CHAPTER 2

Fundamentals

The following statements are all provable in (sometimes just parts of) the Zermelo-Fraenkel set theory with the axiom of choice: ZFC.[1] The collection of these results should *not* be seen as a *complete* introduction to the theory we are using in the upcoming chapters. Most of the following statements will just be cited, anyway. For a detailed survey and proofs the author strongly refers to the standard books, *e.g.*, [ChaKei90, Dev84, Dra74, Jec03, Kan94]. For the conveniency of the reader we are (mostly) using standard notation.

With this in mind, the reader will find in this chapter some important standard facts and even some other (technical) basics we will need later.[2]

Constructing Models

Let us start with some (very) basic set theory. A set X is said to be *extensional* if for all distinct $u, v \in X$ there is an $x \in X$ such that $x \in u$ if and only if $x \notin v$.

Lemma 4 ([Jec03, Theorem 6.15], Mostowski Collapse)**.** *For each extensional set X there is a unique transitive set M and an unique bijection $\pi : X \longleftrightarrow M$ such that $\pi : \langle X, \in \rangle \overset{\sim}{\longleftrightarrow} \langle M, \in \rangle$. Moreover, if $Y \subseteq X$ is transitive, then $\pi \restriction Y = \mathrm{id} \restriction Y$.*

[1] In fact, in most cases we will not need the presence of the Axiom of Choice here. However, in our applications we will have it.

[2] In most cases, the more famous the statement is, the less we are proving it here in this chapter.

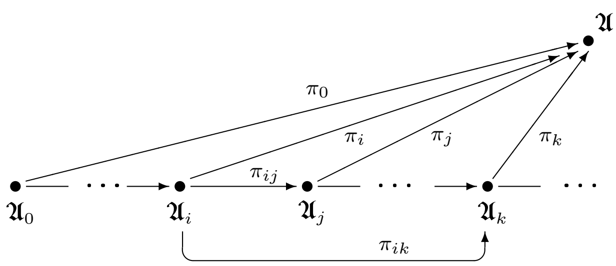
More generally, X has not to be a set but it must be so-called set-like, meaning that for all $x \in X$ the collection of all $\in$-predecessors, $\in{}''\{x\}$, is a set.

Now let us look at models of, say, a countable language $\mathcal{L}$, and elementary embeddings between them.

Definition 5. *A directed system of models $\langle \mathfrak{A}_i \,|\, i < \omega \rangle$ has elementary embeddings $\pi_{ij} : \mathfrak{A}_i \longrightarrow \mathfrak{A}_j$ such that $\pi_{ik} = \pi_{jk} \circ \pi_{ij}$ for all natural numbers $i < j < k$.*

Then we can prove:

Lemma 6 ([Jec03, Lemma 12.2], Direct Limit)**.** *If $\langle \mathfrak{A}_i, \pi_{ij} \,|\, i < j < \omega \rangle$ is a directed system of models, then there exists a model $\mathfrak{A}$, unique up to isomorphism, and elementary embeddings $\pi_i : \mathfrak{A}_i \longrightarrow \mathfrak{A}$ such that $\mathfrak{A} = \bigcup_{i<\omega} \mathrm{rng}(\pi_i)$ and $\pi_i = \pi_j \circ \pi_{ij}$ for all $i < j$.*



The model $\mathfrak{A}$ in the last lemma is called the *direct limit* of the given sequence $\langle \mathfrak{A}_i, \pi_{ij} \mid i < j < \omega \rangle$.

Stationary Sets

Remember, for a regular cardinal κ we call $X \subseteq \kappa$ a closed and unbounded set, *club* for short, if it is closed under limit points and unbounded in κ. We call a set $S \subseteq \kappa$ *stationary* if it meets all club sets.

There are nice properties, *e.g.*, the collection of all club sets is closed under intersections of strictly less than κ many sets. Moreover, it is closed under diagonal intersection of length κ. Using this we can prove the following important and well-known fact:

Lemma 7 ([Jec03, Theorem 8.7], Fodor). *If $S \subseteq \kappa$ is stationary and $\pi : S \longrightarrow \kappa$ is a regressive ordinal function, that is $\pi(\xi) < \xi$ for all $\xi \in S \backslash \{\varnothing\}$, then there is a stationary subset $T \subseteq S$ and an ordinal $\gamma < \kappa$ such that $f(\alpha) = \gamma$ for all $\alpha \in T$.*

Sometimes useful –as we will see later– is also the following collection of well-known facts:

Lemma 8. *Let $\kappa > \omega$ be a regular cardinal.*

- *Let $\mathrm{cf}(\mu) > \kappa$. Then the set $\{\gamma < \mu \,|\, \mathrm{cf}(\gamma) = \kappa\}$ is a stationary subset of μ.*
- *For each function $f : \kappa \longrightarrow \kappa$, the set $\{\alpha < \kappa \mid f''\alpha \subseteq \alpha\}$ is closed and unbounded in κ.*
- *The set of all limit points of a club set of κ is club again.*

⋆ ⋆ ⋆

Now, call a transitive class $W \subseteq \mathbf{V}$ an *inner model* if W contains all ordinal numbers and satisfies ZFC. In fact, the constructible universe $\mathbf{L}$ is an inner model as we will discuss in one of the next sections. Then we can prove the following

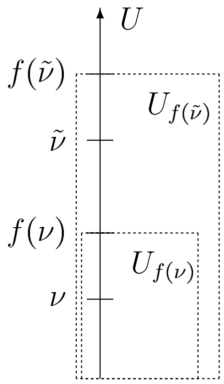
Lemma 9. *Let W be an inner model and $\kappa < \tau$ be infinite cardinals such that $\kappa^{+W} < \tau$ and $\{\ \lambda < \tau \mid W \models \mathrm{cf}(\lambda) = \kappa^+\ \}$ is stationary in the universe $\mathbf{V}$. Moreover let $U := \langle\, U, <_U \rangle$ be a linear order such that $U \subseteq H$ for a suitable $H \in W$, $|H|^W = \tau$ and*

$$U_x := \{\ z \mid z \leqslant_U x\ \} \in W,\ |U_x|^W \leqslant \kappa \text{ for any } x \in U.$$

Then we have $|U| < \tau$.

Proof. Suppose not. Without loss of generality, using a suitable bijection let H be just τ. So, let γ be the cofinality of $\langle\, U, <_U \rangle$, that is the cardinality of a minimal subset of U which lies cofinal. Hence, we have $\gamma \leqslant |U| = \tau$. However, letting $f : \gamma \longrightarrow U$ be a monotone and cofinal enumeration, we have $U = \bigcup_{\nu<\gamma} U_{f(\nu)}$ and so afterall also $\tau = |U| \leqslant \gamma \cdot \kappa \leqslant \tau$. This means we have $\gamma = \tau$.

Define $g(\nu) := \sup(U_{f(\nu)} \cap \nu)$ for $\nu < \tau$. Then g is weakly monotone, that is $g(\nu) \leqslant \nu$, and $\sup_{\nu<\tau} g(\nu) = \sup U = \tau$. By our assumption, the set $S := \{ \nu < \tau \mid W \models \mathrm{cf}(\nu) = \kappa^+ \}$ is stationary in $\mathbf{V}$. For elements ν of S is $g(\nu)$ strictly less than ν because in W we have: $|U_{f(\nu)}| \leqslant \kappa < \kappa^+ = \mathrm{cf}(\nu) \leqslant \nu$, and so $U_{f(\nu)}$ is bounded in ν.



Hence, $g \restriction S$ is a regressive function on a stationary set. Now, Theorem 7 of Fodor implies that there is a stationary subset S' of S such that $g(\nu) = \alpha$ for a suitable $\alpha < \tau$ and arbitrary $\nu \in S'$. But in this case we also have the following contradiction:

$$\tau = \sup_{\nu\in S} g(\nu) = \sup_{\nu\in S'} g(\nu) = \alpha < \tau.$$

Note, the first equality just holds because by definition we have for $x \leqslant_U y$ obviously $U_x \subseteq U_y$ and so

$$\bigcup\{ U_{f(\nu)} \mid \nu < \tau \} = \bigcup\{ U_{f(\nu)} \cap \nu \mid \nu < \tau \},$$

meaning the range of f and g are both cofinal in τ.

Therefore, the lemma is proved. ☒(Lemma 9)

Forcing

Working in a so-called *ground model* M, we consider a partial order $\langle \mathbb{P}, <_{\mathbb{P}} \rangle$ and call it sometimes *notion of forcing* with the so-called *forcing conditions* as its elements. We also say that a condition p *is stronger than* a condition q if $p <_{\mathbb{P}} q$. We call a set $D \subseteq \mathbb{P}$ *dense* in $\mathbb{P}$ if for every $p \in \mathbb{P}$ there is a $q \in D$ such that $q \leqslant_{\mathbb{P}} p$.

Call a non-empty $G \subseteq \mathbb{P}$ a *filter* if firstly, whenever $p < q$ and $p \in G$, then also $q \in G$; and, secondly, if p, q are elements of G, then there is an $r \in G$ such that r is stronger than both, p and q. Moreover, call a filter G *generic over* M (or just M*-generic*) if for every dense D in $\mathbb{P}$ and $D \in M$, the filter G always meets D.

Then we can construct the so-called *forcing extension* or *generic extension*, $M[G]$, of the ground model M that satisfied ZFC, given a generic filter G, such that $M[G] \models$ ZFC, $M \subseteq M[G]$, $G \in M[G]$, $\mathrm{On}^{M[G]} = \mathrm{On}^M$ and it is minimal in the sense that if N is a transitive model of ZF such that $M \subseteq N$ and $G \in N$, then $M[G] \subseteq N$.

The main idea now is that we are able to name elements of the generic extension within the ground model. Moreover, we can define the so-called *forcing relation* '$\Vdash$' within M and so we are able to decide within the ground model what properties hold in the generic extension:

Lemma 10 ([Jec03, Kun80])**.** *For every generic $G \subseteq \mathbb{P}$ over M and every formula φ of the forcing language we have*

$$M[G] \models \varphi \quad \textit{if and only if} \quad \exists p \in G\, p \Vdash \varphi.$$

For a collection of the properties of the forcing relation we refer to [Jec03, Theorem 14.7].

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In Chapter 3 we will use the connection to Boolean algebras. In fact, we can look at the universe $\mathbf{V}$ as collection of functions, ranging into the set $2 = \{0, 1\}$, that is, roughly speaking, the identification of sets with their characteristic functions. So we can identify the universe $\mathbf{V}$ with $\mathbf{V^2}$ where $\mathbf{2}$ is the simplest Boolean algebra. Then the formula $x \dot{\in} y$ has truth value 0 or 1.

Taking now a more complex Boolean algebra $\mathcal{B}$ we can look at atomic formulae $x \dot{\in} y$ and $x \dot{=} y$ where the truth value can be an element of $\mathcal{B}$ strictly between $0_\mathcal{B}$ and $1_\mathcal{B}$. Choosing $\mathcal{B}$ in an appropiate way we can try to decide formulae which we cannot in $\mathbf{V^2}$.

Furthermore, having a Boolean algebra $\mathcal{B}$ we can consider the related partial order $\langle \mathcal{B}', \leqslant_{\mathcal{B}'} \rangle$ defined as $\mathcal{B}' := \mathcal{B} \backslash \{0_\mathcal{B}\}$ and setting $b_0 \leqslant_{\mathcal{B}'} b_1$ if $p \cdot q = p$.

Moreover, we can start from a partial order to construct a related Boolean algebra as follows:

Lemma 11 ([Jec03]). *For every partially ordered set* $\langle \mathbb{P}, <_{\mathbb{P}} \rangle$ *there is a complete Boolean algebra* $\mathcal{B} = \mathcal{B}(\mathbb{P})$ *and a mapping* $\pi : \mathbb{P} \longrightarrow \mathcal{B}'$ *where* $\mathcal{B}'$ *is as above such that*

- *if* $p \leqslant_{\mathbb{P}} q$, *then* $\pi(p) \leqslant_{\mathcal{B}'} \pi(q)$,
- p *and* q *are compatible if and only if* $\pi(p) \cdot \pi(q) \neq 0_{\mathcal{B}}$,
- *the set* $\{ \pi(p) \mid p \in \mathbb{P} \}$ *is dense in* B.

Moreover, the Boolean algebra $\mathcal{B}$ *is unique up to isomorphism.*

Having these complete Boolean algebras we can construct a Boolean-valued model (of the language of set theory) where the Boolean values of $\dot{=}$ and $\dot{\in}$ are given by two functions of two variables $[\![x \dot{=} y]\!]$ and $[\![x \dot{\in} y]\!]$, *cf.* [Jec03] for all details.

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We are now interested in cardinal preserving properties for a given notion of forcing. For a cardinal κ in the ground model, say a partial order yields the κ*-chain condition*, κ*-c.c.* for short, if every dense set has cardinality strictly less than κ.

Lemma 12. *If* $\mathbb{P}$ *yields the* κ*-c.c., then it preserves cofinalities above* κ*, that means, if* λ *is a cardinal such that* $\mathrm{cf}^M(\lambda) \geqslant \kappa$*, then we have* $\mathrm{cf}^M(\lambda) = \mathrm{cf}^{M[G]}(\lambda)$. *Moreover, if* κ *is regular, then cardinals are preserved above* κ.

A partial order $\mathbb{P}$ is λ-*closed* if whenever $\gamma < \lambda$ and $\{ p_\xi \mid \xi < \gamma \}$ is a decreasing sequence of elements of $\mathbb{P}$, that is $p_\xi \leqslant_{\mathbb{P}} p_\eta$ for $\eta < \xi$, then there is a $q \in \mathbb{P}$ such that for each $\xi < \gamma$ we have $q \leqslant_{\mathbb{P}} p_\xi$.

This property ensures that objects with suitable small cardinality within the forcing extension can already be found in the ground model.

Lemma 13 ([Jec03, Kun80]). *If* $\mathbb{P}$ *is* λ*-closed for a cardinal* λ*, then there are no new sets of ordinals of cardinality strictly smaller than* λ *in the forcing extension. Therefore,* $\mathbb{P}$ *preserves cofinalities below* λ. *Moreover,* $\mathbb{P}$ *preserves also cardinals below* λ.

We turn now to the problem of the iterated application of forcing, that is in the easiest case the following two-step product forcing.

Lemma 14. *Let $\mathbb{P}$ and $\mathbb{Q}$ be two notions of forcing in M. In order that $G \subseteq \mathbb{P} \times \mathbb{Q}$ is generic over M, it is necessary and sufficient that $G = G_1 \times G_2$ where $G_1 \subseteq \mathbb{P}$ is generic over M and $G_2 \subseteq \mathbb{Q}$ is generic over $M[G_1]$. Moreover, in this case we have*

$$M[G] \;=\; M[G_1][G_2]\,(=\; M[G_2][G_1]\,).$$

Now, in general, in applications the second forcing might be not an element of the ground model M. The important fact here is that even then, in case of a two-step iteration, we can represent it by a single notion of forcing extension over the ground model.

Let $\mathbb{P}$ be a partial order in M and $\dot{\mathbb{Q}}$ a name for a partial order, that is $\Vdash_{\mathbb{P}} (\dot{\mathbb{Q}}$ is partial order). Define then $\mathbb{P} \star \dot{\mathbb{Q}}$ as the set

$$\{\, (p, \dot{q}) \mid p \in \mathbb{P} \wedge \;\Vdash_{\mathbb{P}} \dot{q} \in \dot{\mathbb{Q}}\,\}$$

and, moreover,

$$(p_1, \dot{q}_1) \leqslant (p_2, \dot{q}_2) \quad \text{if and only if} \quad p_1 \leqslant p_2 \text{ and } p_1 \Vdash \dot{q}_1 \leqslant_{\dot{\mathbb{Q}}} \dot{q}_2.$$

For more details and facts we again refer to [Jec03, Kun80].

Constructible Universe

For the whole section, we strongly refer to [Jen72, Dev84] for proofs and details. The idea of taking constructible sets is easy to describe: When we look at the von Neumann's view of $\mathbf{V}$, taking all subsets in the successor step $\mathbf{V}_{\alpha+1} = \mathcal{P}(\mathbf{V})$, then we realize that we have no idea what does this really mean. So an attempt could be just to take the subsets we really need, meaning all subsets that can be described or constructed in some sense.

Therefore, let us turn to the theory of constructible sets, looking at Gödel's constructible universe $\mathbf{L} := \bigcup_{\alpha \in \mathrm{On}} \mathbf{L}_\alpha$, where $\mathbf{L}_0 := \varnothing$, for limit ordinals λ set $\mathbf{L}_\lambda := \bigcup_{\alpha<\lambda} \mathbf{L}_\lambda$ and, finally, take as $\mathbf{L}_{\alpha+1}$ the collection of all within $\mathbf{L}_\alpha$ from parameters taken from $\mathbf{L}_\alpha$ definable sets. Then $\mathbf{L}$ is an inner model of set theory, in fact, it is the smallest one.

Lemma 15 ([Dev84]). *The following hold:*

(a) *Assume* $\mathbf{V} = \mathbf{L}$*. Then* GCH *and* AC.

(b) $\mathbf{L} \models (\mathbf{V} = \mathbf{L} + \mathsf{ZFC} + \mathsf{GCH})$.

Let ZF^- be all axioms of ZF without the power set axiom. Then sometimes it is useful to have the following

Lemma 16 ([Dev84, Jen72]). *For a regular and uncountable cardinal* κ *we have that* $\mathbf{L}_\kappa$ *is a model of* $\mathbf{V} = \mathbf{L}$ *and* ZFC^-.

This is best possible situation, having $\mathbf{L}_{\kappa+} \models \mathcal{P}(\kappa) \notin \mathbf{V}$ because there are cofinal many ranks of subsets of κ. So, in case of a regular limit of cardinals, an ($\mathbf{L}$-)inaccessible, we would have found a model of full set theory — proving that we cannot expect to prove the existence of such a cardinal in the presence of Gödel's Incompleteness Theorem.

One of the most important results in this area is:

Lemma 17 ([Dev84], Gödel, Condensation Lemma). *Let* α *be an arbitrary limit ordinal. If* $X \prec_1 \mathbf{L}_\alpha$*, that is preserving* $\exists$*-formulae, then there are unique* π *and* β *such that* $\beta \leqslant \alpha$ *and:*

(a) $\pi : \langle X, \in \rangle \stackrel{\sim}{\longleftrightarrow} \langle \mathbf{L}_\beta, \in \rangle$,

(b) *if* $Y \subseteq X$ *transitive, then* $\pi \restriction Y = \mathrm{id} \restriction Y$,

(c) $\pi(x) \leqslant_{\mathbf{L}} x$ *for all* $x \in X$.

Here, $\leqslant_{\mathbf{L}}$ is the canonical well-ordering of the constructible universe, *cf.* [Dev84] for details. And finally on the way to prove the Generalized Continuum Hypothesis we prove that bounded subsets will be caught by the next cardinal level of the constructible hierarchy.

Lemma 18 ([Dev84]). *Assume* $\mathbf{V} = \mathbf{L}$ *and let* κ *be a cardinal. If* x *is a bounded subset of* κ*, or more generally, if* $x \subseteq \mathbf{L}_\alpha$ *for some* $\alpha < \kappa$*, then* $x \in \mathbf{L}_\kappa$.

Using the fine structure theory[3], Jensen was able to prove the important covering property for the constructible universe. For, let us say that $0^{\#}$ exists, if there is a non-trivial elementary embedding $\pi : \mathbf{L} \longrightarrow \mathbf{L}$.

Lemma 19 ([Dev84], Jensen, Covering Lemma)**.** *Assume,* $0^{\#}$ *does not exist. If* X *is an uncountable set of ordinals, then there is a constructible set,* Y*, of ordinals such that* $X \subseteq Y$ *and* $|X| = |Y|$*.*

⋆ ⋆ ⋆

Now, for many applications it is useful to consider a more general version of the constructible universe: for some set A, we can consider the class $\mathbf{L}[A]$, the universe of all sets constructible relative to A, in fact, let the levels $\mathbf{L}_\alpha[A]$ be similarly defined as for the usual hierarchy and let $\mathbf{L}_{\alpha+1}[A]$ be the set of all subsets of $\mathbf{L}_\alpha[A]$ that are definable over $\mathbf{L}_\alpha[A]$ using parameters from $\mathbf{L}_\alpha[A]$ and A itself.

Then we have similiar properties as for the $\mathbf{L}_\alpha$-hierarchy, *cf.* [Dev84]. In particular we have for $\alpha \geqslant \omega$ that $|\mathbf{L}_\alpha[A]| = |\alpha|$ and for $B = A \cap \mathbf{L}[A]$,

$$\mathbf{L}[A] \;=\; \mathbf{L}[B] \;=\; (\,\mathbf{L}[B]\,)^{\mathbf{L}[A]}.$$

The price of having more freedom in the construction of subsets is loosing parts of GCH: One major tool for this assertion was the Condensation Lemma. But now, having an $X \prec_1 \mathbf{L}_\alpha[A]$ we just find π, β and B such that

$$\pi : X \stackrel{\sim}{\longrightarrow} \mathbf{L}_\beta[B],$$

where $B = \pi''(A \cap X)$. Thus, in general this does not lead to a structure of the $\mathbf{L}[A]$-hierarchy. However, we can then prove the following

Lemma 20 ([Dev84])**.** *Let* $A \subseteq \kappa$*. Then* $\mathbf{L}[A]$ *is an inner model of* ZFC *and we have* $\mathbf{L}[A] \models 2^\lambda = \lambda^+$ *for* $\lambda \geqslant \kappa$*.*

Moreover, using a bit more technical tools we can finally prove

Lemma 21 ([Dev84])**.** *Let* $\mathbf{V} = \mathbf{L}[A]$*, where* $A \subseteq \kappa^+$*. Then* $2^\kappa = \kappa^+$ *holds and so if* $\kappa = \aleph_0$*, we have the full* GCH*.*

[3] *cf.* [Dev84, Jen72].

At the very last, let us prove the following statement we are going to use later.

Lemma 22. *For an uncountable and regular cardinal* κ *and a subset* $D \subseteq \kappa$ *we have that* $\mathbf{L}[D] \models 2^{<\kappa} = \kappa$.

Proof. Let $b \subseteq \gamma < \kappa$ be a given small subset of κ. Then it is sufficient to prove the following

Claim. $b \in \mathbf{L}_\kappa[D]$.

With the claim in mind, we are obviously done, having small subsets of κ already within the model $\mathbf{L}_\kappa[D]$ which has cardinality κ as we already know. And so, it remains to prove the claim.

For, let $b \in \mathbf{L}_\xi[D] \models \mathsf{ZFC}^-$ for a suitable ordinal ξ. Heading a condensation argument, define simultanously two sequences $\langle X_i \mid i < \omega \rangle$ of elementary submodels of $\langle \mathbf{L}_\xi[D], \in, D \rangle$ and $\langle \kappa_i \mid i < \omega \rangle$ of ordinals as follows:

Let X_0 be the smallest elementary submodel of $\langle \mathbf{L}_\xi[D], \in, D \rangle$, containing γ as a subset. Define κ_i as the least upper bound of $X_i \cap \kappa$. Moreover, let X_{i+1} be the smallest elementary submodel of $\langle \mathbf{L}_\xi[D], \in, D \rangle$, containing κ_i as a subset. Finally set $X := \bigcup_{i<\omega} X_i$.

Then for $\bar{\kappa} := \sup_{i<\omega} \kappa_i$ we have $\bar{\kappa} < \kappa$. The model X is an elementary substructure of $\langle \mathbf{L}_\xi[D], \in, D \rangle$. Moreover, we have that $\gamma \leqslant \bar{\kappa} \subseteq X$ and also $|X| = |\bar{\kappa}| < \kappa$.

Now, let $\sigma : \bar{X} \xrightarrow{\sim} X$ be the collapsing map. Then by the condensation property we have that $\bar{X}$ is isomorphic to $\langle \mathbf{L}_{\bar{\xi}}[\bar{D}], \in, \bar{D} \rangle$ for suitable $\bar{\xi}$ and $\bar{D}$ such that $\sigma \upharpoonright \bar{\kappa}$ is the identity map. We also have that $\bar{D} \subseteq \bar{\kappa}$ and even more important, $\bar{D} = D \cap \bar{\kappa}$.

However, then we have that $\bar{X} = \langle \mathbf{L}_{\bar{\xi}}[D \cap \bar{\kappa}], \in, D \cap \bar{\kappa} \rangle$ which is clearly an element and especially a subset of $\mathbf{L}_\kappa[D]$ because $\bar{\xi} < \kappa$.

Therefore, $b = \sigma^{-1}(b) \in \bar{X} \subseteq \mathbf{L}_\kappa[D]$ as desired. ⊠(Lemma 22)

Combinatorical Principles and Trees

We will not consider $\diamondsuit$-principles and related subjects like Souslin trees. Here, we are interested in constructing special Aronszajn trees. For, we consider coherent square-sequences, which Jensen introduced in [Jen72].

Definition 23 ($\square_\kappa$-Sequence)**.** *For an infinite cardinal κ call a sequence $\langle C_\alpha \mid \alpha \in \kappa^+ \cap \mathrm{Lim} \rangle$ a $\square_\kappa$-sequence if*

(a) $\forall \alpha \in \kappa^+ \cap \mathrm{Lim}\,(\, C_\alpha \subseteq \alpha$ *club* $)$,

(b) $\forall \alpha \in \kappa^+ \cap \mathrm{Lim}\,(\, \mathrm{cf}(\alpha) < \kappa \longrightarrow \mathrm{otp}(C_\alpha) < \kappa \,)$,

(c) *if $\beta < \alpha$ is a limit point of C_α, then $C_\alpha = \beta \cap C_\alpha$.*

We say, $\square_\kappa$ holds if there is a $\langle C_\alpha \mid \alpha \in \kappa^+ \cap \mathrm{Lim} \rangle$-sequence.

For our purpose it will be interesting another weaker version of this combinatorical principle — the so-called *Weak Square*.

Definition 24 ($\square^*_\kappa$-Sequence)**.** *For an infinite cardinal κ call a sequence $\langle C_\alpha \mid \alpha \in \kappa^+ \cap \mathrm{Lim} \rangle$ a $\square^*_\kappa$-sequence if*

(a) $\forall \alpha \in \kappa^+ \cap \mathrm{Lim}\,(\, C_\alpha$ *is club in* $\alpha\,)$,

(b) $\forall \beta \in \kappa^+ \cap \mathrm{Lim}\,(\, |\{C_\alpha \cap \beta \,:\, \alpha \leqslant \beta\}| \leqslant \kappa \,)$,

(c) $\forall \alpha \in \kappa^+ \cap \mathrm{Lim}\,(\, \mathrm{otp}(C_\alpha) \leqslant \kappa \,)$.

We say, $\square^*_\kappa$ holds if there is a $\square^*_\kappa$-sequence.

Choose for all limit ordinals $\alpha \in \kappa^+$ club sets $C_\alpha \subseteq \alpha$ such that $\mathrm{otp}(C_\alpha) = \kappa$. If there are only κ many bounded subsets of κ, then this forms trivially a $\square_\kappa$-sequence and finally we have

Lemma 25 ([Dev84, Jen72])**.** *The following hold:*

(a) *If $\kappa^{<\kappa} = \kappa$, then $\square^*_\kappa$.*

(b) *If $\square_\kappa$, then $\square^*_\kappa$.*

(c) *Assume $\mathbf{V} = \mathbf{L}[A]$ for an $A \subseteq \kappa^+$ such that for all $\alpha < \kappa^+$,*

$$|\alpha|^{\mathbf{L}[A\cap\alpha]} \leqslant \kappa,$$

then $\square_\kappa$ holds. In particular, if $\mathbf{V} = \mathbf{L}$, then $\square_\kappa$ holds for all infinite cardinals κ.

Now, call a partial order $\langle T, <_T \rangle$ a *tree* if the set of all predecessors of an element of T is well-ordered by $<_T$. Moreover, for a cardinal κ we call a tree T a κ^+-*Aronszajn tree* if T has height κ^+ such that every branch and every level has cardinality at most κ. Let a *special* κ^+-*Aronszajn tree* be an Aronszajn tree T whose nodes are one-to-one functions from ordinals less than κ^+ into κ, ordered by inclusion. Or equivalently, there is a function $\sigma : T \longrightarrow \kappa$ such that $\sigma(x) \neq \sigma(y)$ for all tree elements $x <_T y$.

Call a tree T a κ-*Souslin tree* if T has height κ and every branch and every antichain has cardinality strictly less than κ. Obviously, a κ-Souslin tree is a κ-Aronszajn tree. However, a special κ^+-Aronszajn tree is not Souslin, because $A_\nu := \sigma^{-1}{}''\{\nu\}$ are antichains by the property of σ defined above. But $\bigcup_{\nu<\kappa} A_\nu = T$ and $|T| = \kappa^+$.

Lemma 26 ([Kan94]). *If κ is regular and $2^{<\kappa} = \kappa$, then there is a κ^+-Aronszajn tree.*

There is an important connection to the theory of trees that we will use in Chapter 5 and give a proof in the appendix.

Lemma 27 ([Dev84, Jen72]). *There is a special κ^+-Aronszajn tree if and only if $\square^*_\kappa$ holds.*

The idea of the proof is easy to understand: Having a κ^+-Aronszajn tree we can consider suitable subsets of branches of the tree. The restrictions of the tree, having no cofinal branches and each tree level has cardinality at most κ, helps to prove to get a $\square_\kappa$-sequence.

On the other hand, imitating the proof of an $\aleph_1$-Aronszajn tree, we now need the $\square_\kappa$-sequence to survive the limit points during the construction without taking to many branches on such levels.

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Remember, as in the first chapter, we call a model $\mathfrak{A} = \langle \mathbb{A}; A, \dots \rangle$ a (κ, λ)-model of a language $\mathcal{L} = \{\dot{A}, \dots\}$, if $|\mathbb{A}| = \kappa$ and $|A| = \lambda$. Then we have the following

Lemma 28 ([ChaKei90, Theorem 7.2.11])**.** *There is a sentence φ in a finite language such that for all infinite cardinals κ, φ has a (κ^+, κ)-model if and only if there exists a special κ^+-Aronszajn tree.*

For, let $\mathcal{L} = \{U, T, <, r, f, g, h\}$ where U is a unary relation, T and $<$ are binary relations, r and f are unary functions and, finally, g and h are binary functions.

Then let φ be the sentence of the language $\mathcal{L}$, saying that

(a) the relation T acts like a tree, meaning that the partial order $\langle\, \mathrm{dom}(T) \cup \mathrm{rng}(T), T\, \rangle$ is a tree in the usual sense; the relation $<$ is a linear order; and the relation U is an initial segment for $<$, that is, $\exists x\, \forall y\, (\, U(y) \longleftrightarrow y < x\,)$,

(b) the function r acts like the tree order function or rank function, that is $x\ T\ y \longrightarrow r(x) < r(y)$, $\forall z\, \exists x\, (\, r(x) = z\,)$ and $z < r(y) \longrightarrow \exists x\, (xTy \wedge r(x) = z)$; and the function f acts like the function for a special Aronszajn tree, that is $\forall x\, U(\, f(x)\,)$ and $xTy \longrightarrow f(x) \neq f(y)$,

(c) use the function g to assert that for each x, the set of all predecessors $\{y | y < x\}$ has cardinality at most $|U|$; and finally, use the function h to assert that for each x, the set of all elements of the same rank $\{y \mid r(y) = x\}$ has cardinality at most $|U|$.

It is an easy exercise to show that this sentence φ will work to prove the lemma above.

To round up the theory we remind the reader of the following

Lemma 29 ([Dev84])**.** *Assume* GCH*. Let κ be an uncountable cardinal for which $\square_\kappa$ holds. Then there is a κ^+-Souslin tree.*

Moreover, in [Jen72, p.286, Remark (3)], Jensen showed the following fact which we will use later

Lemma 30 ([Jen72])**.** *If κ^+ is not Mahlo in* **L***, then $\square_\kappa$ holds.*

In fact, we can prove the following generalization

Lemma 31 ([Jen72]). *Suppose κ^+ is not a Mahlo cardinal in $\mathbf{L}[B]$ for a subset $B \subseteq \aleph_1$, then $\square_\kappa$ holds.*

Primitive Recursive Functions

The well-known primitive recursive functions on the natural numbers can be generalized to primitive recursive functions on ordinals. The easiest way here is to consider the canonical functions like successor function, addition, multiplication, taking powers and taking iterated powers.

Then we call an ordinal α *primitive recursive closed* if it is closed under these ordinal functions restricted to α.

On the other hand we can generalize these functions to sets (not only ordinals) as are given by the next

Definition 32 (Primitive Recursive Functions). *A (class) function $f : \mathbf{V}^n \longrightarrow \mathbf{V}$ is said to be* primitive recursive *(p.r.) if and only if it is generated by the following schemata:*

(a) $f(x_1, \ldots, x_n) = x_i$ *for* $1 \leqslant i \leqslant n$,

(b) $f(x_1, \ldots, x_n) = \{x_i, x_j\}$ *for* $1 \leqslant i, j \leqslant n$,

(c) $f(x_1, \ldots, x_n) = x_i \backslash x_j$ *for* $1 \leqslant i, j \leqslant n$,

(d) $f(x_1, \ldots, x_n) = h(\, g_1(x_1, \ldots, x_n), \ldots, g_k(x_1, \ldots, x_n)\,)$, *where* h, g_1,…,g_k *are all p.r.,*

(e) $f(y, x_1, \ldots, x_n) = \bigcup_{z \in y} g(z, x_1, \ldots, x_n)$, *where g is p.r.,*

(f) $f(x_1, \ldots, x_n) = \omega$,

(g) $f(y, x_1, \ldots, x_n) = g(y, x_1, \ldots, x_n, \langle f(z, x_1, \ldots, x_n) | z \in h(y) \rangle)$, *where g and h are p.r. and, moreover,*

$$z \in h(y) \longrightarrow \operatorname{rank}(z) < \operatorname{rank}(y).$$

In fact, in [JenKar71], it is shown that α is closed under ordinal primitive recursive functions if and only if $\mathbf{L}_\alpha$ is closed under the primitive recursive functions on general sets.

Even a rather complex function like $\langle \mathbf{L}_\nu | \nu \in \mathrm{On} \rangle$ is primitive recursive. This means that a level of the constructible universe with height a p.r.-closed ordinal has in most cases enough set theory within to work with. We are going to use such arguments in Chapter 5.

We will now look on cardinals and try to find many p.r.-closed ordinals.

Lemma 33. *Let $\kappa > \omega$ be a regular cardinal. Then κ is p.r.-closed and there are cofinal many p.r.-closed ordinals below κ.*

Proof. Obviously, as a ZF^--model, $\mathbf{L}_\kappa$ is closed under functions of Definition 32. Moreover, let $\gamma_0 < \kappa$ be chosen. Then we can define γ_{i+1} as the smallest γ such that the union of all ranges of functions given by Definition 32 restricted to $\mathbf{L}_{\gamma_i}$ is a subset of $\mathbf{L}_\gamma$. Then $\sup_{i<\omega} \gamma_i < \kappa$ is p.r.-closed. ☒(Lemma 33)

Then the same argument proves that for a ZF^--model $\mathfrak{A}$ there are cofinal many p.r.-closed ordinals in $\mathrm{On}^{\mathfrak{A}}$ and even the following

Corollary 34. *Let $\mathfrak{A}$ be a model ZF^-. Then there are cofinal many limits of p.r.-closed ordinals in $\mathrm{On}^{\mathfrak{A}}$.*

The next rather technical statement will allow us later to find p.r.-closed ordinals in elementary submodels. In fact, we are arguing to get finally Lemma 37 at the end of this section. For, let us define for $i < \omega$ and ordinals ν:

$$g_0(\nu) := \nu + 1,$$
$$g_{i+1}(\nu) := g_i^{\nu+1}(\nu + 1).$$

Here, the iterated power is defined in the usual way by induction on non-empty ordinals: let $g_i^1(\mu) := g_i(\mu)$, $g_i^{\nu'+1}(\mu) := g_i(\ g_i^{\nu'}(\mu)\ )$ and finally $g_i^\lambda(\mu) := \sup_{\nu'<\lambda} g_i^{\nu'}(\mu)$ for limit ordinals λ. Then these functions are obviously primitive recursive. Moreover, we have the following

Lemma 35 ([JenKar71])**.** *Let F be a p.r. set function. Then there is a Σ_1-formula φ such that whenever $x_1, \ldots, x_n \in \mathbf{L}_\alpha[A]$ there is $i < \omega$ such that*

$$y = F(x_1, \ldots, x_n) \iff \mathbf{L}_{g_i(\alpha)}[A] \models \varphi(y, x_1, \ldots, x_n).$$

This statement ensures that each primitive recursive function can be caught by the g_i-functions as rank in the **L**-hierarchy. But then we now have

Corollary 36. $\mathbf{L}_\alpha[A]$ *is p.r.-closed if and only if* α *is closed under the functions* g_i *for* $i < \omega$.

Now, let φ_i be the formula for the function g_i given by Lemma 35. Then we have that α is p.r.-closed if and only if for all $i < \omega$ we have

$$\mathbf{L}_\alpha[A] \models \forall\nu\exists\xi\, \varphi_i(\xi, \nu).$$

Because even the sequence $\langle \varphi_i \mid i < \omega \rangle$ is p.r.-closed, we finally can code altogether in one formula, saying "On is p.r.-closed":

Lemma 37 ([JenKar71])**.** *There is a formula* φ *such that* α *is p.r.-closed if and only if* $\mathbf{L}_\alpha \models \varphi$.

(Small) Large Cardinals

And finally we state an equivalence we will find useful in our construction later. Remember, we call a cardinal κ *(strongly) inaccessible* if κ is regular and for all $\lambda < \kappa$ we have $2^\lambda < \kappa$.

Call κ *Mahlo* if the set $\{ \gamma < \kappa \mid \gamma \text{ is inaccessible} \}$ is stationary in κ, and finally, call a cardinal κ *weakly compact* if the partition relation $\kappa \longrightarrow (\kappa)^2_2$ holds.

Here, $\kappa \longrightarrow (\kappa)^2_2$ means that every partition of $[\kappa]^2$, the set of all unordered pairs of κ, into two pieces has a homogeneous set of size κ. We refer to [Dra74, Jec03, Kan94] for more details.

⋆ ⋆ ⋆

In future facts we will use the following reflecting properties of the constructible universe:

Lemma 38. *The following hold:*

(a) *If κ is a regular cardinal, then (κ is a regular cardinal)$^{\mathbf{L}}$,*

(b) *If κ is inaccessible, then (κ is inaccessible)$^{\mathbf{L}}$,*

(c) *If κ is Mahlo, then (κ is Mahlo)$^{\mathbf{L}}$,*

(d) *If κ is weakly compact, then (κ is weakly compact)$^{\mathbf{L}}$.*

Finally, we state two well-known connections between large cardinals and the (non-)existence of trees:

Lemma 39 ([Dev84, Jen72])**.** *If κ is a regular uncountable cardinal which is not Mahlo in the constructible universe, then there is a constructible special Aronszajn tree of height κ.*

The proof uses arguments about combinatorical principles given by, *e.g.*, Lemma 27 and Lemma 30. Another well-known fact is the following:

Lemma 40 ([Dev84, Dra74, Jec03, Kan94])**.** *Let κ be an uncountable cardinal. The following are equivalent:*

(a) *κ is weakly compact,*

(b) *κ is inaccessible and there is no κ-Aronzsajn tree.*