EXTENDIBILITY OF EMBEDDINGS
— HANDOUT —

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ABSTRACT. Lifting given embeddings is an important tool in core model theory. However, the constructed structure during the lifting process (called pseudo-ultrapower) is not necessarily well-founded. We will consider criteria which ensure well-foundedness. One of this will be a generalization of the well-known *frequent extension of embeddings lemma*.

CONTENTS

1. INTRODUCTION

The simplest core model is the one less than $0^{\#}$. This is GÖDEL's constructible universe $\mathbf{L}$. Already in this basic situation in core model theory one can find problems with the extendibility. Let us make the characterization of this (small) large cardinal axiom $0^{\#}$ our starting point. We will exemplify the notion of the *lifting of an embedding* by giving a special first-order characterization of the axiom $0^{\#}$. One standard way of formulating $0^{\#}$ (among others) is via the

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existence of a (class-sized) embedding from $\mathbf{L}$ into itself. By using the concept of lifting, we can see that we can equivalently characterize it by a first-order formula talking about a set-sized embedding:

EXAMPLE: Suppose we have a non-trivial elementary embedding $\sigma : \mathbf{L}_\alpha \longrightarrow \mathbf{L}_{\alpha'}$, where α and α' are limit ordinals and $\mathrm{crit}(\sigma) < |\alpha|$. We are looking[1] for an elementary embedding $\tilde{\sigma} : \mathbf{L} \longrightarrow \mathbf{L}$ to get $0^\#$.

One possibility is taking the *usual* ultrapower construction, where we take the $\mathbf{L}$-ultrafilter $\{X \subseteq \mathrm{crit}(\sigma) \mid X \in \mathbf{L} \wedge \mathrm{crit}(\sigma) \in \sigma(X)\}$ to get a (well-founded) ultrapower. (In this case the ultrapower consists of the equivalence classes modulo the above ultrafilter of constructible functions $f : \mathrm{crit}(\sigma) \longrightarrow \mathbf{L}$.)

This is a first and rather simple example for lifting an embedding (in this case from $\mathbf{L}_\alpha$ to $\mathbf{L}$). But there are cases in which the application of the lifting technique requires more information about this constructed lifting. A property which seems reasonable is, e.g., $\tilde{\sigma} \supseteq \sigma$. That means we *extend* our given embedding. Thus we will get properties like $\sigma "Y \subseteq \tilde{\sigma} "\tilde{Y}$, where $Y \subseteq \tilde{Y}$ and $Y \subset \mathrm{dom}(\sigma)$ but $\tilde{Y} \subset \mathrm{dom}(\tilde{\sigma}) \setminus \mathrm{dom}(\sigma)$. (This is used in a new proof of the Covering Lemma by Jensen, which is carried out in detail by the present author in his diplom thesis.) We will now construct the pseudo-ultrapower and afterwards see another applications of this property.

Let us switch to the $\mathbf{J}_\alpha$ hierarchy instead of the usual $\mathbf{L}_\alpha$ levels, because the following lemmas will use this approximation[2] of $\mathbf{L}$. For the rest of this note we shall work on the following problems:

> We are given a cofinal function $\sigma : \mathbf{J}_\alpha \underset{\Sigma_0}{\longrightarrow} \mathbf{J}_{\alpha'}$ and an ordinal $\beta > \alpha$ such that α is a cardinal in $\mathbf{J}_\beta$. **Under what circumstances can we extend σ to a cofinal embedding defined on $\mathbf{J}_\beta$?**

[1] This is possible by [Devlin84, p.192].

[2] See [Jensen72] for more details.

If we are in the described situation, we can form the structure $\mathbb{D} := \langle D, \tilde{\in}, \tilde{\doteq} \rangle$ by

$$(1)\quad \begin{cases} \bullet\ D := \{[\xi, f] \mid f \in \mathbf{J}_\beta \ \wedge\ \mathrm{dom}(f) \in \mathbf{J}_\alpha \ \wedge\ \xi \in \sigma(\mathrm{dom}(f))\}. \\ \bullet\ [\xi_0, f_0]\, R\, [\xi_1, f_1] \longleftrightarrow \langle \xi_0, \xi_1 \rangle \in \sigma(\{\langle \eta_0, \eta_1 \rangle \,|\, f_0(\eta_0)\, R\, f_1(\eta_1)\}) \\ \quad \text{for } R \in \{\tilde{\in}, \tilde{\doteq}\}. \end{cases}$$

If $\mathbb{D}$ is well-founded, then there will be a β' (and $\tilde{\sigma}$) such that $\tilde{\sigma} : \mathbf{J}_\beta \underset{\Sigma_0}{\longrightarrow} \mathbf{J}_{\beta'}$ is cofinal. With this construction we get similar properties between the usual and the pseudo-ultrapower. We are able to deduce the theorem of Łoś and the description of the pseudo-ultrapower, in fact we get $\mathbf{J}_{\beta'} = \{\sigma(f)(\xi) \,|\, f \in \mathbf{J}_\beta \wedge \xi \in \sigma(\mathrm{dom}(f))\}$.

$$\begin{tikzcd} \mathbf{J}_\beta \arrow[r, "\tilde{\sigma}", "\text{cofinal},\Sigma_0"'] & \mathbf{J}_{\beta'} \\ \mathbf{J}_\alpha \arrow[u, "\subseteq"] \arrow[r, "\sigma", "\text{cofinal},\Sigma_0"'] & \mathbf{J}_{\alpha'} \arrow[u, "\subseteq"'] \end{tikzcd}$$

Now we are able to show another advantage of the property being an extension. For $R = \tilde{\in}$ we clearly get $[\xi_0, f_0]\tilde{\in}[\xi_1, f_1] \iff \tilde{\sigma}(f_0)(\xi_0) \in \tilde{\sigma}(f_1)(\xi_1)$. But as in the usual construction there are still problems to ensure the well-foundedness of this structure.

A similar construction will produce a so-called *finestructural upward extension* such that we also get similar properties as in the case for the Σ_0-upward extension above. We will take more resonable functions f in the definition (1) of $\mathcal{D}$ to get the finestructural extension. This construction needs a couple of finestructural methods and therefore we will not give any details here. Of course, only in conjunction with this construction we get the *whole usefulness* of the concept of the upward extension construction.

2. The first Criterion

Case: **'uncountable cofinality'**.

This is the easiest case to ensure the extendibility of a given embedding.

Definition 2.1. Say that α *is nice in* β if

(a) $\alpha \leq \beta$.

(b) If $\alpha < \beta$, then $\mathrm{cf}(\alpha) > \omega$.

(c) If $\alpha < \beta$, then $(\xi < \alpha)(\exists \tau \leq \alpha)(\forall \xi < \tau \wedge \mathrm{cf}(\tau) > \omega \wedge \mathbf{J}_\beta \models \tau \text{ is regular})$.

Clearly, if α is nice in β, α is a cardinal in $\mathbf{J}_\beta$. Thus in this situation we can make the pseudo-ultraproduct construction in this situation.

Lemma 2.2. Let α be nice in β. Then the canonical upward extension exists, *i.e.,* the pseudo-ultrapower $\mathfrak{A}$ is well-founded.

A simply strengthening of the property of being nice will ensure the well-foundedness of the finestructural upward extension and it is also possible to get the statement above for the hierarchy of the relative constructiblity, *i.e.*, for $\mathbf{J}_\alpha[E]$ instead of only $\mathbf{J}_\alpha$.

3. The second Criterion

Case: **'Take many and we will get one we want'**.

In the proof of the lemma 2.2 we use the fact that α had uncountable cofinality in an essential way. Moreover, there are counterexamples of ill-founded pseudo-ultrapowers in the countable case. What if we are in a situation where we cannot avoid countable cofinalities? In this situation we cannot hope to get a direct extension of our given embedding, but luckily, in typical applications (like getting $0^\#$) we do not need the foundation of the upward extension of a *special* given embedding, but we need *one* such an extension. The idea is now to consider many embeddings and their upward extensions and to hope that at least one of them is well-founded. So our original problem transforms into a different version:

> Suppose we have different embeddings $\sigma_\alpha : \mathbf{J}_\alpha \longrightarrow \mathbf{J}_{\alpha'}$. **Under what circumstances can we find an α^* and $\beta > \alpha^*$ such that σ_{α^*} is extendible to $\mathbf{J}_\beta$?**

The vague answer to this questions is: If we take **many** embeddings, then the process works. Now we have to ask what we mean by 'many'. To formalize 'many' in this context, we will first consider reasonable embeddings indexed by ordinals such that we can use well-known terms on it. We can show under resonable circumstances that if we start with stationary many such embeddings (in terms of subsets of ordinals), then stationary many of their upward extensions will be well-founded.

More exactly, let γ be regular, $\tau > \gamma$ be uncountable, and $f : \gamma \overset{\text{onto}}{\longrightarrow} \mathbf{J}_\tau$ be a surjection. Set $X_\alpha := f\,"\alpha$ and $\mathcal{C} := \{\alpha < \gamma \mid X_\alpha \prec \mathbf{J}_\alpha \ \wedge\ X_\alpha \cap \gamma = \alpha \ \wedge\ \sup(X_\alpha \cap \mathrm{On}) = \tau \ \wedge\ \gamma \in X_\alpha\}$.

Clearly, $\mathcal{C}$ is a club subset of γ. Let $\sigma_\alpha : \mathbf{J}_{\tau_\alpha} \overset{\sim}{\longleftrightarrow} X_\alpha$ be the (inverse) Mostowski collapse of $\alpha \in \mathcal{C} \cup \{\gamma\}$. A short comment on the technical properties in the definition of $\mathcal{C}$: The first one is necessary for getting σ_α. The second ensures $\mathrm{crit}(\sigma_\alpha) = \alpha$ and therefore the fourth gives $\sigma_\alpha(\alpha) = \gamma$. If we went into details, we would also consider the embeddings $\sigma_{\alpha\beta} := \sigma_\beta^{-1}\sigma_\alpha : \mathbf{J}_{\tau_\alpha} \longrightarrow \mathbf{J}_{\tau_\beta}$ for $\alpha \leq \beta$. Then this function is elementary and because of the third property also cofinal.

Set $\mathcal{D} := \{\alpha \in \mathcal{C} \mid \mathrm{cf}(\alpha) > \omega\}$. Then $\mathcal{D}$ is a stationary subset of γ. Then the following lemma says that if we start with *many* embeddings, then *many* of them will be extendible.

Lemma 3.1 (Frequent Extension Lemma)**.** Let $\mathcal{S} \subseteq \mathcal{D}$ be stationary in γ. For $\alpha \in \mathcal{S}$ let $\mu_\alpha > \tau_\alpha$ be arbitrary chosen, such that τ_α is a cardinal in $\mathbf{J}_{\mu_\alpha}$. In addition let $\tilde{\sigma}_\alpha : \mathbf{J}_{\mu_\alpha} \underset{\Sigma_0}{\longrightarrow} \mathfrak{A}_\alpha$ the canonical upward extension of σ_α. **Then** there is a club set $\mathcal{C} \subseteq \gamma$, such that the pseudo-ultrapower $\mathfrak{A}_\alpha$ is well-founded for every $\alpha \in \mathcal{S} \cap \mathcal{C}$.

4. The Variation of the second Criterion

Why do we need such a surjection? — Only for coding the term of 'stationary many'. But we should be able to speak about this problem without such a surjection. Therefore, let us consider reasonable terms for subsets of $\mathbf{J}_\tau$ in this section. First of all we imitiate the situation of the third section without transforming the objects via a surjection to the language of the ordinals.

Let $\gamma < \tau$ uncountable such that $\mathrm{cf}(\tau) < \gamma$, where γ is regular. Set $\mathcal{C} := \{u \prec \mathbf{J}_\tau \mid u \cap \gamma \text{ is transitive} \ \wedge\ \gamma \in u \ \wedge\ \sup(u \cap \mathrm{On}) = \tau \ \wedge\ |u| < \gamma\}$.

We call a subset $\mathcal{C}$ of $[\mathbf{J}_\tau]^{<\gamma}$ *club (closed and unbounded)*, if it is closed under chains of length $<\gamma$ (*i.e.*, for $\langle u_i \mid i < \delta\rangle$, where $u_i \in \mathcal{C}$ for all $i < \delta$ and $\delta < \gamma$, is $\bigcup\{u_i \mid i < \delta\} \in \mathcal{C}$) and for all $u \in [\mathbf{J}_\tau]^{<\gamma}$ there is a superset $v \supseteq u$ in $\mathcal{C}$.

We call a subset $\mathcal{C}$ of $[\mathbf{J}_\tau]^{<\gamma}$ *club**, if it is closed under chains with length δ, where $\delta < \gamma$ and $\mathrm{cf}(\delta) > \omega$, and also as above for all $u \in [\mathbf{J}_\tau]^{<\gamma}$ there is a supset $v \supseteq u$ in $\mathcal{C}$.

Both terms of being *closed and unbounded* imply a term of stationarity with useful properties, *e.g.*, the theorem of Fodor or the pigeon hole principle.

With the following lemma we will get a *reformulation* of lemma 3.1 in the new terms without any surjection.

Lemma 4.1. Let $\mathcal{S} \subseteq \mathcal{C}$ be stationary* in $[\mathbf{J}_\tau]^{<\gamma}$. Choose for every $u \in \mathcal{S}$ a $\mu_u \geq \tau_u$, such that τ_u is a cardinal in $\mathbf{J}_{\mu_u}$. Let $\tilde{\sigma}_u : \mathbf{J}_{\mu_u} \longrightarrow \mathfrak{A}_u$ be the canonical upward extension of σ_u. **Then** $\mathcal{S}' := \{u \in \mathcal{S} \mid \mathfrak{A}_u \text{ is not well-founded}\}$ is not stationary* in $[\mathbf{J}_\tau]^{<\gamma}$. (With other words: There is an uncountable closed set $\mathcal{T}$, such that the elements of the stationary* $\mathcal{S} \cap \mathcal{T}$ are only indices of well-founded pseudo-ultrapowers.)

We conclude for example the following corollary taking $\mathcal{S} := \mathcal{C}$:

Corollary 4.2. Choose for every $u \in \mathcal{C}$ the ordinals τ_u und μ_u as above and also the functions $\tilde{\sigma}_u$. Then there is a uncountable closed and unbounded subset of $\mathcal{C}$, such that it's elements are only indices of well-founded pseudo-ultrapowers.

These statements are enough for typical applications like the proof of the Covering Lemma. They reflect lemma 3.1 in the new context, because for subsets $\mathcal{T}$ of $\mathcal{D} := \{\alpha \in \mathcal{C} \mid \mathrm{cf}(\alpha) > \omega\}$ being unbounded and uncountably closed and being the intersection with a club subset are the same. Thus considering the stationary and the stationary* sets we will have the same subsets of $\mathcal{D}$ in the case of the ordinals.

5. Counterexamples

After we gave conditions under which embeddings are in fact extendible to larger initial segments of the $\mathbf{L}$-hierarchy, we now conclude this note by saying that these conditions are in a sense optimal: We can show by a forcing argument (using Shelah's RCS forcing construction, *cf.* [Shelah98]) that the statements in these lemmas are optimal proven, *i.e.,* we can neither give up the restriction of subsets of $\mathcal{D}$ in lemma 3.1 nor the term 'stationary*' by 'stationary' in lemma 4.1.

Roughly speaking we can say that we take a model of ZFC in which $0^{\#}$ exists. Therefore we can use the SILVER indiscernibles to construct a reasonable forcing over $\mathbf{L}$. A generic extension will content a stationary subset of $\mathcal{C}$ (from section 3) without well-founded pseudo-ultrapowers. Moreover, assuming a surjection f as above in the second section we can then show that there are stationary many ordinals α such that $X_\alpha = f\,"\alpha$ implies[3] an ill-founded pseudo-ultrapower. By the constructed forcing we know that this stationary set is disjoint of $\mathcal{D}$ from section 2.

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[3]That means if we take this substructure $X_\alpha \prec \mathbf{J}_\tau$ and the associated embedding σ_α as in the third section, then there are liftings $\tilde{\sigma}_\alpha$ with an ill founded pseudo-ultrapower.