APPENDIX

Weak Square and special Aronszajn Trees

Introduction

In this chapter we will give a proof of a useful statement, Jensen has originally proved although he has never published it. In fact, he mentioned the equivalence we are going to prove now in his well-known paper [Jen72]. In Chapter 5 we were using this theorem to prove the main Theorem 59. The statement says the following

Theorem 81 (Jensen). *There is a special κ^+-Aronszajn tree if and only if $\square^*_\kappa$ holds.*

To fix notation, given a tree $\langle T, <_T \rangle$, let T_α be the set of all elements of T with tree level α. Call $T \restriction \alpha$ the (sub-)tree of all elements of T with a tree height strictly less than α. Moreover, denote with $\mathrm{rk}_T(x)$ the tree level of an element x of the tree T.

We already defined in Chapter 2 what we mean with the combinatorical principle weak square, $\square^*_\kappa$, and with a special κ^+-Aronszajn tree. However, because we will extend the definitions, let us repeat the following

Definition 82 (Aronszajn Tree). *We call $\langle T, <_T \rangle$ a κ-Aronszajn tree if the following hold:*

- *T is a tree of height κ,*
- *for $\alpha < \kappa$, all levels T_α have cardinality strictly less than κ,*
- *all branches in T have cardinality strictly less than κ,*
- *T is normal, that is*
    - *T has exactly one root,*
    - *for each $x \in T_\gamma$ and all level $\gamma' > \gamma$ there is $y \in T_{\gamma'}$ such that $x <_T y$,*

– *every element of T has at least two distinct successors in the next tree level,*
– *above each branch in $T \upharpoonright \lambda$ there is exactly one element at limit height λ in T.*

The property of being a normal tree is sometimes useful. However, the existence of a tree yielding just the first three conditions is equivalent to the existence of a tree given by the definition above.

We are interested in special Aronszajn trees given by the following

Definition 83. *A κ^+-Aronszajn tree is called* special*, if there is a function $\sigma : T \longrightarrow \kappa$ such that for all tree elements x and y we have: if $x <_T y$, then $\sigma(x) \neq \sigma(y)$.*

Trivially, a κ-Aronszajn tree has cardinality κ. And a κ^+-Aronszajn tree is special if and only if it is union of κ-many antichains: on the one hand take $A_\alpha := \sigma^{-1}{''}\{\alpha\}$ and on the other hand let $\sigma(x)$ be defined as the smallest $\alpha < \kappa$ such that $x \in A_\alpha$ for $T = \bigcup_{\alpha<\kappa} A_\alpha$.

Moreover, special κ^+-Aronszajn trees have antichains of cardinality κ^+. In particular, such trees are not Souslin trees.

$\aleph_1$-Aronszajn Trees

Constructing an $\aleph_1$-Aronszajn tree is an easy exercise. To make clear where the problems occur when we try to construct Aronszajn trees for larger cardinals, we will remind the reader of the proof idea in case of the first uncountable cardinal.

By induction on countable ordinals we construct the tree levels T_α we are looking for such that its elements will be strictly monotone functions $f : \alpha \longrightarrow \mathbb{Q}$.

Here, with $\mathbb{Q}$ we denote the well-known linear order of the rational numbers. Moreover, we will have a function $\sup : T_\alpha \longrightarrow \mathbb{Q}$ such that $\sup(f) := \sup \operatorname{rng}(f)$ and whenever $x <_T y$, then $\sup(x) <_{\mathbb{Q}} \sup(y)$.

The induction is very easy. To start with, we take $T_0 := \{\varnothing\}$ and $\sup(\varnothing) := 0$. For the successor step set

$$T_{\beta+1} := \{\, f^\frown q \mid f \in T_\beta,\ \sup(f) < q,\ q \in \mathbb{Q} \,\},$$

and $\sup(f^\frown q) := (f^\frown q)(\beta) = q$ where $f^\frown q := f \cup \{\langle q, \beta \rangle\}$ for $f \in T_\beta$.

Now, in case where α is a countable limit ordinal and for $f \in T \upharpoonright \alpha$ and $q \in \mathbb{Q}$ such that $\sup(f) < q$, we are going to define an extended function $g : \alpha \longrightarrow \mathbb{Q}$ such that $f \subsetneq g$ and $\sup(g) = q$ as follows:

Let $f \in T_\beta$ for a $\beta < \alpha$. Choose an unbounded $C \subseteq \alpha$ above β of length ω. Then C is trivially a club subset of α. Moreover, choose $\langle q_i \mid i \leqslant \omega \rangle$ such that $q_i < q_j$ for $i < j \leqslant \omega$, $q_0 := \sup(f)$ and $q_\omega = q$. Both can be done easily.

Now, along the (club) set $C = \langle c_i \mid i < \omega \rangle$ starting from f let us look at a branch of the so far defined tree given by tree elements on levels indexed by elements of C: For each $c_i \in C$ choose an $f_i \in T_{c_i}$ such that $\sup(f_i) = q_i$. By the successor step, this is trivially possible because there are no limit stages within the enumeration of C. Then let $g_{f,q}$ be such that $f_i \subsetneq g_{f,q}$, $\mathrm{dom}(g_{f,q}) = \alpha$ and $\sup(g_{f,q}) = q$. Finally define

$$T_\alpha := \{\, g_{f,q} \mid f \in T \upharpoonright \alpha,\ \sup(f) < q,\ q \in \mathbb{Q} \,\}.$$

It is an easy exercise to show that $T := \bigcup_{\alpha < \aleph_1} T_\alpha$ is an $\aleph_1$-Aronszajn tree. In fact, it is even a special one, considering the map $\sigma : T \longrightarrow \aleph_0$ defined by $\sigma := (f \circ \sup)$ where f is an arbitrary but fixed bijection between $\mathbb{Q}$ and $\aleph_0$.

The combinatorical Principle $\square_\kappa^*$

Combinatorical Principles, as small fragments of the constructible universe extracted in a useful assertion, have applications in many areas of mathematics.

We will in fact look at sequences with relatively weak properties of coherency. To start with, let us consider the next lemma speaking about equivalent statements, each asserting suitable weak square-sequences.

Lemma 84. *For an infinite cardinal and arbitrary closed and unbounded subsets Γ and Γ' of limit ordinals below κ^+, the following statements are equivalent:*

(a) *There is a sequence $\langle C_\alpha \mid \alpha \in \kappa^+ \cap \mathrm{Lim} \rangle$ such that*
  (i) $\forall \alpha \in \kappa^+ \cap \mathrm{Lim}\ (C_\alpha \subseteq \alpha$ *club* $)$,
  (ii) $\forall \beta \in \kappa^+ \cap \mathrm{Lim}\ (|\{C_\alpha \cap \beta \ :\ \alpha \leqslant \beta\}| \leqslant \kappa)$,
  (iii) $\forall \alpha \in \kappa^+ \cap \mathrm{Lim}\ (\mathrm{otp}(C_\alpha) \leqslant \kappa)$.

(b) *There is a sequence $\langle \mathcal{C}_\alpha \mid \alpha \in \kappa^+ \cap \mathrm{Lim} \rangle$ such that*
  (i) $\forall \alpha \in \kappa^+ \cap \mathrm{Lim}\ (\mathcal{C}_\alpha \subseteq \mathcal{P}(\alpha),\ |\mathcal{C}_\alpha| \leqslant \kappa)$,
  (ii) $\forall C \in \mathcal{C}_\alpha\ (C$ *is closed in* $\alpha)$,
  (iii) $\exists C \in \mathcal{C}_\alpha\ (C$ *is unbounded in* $\alpha)$,
  (iv) $\forall \alpha, \beta \in \mathrm{Lim}\ \forall C \in \mathcal{C}_\beta\ (\alpha < \beta < \kappa^+ \longrightarrow \alpha \cap C \in \mathcal{C}_\alpha)$,
  (v) $\forall C \in \mathcal{C}_\alpha\ (\mathrm{otp}(C) \leqslant \kappa)$.

(c) *There is a sequence $\langle C_\alpha \mid \alpha \in \Gamma \rangle$ such that*
  (i) $\forall \alpha \in \Gamma\ (C_\alpha \subseteq \alpha$ *club* $)$,
  (ii) $\forall \beta \in \Gamma\ (|\{C_\alpha \cap \beta \ :\ \alpha \leqslant \beta\}| \leqslant \kappa)$,
  (iii) $\forall \alpha \in \Gamma\ (\mathrm{otp}(C_\alpha) \leqslant \kappa)$.

(d) *There is a sequence $\langle \mathcal{C}_\alpha \mid \alpha \in \Gamma' \rangle$ such that*
  (i) $\forall \alpha \in \Gamma'\ (\mathcal{C}_\alpha \subseteq \mathcal{P}(\alpha),\ |\mathcal{C}_\alpha| \leqslant \kappa)$,
  (ii) $\forall C \in \mathcal{C}_\alpha\ (C$ *is closed in* $\alpha)$,
  (iii) $\exists C \in \mathcal{C}_\alpha\ (C$ *is unbounded in* $\alpha)$,
  (iv) $\forall \alpha, \beta \in \Gamma'\ \forall C \in \mathcal{C}_\beta\ (\alpha < \beta < \kappa^+ \longrightarrow \alpha \cap C \in \mathcal{C}_\alpha)$,
  (v) $\forall C \in \mathcal{C}_\alpha\ (\mathrm{otp}(C) \leqslant \kappa)$.

Proof. Considering $\mathcal{C}_\alpha := \{C_\beta \cap \alpha \mid \beta \geqslant \alpha\}$, we can conclude the implication from (a) to (b). Similarly, we show that (c) implies (d).

The other implication, to get (a) from (b) and (c) from (d), respectively, we simply choose as the desired C_α a closed and unbounded subset of the given $\mathcal{C}_\alpha$.

Take the restriction to Γ of the sequence given by (a) and we get a sequence asserted by (c).

Now, let $\langle C^*_\alpha | \alpha \in \Gamma \rangle$ be a sequence in the sense of (c). Let $\langle \gamma_\beta | \beta < \kappa^+ \rangle$ be a monotone enumeration of Γ. Define $\Gamma^* := (\kappa^+ \cap \mathrm{Lim}) \backslash \Gamma$. If Γ^* is empty, we would be done. Otherwise choose for $\alpha \in \Gamma^*$ a set C_α as follows:

Let $\beta < \kappa^+$ be such that $\gamma_\beta < \alpha < \gamma_{\beta+1}$. Then choose C_α as a closed and unbounded subset of α such that $\min C_\alpha > \gamma_\beta$ and $\mathrm{otp}(C_\alpha) \leqslant \kappa$. This can be done easily. For $\alpha \in \Gamma$ just take $C_\alpha := C^*_\alpha$ and we have constructed a sequence in the sense of (a).

Finally, the lemma is proved. $\boxtimes$(Lemma 84)

In [Jen72], Jensen called the equivalent assertions of the last lemma *weak square*, $\square^*_\kappa$. The described sequence is called (in any of the four cases) a $\square^*_\kappa$-sequence.

As we already have seen in Chapter 2, the following hold:

(a) If $\kappa^{<\kappa} = \kappa$, then $\square^*_\kappa$.
(b) If $\square_\kappa$, then $\square^*_\kappa$.

A special κ^+-Aronszajn Tree implies $\square^*_\kappa$

Let $\langle T, <_T \rangle$ be a special κ^+-Aronszajn tree and $\sigma : T \longrightarrow \kappa$ such that whenever $x <_T y$, then $\sigma(x) \neq \sigma(y)$. We can assume, without loss of generality, that T is just κ^+.

We are going to construct a weak square sequence $\langle C_\alpha \mid \alpha \in \Gamma \rangle$ in a sense of (c). For, consider the function $f : \kappa^+ \longrightarrow \kappa^+$, defined by $f(\alpha) := \max\{ \bigcup T_\alpha, \mathrm{rk}_T(\alpha) \}$. Then, using Lemma 8, we know that the set $\Delta := \{\alpha < \kappa^+ \mid f''\alpha \subseteq \alpha\}$ is a club subset of κ^+.

Now, take an element α of Δ. Then we have $\alpha = T \restriction \alpha$, because, for the first inclusion, let $\beta < \alpha$ and γ be the tree rank of β. Then $\beta \in T_\gamma$ and so $\gamma \leqslant f(\beta) \in f''\alpha \subseteq \alpha$. Hence, $\beta \in T_\gamma \subseteq T \restriction \alpha$.

For the second inclusion let β be in T_γ for its tree level $\gamma < \alpha$. Then we conclude $\beta \leqslant f(\gamma) \in f''\alpha \subseteq \alpha$.

Therefore, setting $\Gamma^* := \{\alpha < \kappa^+ : T \upharpoonright \alpha = \alpha\}$, we know that the club set Δ is a subset of Γ^* and because it is obviously closed, Γ^* is a closed and unbounded subset of κ^+ as well.

Now, let Γ be the set of all limit points of Γ^*. Then by Lemma 8, this set Γ is still a closed and unbounded subset of κ^+.

Furthermore, choose for each $\alpha \in \Gamma$ an element x_α with tree level α and define branches $b_\alpha := \{z \in T \mid z <_T x_\alpha\}$ below each chosen x_α. Then we can conclude that

$$b_\alpha \subseteq \alpha \text{ is an unbounded subset.} \tag{6}$$

This is easy to check: Obviously, $b_\alpha \subseteq T \upharpoonright \alpha = \alpha$. So, let $\beta < \alpha$. We will show that $b_\alpha \nsubseteq \beta$. Because of $\alpha \in \Gamma$ we have an α^* such that $\beta < \alpha^* < \alpha$ and $\alpha^* \in \Gamma^*$. Therefore, there is a $y \in T_{\alpha^*} \cap b_\alpha$ and so $b_\alpha \nsubseteq T \upharpoonright \alpha^* = \alpha^*$.

We now can fix an arbitrary $\alpha < \kappa^+$ and define C_α as a cofinal subset in b_α of order type at most κ by choosing increasing elements $t_\beta^{(\alpha)}$ of the branch b_α minimal with respect to the given function σ by induction as follows:

Let $t_0^{(\alpha)} \in b_\alpha$ be such that $\sigma(t_0^{(\alpha)}) = \min \sigma'' b_\alpha$ and let $t_\beta^{(\alpha)} \in b_\alpha$ be such that $\sigma(t_\beta^{(\alpha)}) = \min\{\, \sigma(z) \mid (\forall \gamma < \beta)(t_\gamma^{(\alpha)} <_T z <_T x_\alpha) \,\}$. This is well-defined because of the one-to-one property of σ on the branch b_α.

We go on with the construction till it breaks down. Let γ_α be minimal such that t_{γ_α} does not exist. Then γ_α is a limit ordinal because otherwise, the set $\{z \mid t_\beta^{(\alpha)} <_T z <_T x_\alpha\}$ would be non-empty for $\gamma_\alpha = \beta + 1$ and so the definition would not break.

Moreover, the set $C'_\alpha := \{t_\beta^{(\alpha)} \mid \beta < \gamma_\alpha\}$ is cofinal in b_α. Otherwise we would have that the set $\{z \mid (\forall \beta < \gamma)(t_\beta <_T z <_T x_\alpha)\}$ is non-empty and so t_{γ_α} would be defined again. Therefore, we also have by (6) that

$$C'_\alpha \text{ is cofinal in } \alpha. \tag{7}$$

Trivially, by definition, the sequence $\langle\, \sigma(t^{(\alpha)}_\beta)\,|\, \beta < \gamma_\alpha \,\rangle$ is strictly monotone in κ. And so, we conclude that the order type of C'_α is at most κ and hence $\gamma_\alpha \leqslant \kappa$.

Finally, let C_α be the closure of C'_α by taking all limit points below α such that C_α is a closed and unbounded subset of α of order type at most κ.

The important property of a (weak) square sequence is its coherency we still have to prove with the next

Lemma 85. *For all $\alpha < \kappa^+$ we have $|\,\{\, C_\alpha \cap \beta \,:\, \alpha \geqslant \beta \,\}\,| \leqslant \kappa$.*

Proof. Define for every $x \in T_\xi$ where ξ is a limit ordinal, the branch $b_x := \{z \mid z <_T x\}$ and the sequence $t^{(x)} := \{t^{(x)}_\beta \mid \beta < \gamma_x\}$ as we did for the x_α's above as follows:

Let $t^{(x)}_0$ be such that $\sigma(t^{(x)}_0) = \min \sigma'' b_x$ and again $t^{(x)}_\beta \in b_x$ satifying

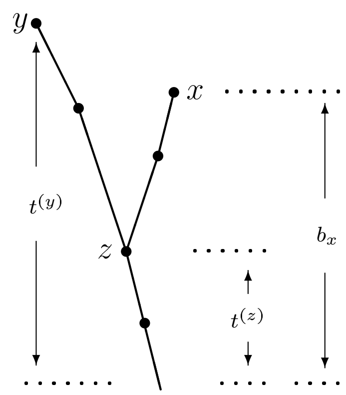
$$\sigma(t^{(x)}_\beta) = \min\{\, \sigma(z) \,|\, (\forall \gamma < \beta)(t^{(x)}_\gamma <_T z <_T x) \,\}. \tag{8}$$

Then for each limit ordinal $\alpha < \kappa^+$ we have $t^{(\alpha)} = t^{(x_\alpha)}$, and moreover, as above, that $t^{(x)}$ is cofinal in b_x. Further, $t^{(x)}$ is strictly monotone with respect to the tree relation $<_T$ and the order type of $t^{(x)}$ is $\gamma_x \leqslant \kappa$.

Consider now the sequence $t^{(y)} \cap b_x$ for a fixed tree element y at limit height and let $z := \sup_T t^{(y)} \cap b_x$. Then we obviously have

$$t^{(y)} \cap b_x = t^{(z)} \tag{9}$$

by the definition given in (8) using the map σ that is one-to-one on a fixed branch.



Therefore, for each limit ordinal $\beta < \kappa^+$ we then have that the interesting set $\{t^{(\alpha)} \cap \beta \mid \alpha \geqslant \beta\}$ is a subset of all small branches of the set $\{t^{(z)} \mid z \in T \restriction (\beta+1)\}$. However, the cardinality of the initial segment $T \restriction (\beta+1)$ of the tree is at most κ. And hence, the cardinality of the set $\{C_\alpha \cap \beta \,:\, \alpha \geqslant \beta\}$ is at most κ for all $\beta < \kappa^+$ and so the lemma is proved. $\boxtimes$(Lemma 85)

Therefore, with the sequence $\langle C_\alpha \mid \alpha \in \Gamma \rangle$ we finally have found the desired $\square^*_\kappa$-sequence.

Construction of the partial Order

We now try to imitate the desired behavior of the structure of the rational numbers with a new partial order of cardinality κ. Therefore let us prove the following

Lemma 86. *There is a partial order $\langle \mathbb{P}, <_\mathbb{P} \rangle$ and a subset $\mathcal{S}$ of the set of all sequences with elements of $\mathbb{P}$ such that:*

(a) *$\mathbb{P}$ is partial order, ${}^1\mathbb{P} \subseteq \mathcal{S}$, $|\mathcal{S}| \leqslant |\mathbb{P}| = \kappa$,*
(b) *$\mathbf{0}, \mathbf{1} \in \mathbb{P}$ such that for every $p \in \mathbb{P}$ we have $\mathbf{0} \leqslant_\mathbb{P} p \leqslant_\mathbb{P} \mathbf{1}$,*
(c) *for all $p, q \in \mathbb{P}$ there is an element $q' \in \mathbb{P}$ such that whenever $p <_\mathbb{P} q$, then $p <_\mathbb{P} q' <_\mathbb{P} q$,*
(d) *for every $s \in \mathcal{S}$ there is a limit ordinal $\alpha \in \kappa+1$ or $\alpha = 0$ such that $\mathrm{dom}(s) = \alpha + 1$ and s is strictly monotone with respect to the relation $<_\mathbb{P}$,*
(e) *for every $s \in \mathcal{S}$ and for all limit ordinals $\alpha \in \mathrm{dom}(s)$ we have $s \restriction (\alpha+1) \in \mathcal{S}$,*
(f) *for every limit ordinal $\alpha \in \kappa + 1$ and all $p, q \in \mathbb{P}$, $p < q$, there is an $s \in \mathcal{S}$ such that $\mathrm{dom}(s) = \alpha + 1$, $s(0) = p$ and $s(\alpha) = q$.*

Proof. Consider an elementary submodel H of $\mathbf{H}_{\kappa^+}$ of cardinality κ such that $\kappa \subseteq H$. Then H is transitive because for each element x of H there is a surjection from κ onto x within the elementary submodel H of $\mathbf{H}_{\kappa^+}$ and therefore, with the domain also the whole range x is a subset of H.

Let $\mathbb{P} := \{\, X \subseteq \kappa \mid X = \kappa \ \vee\ (X \in H \ \wedge\ |\kappa \backslash X| = \kappa\,)\,\}$ and define the following relation on $\mathbb{P}$ as follows

$$X \sqsubset Y \quad :\Longleftrightarrow \quad X \subseteq Y \subseteq \kappa \ \wedge\ |Y \backslash X| = \kappa.$$

We still have to work to define the set of sequences $\mathcal{S}$. Therefore, let Θ be a function such that for all α such that $0 < \alpha \leqslant \kappa$ and $X, Y \subseteq \kappa$ where $X \sqsubset Y$ we have $\Theta(\alpha, X, Y) := \langle\, Z_\gamma \mid \gamma \leqslant \alpha \,\rangle \in H$ such that

- $Z_0 = X$, $Z_\alpha = Y$;
- for every $\gamma < \gamma' \leqslant \alpha$ we have $Z_\gamma \sqsubset Z_{\gamma'}$;
- for every limit ordinal $\lambda \in \alpha + 1$ we have $Z_\lambda = \bigcup_{\gamma<\lambda} Z_\gamma$.

For, let α and X, Y be given and fix a bijection $f : \alpha \times \kappa \longrightarrow Y \backslash X$ in H. Let Z_γ be $X \cup f''(\gamma \times \kappa)$ for $\gamma \leqslant \alpha$. Then Z_0 and Z_α have the desired properties and because of $f''(\lambda \times \kappa) = \bigcup_{\gamma<\kappa} f''(\gamma \times \kappa)$ we have the property for Z_λ where λ is a limit ordinal. Moreover, the missing property is given by the following

$$|Z_{\gamma'} \backslash Z_\gamma| = |f''(\gamma' \times \kappa) \backslash f''(\gamma \times \kappa)| = |(\gamma' \backslash \gamma) \times \kappa| = \kappa.$$

Finally, let

$$\begin{aligned} \mathcal{S} \ := \ \{\, \Theta(\gamma, p, q) \restriction (\gamma' + 1) \ \mid & \text{ limit ordinals } \gamma' \leqslant \gamma \leqslant \kappa; \\ & \text{or } \gamma' = 0;\ p,\ q \in \mathbb{P};\ p < q\,\}. \end{aligned}$$

Moreover, for given p and q, elements of $\mathbb{P}$, such that $p <_{\mathbb{P}} q$ we have for $q' := \Theta(2, p, q)(1)$ that $p <_{\mathbb{P}} q' <_{\mathbb{P}} q$.

Then the partial order $\langle\, \mathbb{P}, \sqsubset \,\rangle$ and the set $\mathcal{S}$ have the desired properties and so the lemma is proved. $\boxtimes$(Lemma 86)

$\square^*_\kappa$ implies a special κ^+-Aronszajn Tree

Having the partial order $\langle\, \mathbb{P}, <_{\mathbb{P}} \,\rangle$ we constructed in the last section, we will now build up a κ^+-Aronszajn tree using a suitable coherent sequence. For, let $\langle\, \mathcal{C}_\alpha \mid \alpha \in \kappa^+ \cap \mathrm{Lim} \,\rangle$ be a $\square^*_\kappa$-sequence in the sense of (b) of the defining Lemma 84.

By induction on $\alpha < \kappa^+$, we are going to define the following objects

(a) the tree levels T_α consisting of suitable strictly monotone functions $f : \alpha \longrightarrow \mathbb{P}$,

(b) a function $\sup : T_\alpha \longrightarrow \mathbb{P}$ such that the following hold:
- $\sup(\varnothing) = 0$,
- if α is a limit ordinal, then $\sup(f)$ is a supremum of the range of f in the partial order $\mathbb{P}$,
- if $\alpha = \beta + 1$, then $\sup(f)$ is just $f(\beta)$,

(c) a partial function $\mathsf{Ex}_\alpha : T_\alpha \times \mathbb{P} \longrightarrow T_\alpha$ such that $\mathsf{Ex}_\alpha(f, q)$ is defined if and only if $\sup(f) < q$, and $\mathsf{Ex}_\alpha(f, q) = g$ where $f \subseteq g$, $\sup(g) = q$ and $\mathrm{dom}(g) = \alpha$,

We simply can start the induction, letting $T_0 := \{\varnothing\}$. Even in the successor step $\alpha = \beta + 1$ we will extend the so far defined tree $T \restriction \alpha$ maximally possible in the following sense:

$$\begin{aligned} T_\alpha &:= \{ f^\frown q \mid f \in T_\beta,\ \sup(f) < q,\ q \in \mathbb{P} \}; \\ \sup(f^\frown q) &:= (f^\frown q)(\beta) = q; \\ \mathsf{Ex}_{\beta+1}(f, q) &:= \begin{cases} f^\frown q & : \text{if } f \in T_\beta,\ \sup(f) < q; \\ \mathsf{Ex}_\beta(f, q'_{f,q})^\frown q & : \text{if } f \in T \restriction \beta,\ \sup(f) < q. \end{cases} \end{aligned}$$

Here, we let $q'_{f,q}$ arbitrary be chosen such that $\sup(f) < q' < q$. The existence of such a $q'_{f,q}$ follows from the properties of $\mathbb{P}$.

And finally, for limit ordinals $\lambda < \kappa^+$, let a club set $C \in \mathcal{C}_\lambda$, a tree element $f \in T_\beta$ where $\beta < \lambda$, and a $q \in \mathbb{P}$ where $\sup(f) < q < \mathbf{1}_\mathbb{P}$ be given. Furthermore, define $X := \{\beta\} \cup C \backslash \beta \cup \{\lambda\}$. Then let δ be the order type of X and $t : \delta \longrightarrow X$ be the monotone enumeration of X. Last but not least, choose an $s \in \mathcal{S}$ be such that $s(0) = \sup(f)$, $s(\delta - 1) = q$ and $\mathrm{dom}(s) = \delta$.

Define then a branch through the segment $T \restriction \lambda$ of the tree as follows: set $f_0 := f$, $f_{\gamma+1} := \mathsf{Ex}_{t(\gamma+1)}(f_\gamma, s(\gamma + 1))$, and $f_{\lambda'} := \bigcup_{\gamma<\lambda'} f_\gamma$. Here we use the coherency of the $\square^*_\kappa$-sequence that $C \cap t(\lambda')$ is an element of $\mathcal{C}_{t(\lambda')}$, and moreover, that $s \restriction (\lambda' + 1)$ is an element of $\mathcal{S}$. Hence, we indeed used $f_{\lambda'}$ for the definition of the tree level $T_{\lambda'}$.

Further, let $g(f,q,C,s)$ be $f_{\delta-1}$. Note, $t(\delta-1)=\lambda$. Then define

$$\begin{aligned} T_\lambda := \{\, g(f,q,C,s) \mid\ & f \in T \restriction \lambda,\ \sup(f) < q \in \mathbb{P},\ C \in \mathcal{C}_\lambda, \\ & \delta \text{ defined as above}, \\ & s \in \mathcal{S},\ s(0) = \sup(f),\ s(\delta-1) = q \,\} \end{aligned}$$

And finally, $\mathsf{Ex}_\lambda(f,q) := g(f,q,C,s)$ for arbitrary choosen $C \in \mathcal{C}_\lambda$ and $s \in \mathcal{S}$ such that $s(0) = \sup(f)$ and $s(\delta-1) = q$ where δ is defined as above.

The exact choice of the club set C and the sequence s is not necessary to determine. What we need here is that we extend f to the level λ and the supremum q. Which way we exactly choose through the already defined tree $T \restriction \lambda$ is not important because of the coherency of the $\square^*_\kappa$-sequence and $\mathcal{S}$.

Last but not least, let $\sup_\lambda(\, g(f,q,C,s)\,) := q$ and the tree relation is then given by

$$f <_T g \quad \text{if and only if} \quad g \restriction \mathrm{dom}(f) = f \text{ and } f \neq g.$$

Then the set T defined as the union of the defined levels T_α for $\alpha < \kappa^+$ is obviously a κ^+-Aronszajn tree. Moreover, the function $\sup : T \longrightarrow \mathbb{P}$ yields the property that for $f <_T g$ we have $\sup(f) <_{\mathbb{P}} \sup(g)$. That means, together with a fixed bijection between $\mathbb{P}$ and κ we easily can find a map witnessing that T is even special.

Finally, the theorem is completely proved. $\boxtimes$(Theorem 81)