CHAPTER 3

The Forcing

We now define the notion of forcing we are going to use. In fact, it will be Mitchell's forcing that he introduced in [Mit72]. There he used it to prove the statement we are trying to improve with Theorem 59.

To start with, fix κ an uncountable regular cardinal and $\tau > \kappa$ an inaccessible one in a (suitable) ground model for the remaining part of this chapter.

Definition 41. *Let* $\mathbb{P} := \mathbb{P}(\tau)$ *be* $\{\, p \,:\, \exists x \, (p \in {}^{x}2 \,\wedge\, |x| < \omega \,\wedge\, x \subseteq \tau \,\}$, *ordered by the usual reverse inclusion.*

Then the application of $\mathbb{P}$ adjoints in the usual way τ-many (Cohen) reals. Now, the second forcing looks a bit more technical.

For $\alpha < \tau$, let $\mathbb{P}_\alpha := \{\, p \in \mathbb{P} \mid p \restriction \alpha = p \,\}$. If $s \subseteq \mathbb{P}$, then define

$$b_s := \{\, p \in \mathbb{P} \mid \forall q \leqslant_{\mathbb{P}} p \, \exists r \in s \, (\, r \text{ and } q \text{ are compatible} \,) \,\},$$

or equivalently, $b_s = \{\, p \in \mathbb{P} \mid p \Vdash_{\mathbb{P}} \bar{G} \cap s \neq \varnothing \,\}$ for a $\mathbb{P}$-generic $\bar{G}$ over the ground model M.

Let $\mathcal{B}$ be the boolean algebra associated with $\mathbb{P}$. Define then $\mathcal{B}_\alpha \subseteq \mathcal{B}$ by $\mathcal{B}_\alpha := \{\, b_s \mid s \subseteq \mathbb{P}_\alpha \,\}$. Then $\mathcal{B}_\alpha$ is canonically isomorphic to the complete boolean algebra associated with $\mathbb{P}_\alpha$.

Call a function $f \in M$ *acceptable*, if the following conditions hold:

(a) $\mathrm{dom}(f) \subseteq \tau$ and $\mathrm{rng}(f) \subseteq \mathcal{B}$,
(b) $|\mathrm{dom}(f)| < \kappa$,
(c) for all $\gamma < \tau$ we have $f(\gamma) \in \mathcal{B}_{\gamma+\omega}$.

The bounding property given by the last property will give us control of the time when we will have collapsed cardinals with the second forcing. This important fact was used by Mitchell in [Mit72] as we will see later.

Let $\mathcal{A}$ be the set of all acceptable functions in M. Moreover, let $\bar{G}$ be $\mathbb{P}$-generic and let M' be $M[\bar{G}]$, the forcing extension by $\mathbb{P}$ and also ground model for second forcing $\dot{\mathbb{Q}}$. Then for $f \in \mathcal{A}$ define $\bar{f} : \mathrm{dom}(f) \longrightarrow 2$ in M' by $\bar{f}(\gamma) = 1$ if and only if $f(\gamma) \cap \bar{G} \neq \varnothing$.

Definition 42. *With the notation above let $\dot{\mathbb{Q}} := \dot{\mathbb{Q}}(\kappa, \tau)$ be defined over the model M' by letting the field be the set $\mathcal{A}$ of acceptable functions and letting $f \leqslant_{\dot{\mathbb{Q}}} g$ if and only if $\bar{f} \supseteq \bar{g}$.*

Note, although the field of $\dot{\mathbb{Q}}$ is a subset of the ground model M, the definition of the order $\leqslant_{\dot{\mathbb{Q}}}$ is using the $\mathbb{P}$-generic object G.

Finally, we can now put both forcings together. Note, $\dot{\mathbb{Q}}$ is defined in M', a generic extension of $\mathbb{P}$. We, therefore, denote this partial order with a dot, to signify that we are using a name for it.

We then define the notion of forcing we are interested in:

Definition 43 (Mitchell)**.** *Let $\mathbb{M}(\kappa, \tau)$ be the usual two-step product $\mathbb{P} \star \dot{\mathbb{Q}}$ of the forcings $\mathbb{P}$ and $\dot{\mathbb{Q}}$ defined above, that is*

$$\begin{aligned} \mathbb{M}(\kappa, \tau) \quad &:= \quad \mathbb{P} \times \mathcal{A}, \\ (p, f) \leqslant_{\mathbb{M}(\kappa,\tau)} (q, g) \quad &:\Longleftrightarrow \quad p \leqslant_{\mathbb{P}} q \ \text{ and } p \Vdash_{\mathbb{P}} f \leqslant_{\dot{\mathbb{Q}}} g. \end{aligned}$$

In the following we will cite a few lemmas proven in [Mit72]. Actually, Mitchell defines simultanously two such forcings depending which problem they should solve. We are using the second one, in fact, our $\mathbb{M}(\kappa, \tau)$ is his $R_2(\omega, \kappa, \tau)$ and we are stating the lemmas in our terminology.

Fix apart from $\bar{G}$ now also a $\dot{\mathbb{Q}}$-generic $\tilde{G}$ over M' and set N be the forcings extension $M[\bar{G}][\tilde{G}]$. We already know by elementary forcing arguments that then $\bar{G} \times \tilde{G}$ is $\mathbb{M}$-generic over the ground model M, *cf.* [Jec03, Kun80] for details.

Lemma 44 ([Mit72]). *In the first step of the forcing using* $\mathbb{P}$*, we adjoin* τ*-many reals to* M*, however, cardinals are preserved. In the second step we collapse* τ *to* κ^+*.*

One of the major observations for our desired preserving properties is the following:

Lemma 45 ([Mit72, Lemma 3.3]). *The notion of forcing* $\mathbb{M}(\kappa, \tau)$ *has the* τ*-chain condition.*

Therefore we get the following consequence:

Corollary 46 ([Mit72, Lemma 3.4]). *For all ordinals* δ *such that* $\delta \leqslant \kappa$ *or* $\tau \leqslant \delta$ *we have* $|\delta|^M = |\delta|^N$*. Hence cardinals below* κ *and above* τ *are preserved.*

And finally we can conclude:

Corollary 47 ([Mit72, Corollary 3.5]). *In the forcing extension* N *we have* $2^\omega = 2^\kappa = \tau$*.*

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We will now try to look on the forcing construction in a rather different way. In fact, we will split it off in τ-many parts $\mathbb{M}_\nu$ and $\mathbb{M}^\nu$ for $\nu < \tau$ where $\mathbb{M}_\nu$ will consists of conditions p of the forcing $\mathbb{M}$ such that "p works below ν" and similarly for $\mathbb{M}^\nu$. In fact, $\mathbb{M}_\nu$ will add subsets of ω which can be described with conditions below ν and then we will collapse all ordinals below ν to κ. Of course this is only interesting in the case that ν is greater than κ.

Therefore we set

$$\mathbb{P}_\nu := \{\, p \in \mathbb{M} \;:\; p \restriction \nu = p \,\}, \qquad \mathbb{P}^\nu := \{\, p \in \mathbb{M} \;:\; p \restriction \nu = 0 \,\},$$
$$\mathcal{A}_\nu := \{\, f \in \mathcal{A} \;:\; f \restriction \nu = f \,\}, \qquad \mathcal{A}^\nu := \{\, f \in \mathcal{A} \;:\; f \restriction \nu = 0 \,\}.$$

And moreover

$$\mathbb{M}_\nu := \mathbb{P}_\nu \times \mathcal{A}_\nu, \qquad \mathbb{M}^\nu := \mathbb{P}^\nu \times \mathcal{A}^\nu,$$
$$G_\nu := G \cap \mathbb{M}_\nu, \qquad G^\nu := G \cap \mathbb{M}^\nu.$$

As noticed above we finally can prove the following

Lemma 48 ([Mit72, Lemma 3.6])**.** *Let $\nu < \tau$ be a limit ordinal. Then G_ν is $\mathbb{M}_\nu$-generic over M, K^ν is $\mathbb{M}^\nu$-generic over $M[G_\nu]$, and $M[G_\nu][G^\nu] = M[G]$.*

The by far most important tool in analyzing Mitchell's notion of forcing is the following, providing that sequences of length with an uncountable cofinality such that their initial segments can be found in an initial segment of the forcing construction, in fact, are already an element of this segment:

Lemma 49 ([Mit72, Lemma 3.8])**.** *Suppose that γ has uncountable cofinality in the ground model M and let $t : \gamma \longrightarrow M$ be such that $t \in M[G]$ and $t \restriction \alpha \in M[G_\nu]$ for every $\alpha < \nu$. Then $t \in M[G_\nu]$.*

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Now we turn back to the question *when* exactly we add new reals. We already know that with $\mathbb{M}(\kappa, \tau)$ we add τ-many Cohen reals because of the first forcing part. However, we can ask whether the second forcing is changing the powerset of ω again. In fact, it will not as we will prove with a tool that Mitchell has proved with the following

Lemma 50 ([Mit72, Lemma 3.1])**.** *Suppose that $\dot{D}$ is a term such that*

$$\Vdash_{\mathbb{P}} (\ \dot{D} \text{ is strongly dense in } \dot{\mathbb{Q}}\ ),$$

and $f \in \mathcal{A}$. Then there is $g \in \mathcal{A}$ such that $g \supseteq f$ and $p \Vdash_{\mathbb{P}} (\ g \in \dot{D}\ )$.

And so, we are able to prove finally

Lemma 51. $\mathcal{P}(\omega) \cap M[\bar{G}] = \mathcal{P}(\omega) \cap M[G]$.

Proof. The first inclusion $\subseteq$ is obvious. Suppose now the other one does not hold. We are going to deduce a contradiction.

So suppose there is a subset a of ω such that a is an element of $M[G]$ but not in $M[\bar{G}]$, then using the theory of forcing there is a condition $f \in \dot{\mathbb{Q}}$ such that

$$M[\bar{G}] \models (\ \check{f} \Vdash_{\dot{\mathbb{Q}}} (\exists a \subseteq \omega)(a \notin \check{\mathbf{V}})\ ).$$

Then there is already a condition $p \in G \subseteq \mathbb{P}$ such that

$$p \Vdash_{\mathbb{P}} (\check{f} \Vdash_{\dot{\mathbb{Q}}} (\exists a \subseteq \omega)(a \notin \check{\mathbf{V}})).$$

So we can find a $\mathbb{P}$-name for a $\dot{\mathbb{Q}}$-name $\dot{a}$ such that

$$p \Vdash_{\mathbb{P}} (\check{f} \Vdash_{\dot{\mathbb{Q}}} (\dot{a} \subseteq \omega \ \wedge \ \dot{a} \notin \check{\mathbf{V}})).$$

Now consider the set

$$\{ g \in \dot{\mathbb{Q}} \mid (g \leqslant_{\dot{\mathbb{Q}}} \check{f} \ \wedge \ (g \Vdash \check{i} \in \dot{a} \vee g \Vdash \check{i} \notin \dot{a})) \ \vee \ g \perp \check{f} \}$$

and let $\dot{D}_i$ be a $\mathbb{P}$-name for it. Then we can conclude that

$$p \Vdash_{\mathbb{P}} (\dot{D}_i \text{ is dense in } \dot{\mathbb{Q}}).$$

Now, step by step using Lemma 50 we can find $f_i \in \mathcal{A}$ for $i < \omega$ such that $f_0 := f$, $f_{i+1} \supseteq f_i$ and $p \Vdash_{\mathbb{P}} \check{f}_{i+1} \in \dot{D}_i$. Finally define $f^* := \bigcup_{i<\omega} f_i$. Then we obviously have $p \Vdash \check{f}^* \leqslant_{\dot{\mathbb{Q}}} \check{f}_i$ for all $i < \omega$. Hence

$$M[G] \models (\check{f}^* \leqslant_{\dot{\mathbb{Q}}} \check{f} \ \wedge \ \check{f}^* \in \bigcap_{i<\omega} \dot{D}_i^G \ \wedge \ \check{f}^* \in \dot{\mathbb{Q}}).$$

Therefore, for each $i < \omega$ in $M[\bar{G}]$, already $\check{f}^*$ knows about whether $\check{i}$ is in $\dot{a}^G$ or not, that is

$$M[\bar{G}] \models (\check{f}^* \Vdash_{\dot{\mathbb{Q}}} \check{i} \in \dot{a}^G \ \vee \ \check{f}^* \Vdash_{\dot{\mathbb{Q}}} \check{i} \notin \dot{a}^G).$$

Define then in the ground model M,

$$b := \{i < \omega \mid M[\bar{G}] \models (\check{f}^* \Vdash_{\dot{\mathbb{Q}}} \check{i} \in \dot{a}^G)\} \in M.$$

But then we have $\check{f}^* \Vdash \dot{a}^G = b \ \wedge \ \check{b} \in \check{\mathbf{V}}$. Hence $\check{f}^* \Vdash \dot{a}^G \in \check{\mathbf{V}}$.

However, we also have $\check{f}^* \Vdash \dot{a}^G \notin \check{\mathbf{V}}$ because of $\check{f}^* \leqslant_{\dot{\mathbb{Q}}} \check{f}$ and so we have deduced the desired contradiction. ⊠(Lemma 51)

Note, it is a similiar argument like using the property of a partial order being $\aleph_1$-closed. However, our forcing $\dot{\mathbb{Q}}$ does, in fact, not bear this property in $M[\bar{G}]$ where we need it. Although the conditions, functions within $\mathcal{A}$, live in the ground model, the order is defined using the generic object $\bar{G}$ and this causes the problems together with the fact that we added many countable subsets to M when we got $M[\bar{G}]$.

Finally let us turn to a property of Mitchell's forcing that it does not kill stationary subsets of τ.

Lemma 52. *Let $\mathbb{M} = \mathbb{M}(\kappa, \tau)$ be Mitchell's forcing and let G be $\mathbb{M}$-generic over a ground model M. Then, in the extension $N = M[G]$, S remains a stationary subset of τ, where*

$$S := \{ \lambda < \tau \mid M \models \mathrm{cf}(\lambda) = \kappa^+ \}.$$

Proof. Let $\dot{C} \subseteq \check{\tau}$ be closed and unbounded. We are going to prove that

$$\dot{C}^G \cap \{ \lambda < \tau \mid M \models \mathrm{cf}(\lambda) = \kappa^+ \} \neq \varnothing.$$

For, let $\dot{\gamma}$ be a name of a monotone enumeration of $\dot{C}$, that is

$$\Vdash_{\mathbb{M}} \dot{C} = \langle \dot{\gamma}(\xi) \mid \xi < \check{\tau} \rangle \wedge \dot{\gamma} \text{ is monotone.}$$

Moreover, let D_ν be a maximal antichain in $\{ p \mid \exists \alpha \, p \Vdash_{\mathbb{M}} \dot{\gamma}(\check{\nu}) = \check{\alpha} \}$. Because of Lemma 45 we know that $|D_\nu| < \tau$. Now, for conditions $p \in D_\nu$ define $\gamma_{\nu,p}$ as the unique α such that $p \Vdash_{\mathbb{M}} \dot{\gamma}(\check{\nu}) = \check{\alpha}$ and, finally, $\Gamma_\nu := \{ \gamma_{\nu,p} \mid p \in D_\nu \}$. Then

$$\dot{\gamma}^G(\nu) \in \Gamma_\nu \in M. \tag{1}$$

In the ground model M, define a sequence $\langle \beta_\xi \mid \xi \leqslant \kappa^+ \rangle$ by setting $\beta := 0$, $\beta_\lambda := \bigcup_{\xi<\lambda} \beta_\xi$ and more interesting

$$\beta_{\xi+1} := \mathrm{lub}(\bigcup\{\Gamma_\nu \mid \nu < \beta_\xi\}).$$

Here, 'lub' means 'least upper bound'. Finally we have for $\beta := \beta_{\kappa^+}$ that $\bigcup_{\nu<\beta} \Gamma_\nu \subseteq \beta$ and $\mathrm{cf}(\beta) = \kappa^+$.

Then, in $M[G]$, the defined β is an element of $\dot{C}^G$. For, let $\xi < \beta$. By our construction we have

$$\xi \leqslant \dot{\gamma}^G(\xi) < \Gamma_\xi \subseteq \bigcup\{\Gamma_\nu \mid \nu < \beta_{\xi+1}\} \leqslant \beta_{\xi+2} < \beta.$$

Hence, β is a limit point of $\dot{C}^G$ because of (1) and so also an element of the closed set $\dot{C}^G$.

Therefore, $\Vdash_{\mathbb{M}} \check{\beta} \in \dot{C} \cap S$ and the lemma is proved. ⊠(Lemma 52)