu \overset{\text{onto}}{\longrightarrow} \nu$. Hence, for this f we have $\text{dom}(f) = \alpha_\nu \subseteq X$ and so finally $\nu = \text{rng}(f) \subseteq X$.

Now, let $\sigma : \mathbf{L}_{\bar{\beta}} \overset{\sim}{\longleftrightarrow} X$ be the collapsing map. We just showed that ν is a subset of X and therefore σ restricted to $\nu + 1$ is the identity map. We also have $\bar{\beta} > \alpha_\nu$ is p.r.-closed and $\mathbf{L}_{\bar{\beta}} \models |\nu| \leqslant \alpha_\nu$ and so we have, because of the minimality property, that $\beta_\nu \leqslant \bar{\beta}$. However, trivially we have by our construction that $\bar{\beta} \leqslant \beta_\nu$ and the claim is proved.

With the claim in mind, there is everything within $\mathbf{L}_{\beta_\nu}$ definable with parameters taken of $\{\nu, \alpha_\nu\} \cup \alpha_\nu$. However, these parameters are fixed by $\pi : \mathbf{L}_{\beta_{\bar{\nu}}} \longrightarrow \mathbf{L}_{\beta_\nu}$. And so, π is uniquely given by them. Therefore, $\pi_{\bar{\nu}\nu} = \pi \upharpoonright \mathbf{L}_{\bar{\nu}}$ and the lemma is proved. ⊠(Lemma 55)

We now look at the above defined relation more closely, proving

Lemma 56. *The relation $\lhd$ forms a tree on S.*

Proof. It is an easy exercise to verify that the relation $\lhd$ is non-reflexive and transitive.

Moreover, consider a set P of predecessors of an element of the tree. Then $\langle P, \lhd \rangle$ is obviously well-founded, because if $\bar{\nu} \lhd \nu$, then also $\alpha_{\bar{\nu}} < \alpha_\nu$.

It is left to prove that such a set P is also linearly ordered by $\lhd$. If we showed this, we would even have the missing well-ordering property of a set of predecessors of an element of the tree.

Claim. *If* $\bar{\nu}, \nu \lhd \nu$, *then* $\bar{\nu} \unlhd \nu'$ *or* $\nu' \unlhd \bar{\nu}$.

For, let $\bar{\nu} \lhd \nu$ and $\nu' \lhd \nu$. Consider the two maps $\pi : \mathbf{L}_{\beta_{\bar{\nu}}} \longrightarrow \mathbf{L}_{\beta_\nu}$ and $\pi' : \mathbf{L}_{\beta_{\nu'}} \longrightarrow \mathbf{L}_{\beta_\nu}$ given by the definition of the tree where $\pi_{\bar{\nu}\nu} := \pi \restriction \mathbf{L}_{\bar{\nu}}$ and $\pi_{\nu'\nu} := \pi' \restriction \mathbf{L}_{\nu'}$. Then we have by construction (and well-known arguments, *e.g.*, condensation property) the following:

$$\begin{aligned} \mathrm{rng}(\pi) &= \text{the smallest } \bar{X} \prec \mathbf{L}_{\beta_\nu} \text{ such that} \\ &\quad\; A \cap \bar{X} \text{ is transitive and } \alpha_{\bar{\nu}} = A \cap \bar{X}, \\ \mathrm{rng}(\pi') &= \text{the smallest } X' \prec \mathbf{L}_{\beta_\nu} \text{ such that} \\ &\quad\; A \cap X' \text{ is transitive and } \alpha_{\nu'} = A \cap X'. \end{aligned}$$

Without loss of generality, let $\alpha_{\bar{\nu}} \leqslant \alpha_{\nu'}$. But then, $\bar{X}$ is a subset of X'. Therefore, $\pi'^{-1} \circ \pi : \mathbf{L}_{\beta_{\bar{\nu}}} \longrightarrow \mathbf{L}_{\beta_\nu}$ is an elementary embedding with the needed properties to conclude that $(\pi'^{-1} \circ \pi) \restriction \mathbf{L}_{\bar{\nu}} = \pi_{\bar{\nu}\nu}$ because of the uniqueness of the embeddings given by Lemma 55. And so, we finally have $\bar{\nu} \unlhd \nu$.

With the claim also the lemma is proved. ⊠(Lemma 56)

There are two more properties we will find later useful. One of them says that there are, in fact, many limit points within the tree relation.

Lemma 57. *For* $\alpha \leqslant A$ *let* $\xi, \nu \in S_\alpha$ *where* $\xi < \nu$. *Then*

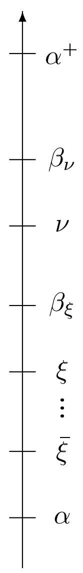
(a) $\beta_\xi < \nu$,
(b) $\sup \{ \bar{\xi} \mid \bar{\xi} \lhd \xi \} = \alpha$.

Proof. Let α, ξ and ν be as above. Then by definition, α is the largest cardinal in $\mathbf{L}_\nu$ and therefore we trivially have $\mathbf{L}_\nu \models |\xi| \leqslant \alpha$.

But this holds even below ν, say at stage $\xi' < \nu$, and so $\mathbf{L}_{\xi'} \models |\xi| \leqslant \alpha$. We then are able to find an $\eta < \nu$ but above ξ' such that η is p.r.-closed. However, this is the condition β_ξ should satisfy. Because of the minimality we finally have $\beta_\xi \leqslant \eta < \nu$. This proves the first fact.

For the second property, note that for $\bar{\xi} \lhd \xi$ we have that the model $\mathbf{L}_\xi$ thinks that $\alpha = \alpha_\xi$ is the largest cardinal and $\mathbf{L}_{\bar{\xi}}$ thinks the same about $\alpha_{\bar{\xi}} < \alpha$. Therefore, $\bar{\xi}$ cannot be greater than α because $\mathbf{L}_{\bar{\xi}}$ is a subset of $\mathbf{L}_\xi$.

On the other hand, let $\gamma < \alpha$. We will find a $\bar{\xi} < \alpha$ such that $\gamma \leqslant \bar{\xi}$ and $\bar{\xi} \lhd \xi$ as follows, working in $\mathbf{L}_\nu$: Starting from $X_0 := \gamma \cup \{\xi\}$ we set $X := \bigcup_{i<\omega} X_i$ where X_{i+1} is the smallest X' such that $X' \prec \mathbf{L}_{\beta_\xi}$ and $X_i \cap \alpha \in X'$. Then X will be the smallest $X \prec \mathbf{L}_{\beta_\xi}$ such that $\gamma \cup \{\xi\} \subseteq X$ and $X \cap \alpha$ transitive.



Moreover, α looks like a regular cardinal in $\mathbf{L}_\nu$, so the cardinality of X is strictly smaller than α even in this model. Consider then the collapse map $\sigma : \mathbf{L}_{\bar{\beta}} \overset{\sim}{\longleftrightarrow} X$ and we have $\sigma(\bar{\alpha}) = \alpha$ for the critical point $\bar{\alpha}$ of the embedding σ. Furthermore, even the map σ is an element of $\mathbf{L}_\nu$. Now let $\bar{\xi}$ such that $\sigma(\bar{\xi}) = \xi$. Then we finally have

Claim. $\bar{\xi} \in S_{\bar{\alpha}}$.

To see this we have to look at the properties in the definition of $S_{\bar{\alpha}}$. Because α is strictly less than ξ we trivially have $\bar{\alpha} < \bar{\xi}$. Also we know that ξ is limit of p.r.-closed ordinals and so is $\bar{\xi}$ by the elementary preserving property of σ and Lemma 37. The same reason shows the regularity and the property 'being the largest cardinal' of $\bar{\alpha}$ within $\mathbf{L}_{\bar{\xi}}$.

Moreover, because of $\bar{\xi} < \bar{\beta}$ we know that $\bar{\xi}$ is strictly smaller than $\bar{\alpha}$. This finishes the proof of the claim.

Trivially, we also have $\gamma \leqslant \bar{\xi} < \alpha$ by our construction and for the elementary embedding $\sigma : \mathbf{L}_{\bar{\beta}} \longrightarrow \mathbf{L}_{\beta_\xi}$ we know by definition and its properties that $\bar{\beta} = \beta_{\bar{\xi}}$ and $\bar{\alpha} = \alpha_{\bar{\xi}}$ holds and so by the uniqueness of the tree embeddings, given by Lemma 55, also $\sigma \upharpoonright \mathbf{L}_{\bar{\xi}} = \pi_{\bar{\xi}\xi}$.

And so, we finally have shown everything for $\bar{\xi} \lhd \xi$. ⊠(Lemma 57)

Note, the second part (b) of the last lemma claims that ξ is a limit point within the tree relation $\lhd$. Moreover, together with Lemma 53 we have finally shown that each $\nu \in S_A$ is a limit point within the tree relation, that is

$$\sup\{ \bar{\nu} \mid \bar{\nu} \lhd \nu \} = A.$$

Considering the figure that might help to understand the structure, we are now ready to define the complete structure we are aiming to:

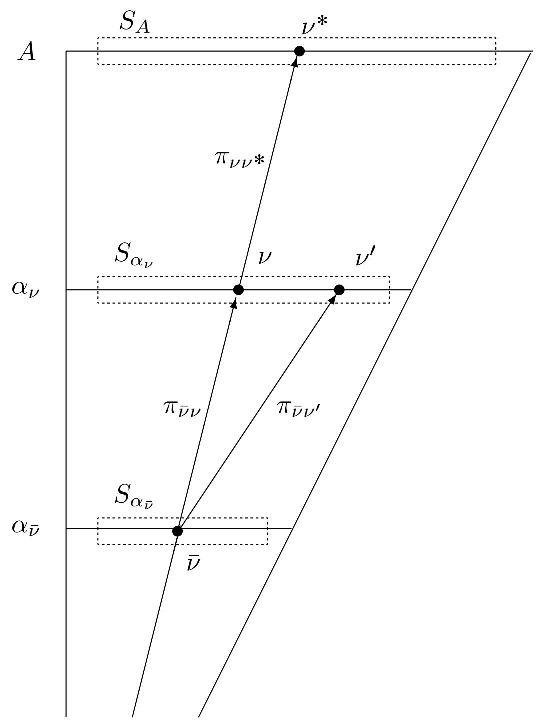


FIGURE 1. The coarse A-morass

Definition 58 (The coarse A-Morass)**.** *Let the cardinal A, the sequence $\langle S_\alpha \mid \alpha \leqslant A \rangle$, the tree relation $\lhd$ with the sequence $\langle \pi_{\bar{\nu}\nu} \mid \bar{\nu} \lhd \nu \rangle$ of embeddings be defined as above. Then we call the structure*

$$\mathfrak{M} := \langle B, A, \langle S_\alpha \mid \alpha \leqslant A \rangle, \lhd, \langle \pi_{\bar{\nu}\nu} \mid \bar{\nu} \lhd \nu \rangle \rangle$$

the coarse *A*-morass *with the universe B such that $A \subseteq B \subseteq \mathrm{On}$.*