Motivation

Let $L/K$ be a finite, totally ramified extension of complete discretely valued fields of characteristic $(0, p)$ with perfect residue field $k$. A smooth projective variety $X$ over $L$ comes with the following linear algebraic data.

- The crystalline cohomology groups of the special fiber $X_0$ become $K$-vector spaces after inverting $p$.
- These carry a natural Frobenius action, compatible with a fixed lift $\phi_0$ of the Frobenius of $k$ to $K$.
- By comparison with the de Rham cohomology of $X$, we inherit a (Hodge) filtration over $L$.

**Definition 1:** Let $\text{isoc}_L$ denote the category of $F$-isocrystals over $K$, that is, the category of pairs $(V, \rho)$, where $V \in \text{Vec}_K$ is a finite dimensional vector space over $K$ and $\rho : \text{Gal}(L/K) \to \text{GL}(V)$ a $\mathbb{Q}$-semilinear automorphism of $V$. The category of filtered $F$-isocrystals over $L/K$ is defined to be the fiber product

$$\text{Fil}_L^{\bullet} \times_{\text{Vec}_K} \text{isoc}_L \times_{\text{isoc}_K} \text{Fil}_K^{\bullet},$$

where $\text{Fil}_K^{\bullet}$ is the category of finite dimensional $K$-vector spaces together with a Z-filtration over $L$.

In fact, the p-adic analogue of a period domain over $L$, parametrizing Hodge structures, is a moduli space for semistable filtered $F$-isocrystals over $K$, cf. [Definition 1]. They form an abelian subcategory of $\mathcal{A}$, which Calmes and Fontaine describe in terms of certain representations of the absolute Galois group, $\text{rep}_{\text{Gal}}(G_L) \xrightarrow{\sim} \text{Fil}_L^{\bullet} \times_{\text{Vec}_K} \text{isoc}_L \times_{\text{isoc}_K} \text{Fil}_K^{\bullet}$.

The cohomology of p-adic period domains is studied in [1]. A similar strategy is pursued in various settings.

- Originally, by Harder and Narasimhan, in the context of moduli spaces of vector bundles on curves.
- Recently, $\text{Fil}_L^{\bullet}$ counts $\mathbb{F}_p$-points of quiver moduli spaces in order to infer their Betti numbers (over $\mathbb{C}$).
- Joyce [2] defines this point count to motivic measures of more general moduli spaces, over any field $K$.

Our goal is to generalize this approach to accommodate for the equivariant setting of $\text{Definition 1}$. Consider three quasi-abelian $K$-linear categories $\mathcal{E}$, $\mathcal{B}$, and $\mathcal{D}$, and assume that $\mathcal{B}$ and $\mathcal{D}$ are semisimple. Let $\mathcal{E} \xrightarrow{\phi} \mathcal{B} \xrightarrow{\psi} \mathcal{D}$ be a $K$-linear exact isofiltration. Then for a field extension $L/K$, we replace $\mathcal{B}$ by the fibre product $(\mathcal{E}_L \times_{\mathcal{B}_L} \mathcal{B}_L) \times_{\mathcal{D}_L} \mathcal{D}_L$, where $\mathcal{E}_L = \mathcal{E} \otimes_L K$.

**Example 1:** (a) For a quiver $Q$, the fibre functor $\text{rep}_Q(Q) \xrightarrow{\phi} \mathcal{B} \xrightarrow{\psi} \mathcal{D}_L$ is an exact isofiltration.

(b) Similarly, this applies to the functorial context on categories of representations in $\mathcal{B}$ of a group $G$. By arguing pointwise, this extends to (pre-)sheaves over $K$. In fact, it follows that $\text{Fil}_L^{\bullet} \times_{\text{Vec}_K} \text{isoc}_L \times_{\text{isoc}_K} \text{Fil}_K^{\bullet}$, and indeed, the fibre functor of a quasi-Tannakian category over $K$ is an exact $K$-linear isofiltration.

(c) In the same vein, this works for the fibre functor $\phi_L : \mathcal{E}_L \to \mathcal{B}_L$, where $\mathcal{A}$ is a totally ordered abelian group.

Equivaraint motivic Hall algebras

Over an arbitrary field $k$, the idea is to replace the number of points $q = \# \text{of} \phi(L/F_p)$ by the affine line itself.

**Definition 3:** Let $L$ be a stack in groupoids on the big fpd-site $\mathbf{A}_{\text{fpd}}$ of affine schemes over $K$. Then the (relative) Grothendieck ring of stacks $K_0(\mathbf{S}h(L))$ is the free $\mathbb{Z}$-module on geometric equivalence classes of algebraic stacks over $L$, of finite type and with affine stabilizers over $K$. This motivic version of $\text{KKG}(L)$ defines a ring $\mathbf{H}(L)$ and is essentially the Grothendieck ring of motives.

The **motivic Hall algebra** $\mathbf{H}(L)$ is then the convolution algebra along the correspondence

$$\mathbf{M} \times \mathbf{M} \xrightarrow{\mathbf{L} \circ \mathbf{R}} \mathbf{M} \times \mathbf{M} \xrightarrow{\mathbf{R} \circ \mathbf{L}} \mathbf{M},$$

where $\mathbf{L} \circ \mathbf{R}$ is the moduli stack of short exact sequences in $\mathcal{E}$, which is mapped in $\mathbf{M}$ to their outer terms and their middle term, respectively. That is, multiplication in $\mathbf{H}(L)$ is defined as the composition

$$K_0(\mathbf{S}h(M)) \otimes \mathbf{M}(\mathbf{S}h(M)) \xrightarrow{\text{nat}} K_0(\mathbf{S}h(M) \times \mathbf{M}(\mathbf{S}h(M)) \xrightarrow{\text{iso}} K_0(\mathbf{S}h(M) \times \mathbf{M}(\mathbf{S}h(M))).$$

Let $\mathbf{M}$ be the moduli stack of $B$. By replacing $K_0(\text{S}(M))$ by its $\mathbb{Q}$-equivariant variant $K_0^{\mathbb{Q}}(\text{S}(M))$, we get

$$\mathbf{M} \times \mathbf{M} \xrightarrow{\mathbf{L} \circ \mathbf{R}} \mathbf{M} \otimes \mathbf{M},$$

where $\mathbf{L} \circ \mathbf{R}$ is the moduli stack of short exact sequences in $\mathcal{E}$, which is mapped in $\mathbf{M}$ to their outer terms and their middle term, respectively. That is, multiplication in $\mathbf{H}(L)$ is defined as the composition

$$K_0(\mathbf{S}h(M)) \otimes K_0(\mathbf{S}h(M)) \xrightarrow{\text{nat}} K_0(\mathbf{S}h(M) \times \mathbf{M}(\mathbf{S}h(M)) \xrightarrow{\text{iso}} K_0(\mathbf{S}h(M) \times \mathbf{M}(\mathbf{S}h(M))).$$

**Theorem 2:** There is a natural map of simplicial stacks $\mathbf{H}(L) \rightarrow K_0(\mathbf{S}h(M))$, whose pushforward

$$\int \mathbf{H}(L) \rightarrow K_0(\mathbf{S}h(M))/[K_0(\mathbf{S}h(M))]$$

is an algebra morphism if $L$ is hereditary. For $D = 0$, this recovers the motivic version of $\text{Definition 2}$. Further directions

If $L$ carries a duality structure, there is a module over the Hall algebra of $L$ on isometry classes of selloid objects, due to M. Young. We have an analogue of Theorem 1 in the equivariant motivic Hall module. In general, we replace $K_0(\mathbf{S}h(M))$ with a ring of analytic stacks (on affine schemes) over a non-commutative field. This again yields a Hall algebra, since Waldhausen’s S-construction defines a 2-Segal stack [4].

**Definition 4:** Let $k \geq 0$. The $k$-coeffs $\mathbf{S}^k(L)$ of the higher Waldhausen $S$-construction are defined as the full subcategory of the category of diagrams $\mathbf{E}$: Fun(\{\emptyset, [k]\} \rightarrow \mathbf{E}, (\emptyset \rightarrow [k]) \rightarrow \{0 \rightarrow \emptyset\})$ with

- (degeneracies) for every function $f: \{k + 1 \rightarrow [0\rightarrow \emptyset]\}$, we have $E_{f(0)} \rightarrow \cdots \rightarrow E_{f(k)} = 0$, and
- (faces) for every $g: \{k \rightarrow [k] \rightarrow \emptyset\}$, the sequence $E_{f(\emptyset)} \rightarrow E_{f(1)} \rightarrow \cdots \rightarrow E_{f(k)}$ is exact.

Benson and Madsen introduced $\mathbf{S}^k(L)$ in the context of real algebraic $K$-theory. We illustrate an element of its 4-skeleton, with image under the upper 4-Segal map $\mathbf{S}^3(L) \rightarrow \mathbf{S}^2(L)$, $\phi_0^3 : \mathbf{S}^3(L) \rightarrow \mathbf{S}^2(L)$ in red.

If $L$ is abelian, $n$ is an equivalence, but this case is an outlier; the general result is as follows.

**Theorem 3:** The simplicial category $\mathbf{S}^k(L)$ is a 2-Segal object.

References