Equivariant motivic Hall algebras

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Motivation

Let $L|K$ be a finite, totally ramified extension of complete discretely valued fields of characteristic $(0,p)$ with perfect residue field k. A smooth projective variety X over L comes with the following linear algebraic data.

- The crystalline cohomology groups of the special fibre X_k become K-vector spaces after inverting p.
- These carry a natural Frobenius action, compatibly with a fixed lift σ of the Frobenius of k to K.
- By comparison with the de Rham cohomology of X, they inherit a (Hodge) filtration over L.

Definition 1: Let isoc_K denote the category of F-isocrystals over K, that is, the category of pairs (V,φ), where $V \in \mathrm{vect}_K$ is a finite dimensional vector space over K, and $\varphi\colon V \otimes_{K,\sigma} K \xrightarrow{\sim} V$ a σ-semilinear automorphism of V. The category of ***filtered F-isocrystals*** over $L|K$ is defined to be the fibre product

$$\mathrm{Fil}^{\mathbb{Z}}_{L|K} \times_{\mathrm{vect}_K} \mathrm{isoc}_K = \mathrm{Fil}^{\mathbb{Z}}_L \times_{\mathrm{vect}_L} \mathrm{isoc}_K, \tag{1}$$

where $\mathrm{Fil}^{\mathbb{Z}}_{L|K}$ is the category of finite dimensional K-vector spaces together with a $\mathbb{Z}$-filtration over L.

In fact, the p-adic analogue of a ***period domain*** over $\mathbb{C}$, parametrizing Hodge structures, is a moduli space for ***semistable*** filtered F-isocrystals over K, cf. Definition 2. They form an abelian subcategory of (1), which Colmez and Fontaine describe in terms of certain representations of the absolute Galois group,

$$\mathrm{rep}^{\mathrm{cris}}_K(G_L) \xrightarrow{\sim} (\mathrm{Fil}^{\mathbb{Z}}_L \times_{\mathrm{vect}_L} \mathrm{isoc}_K)^{\mathrm{ss}}_0.$$

The cohomology of p-adic period domains is studied in [2]. A similar strategy is pursued in various settings.

- Originally, by Harder and Narasimhan, in the context of moduli spaces of vector bundles on curves.
- Reineke [5] counts $\mathbb{F}_q$-points of quiver moduli spaces in order to infer their Betti numbers (over $\mathbb{C}$).
- Joyce [4] refines this point count to motivic measures of more general moduli spaces, over any field K.

Our goal is to generalize this approach to accommodate for the equivariant setting of [2]. Consider three quasi-abelian K-linear categories $\mathcal{E}$, $\mathcal{B}$, and $\mathcal{D}$, and assume that $\mathcal{B}$ and $\mathcal{D}$ are semisimple. Let

$$\mathcal{E} \xrightarrow{\omega} \mathcal{B} \xleftarrow{\nu} \mathcal{D}$$

be two K-linear ***exact isofibrations***. Then for a field extension $L|K$, we replace (1) by the fibre product

$$(\mathcal{E}_L \times_{\mathcal{B}_L} \mathcal{B}) \times_{\mathcal{B}} \mathcal{D} = \mathcal{E}_L \times_{\mathcal{B}_L} \mathcal{D}, \quad \text{where } \mathcal{E}_L := \mathcal{E} \otimes_K L. \tag{2}$$

Example 1: (a) For a quiver Q, the fibre functor $\mathrm{rep}_{\mathcal{B}}(Q) \to \mathcal{B}$, $M \mapsto \bigoplus_{i\in Q_0} M_i$, is an exact isofibration.

(b) Similarly, this applies to the forgetful functor on the category of representations in $\mathcal{B}$ of a group G. By arguing pointwise, this extends to (pro-)group schemes over K. In fact, it follows that

$$\mathrm{Fil}^{\mathbb{Z}}_K \xrightarrow{\omega} \mathrm{vect}_K \xleftarrow{\nu} \mathrm{isoc}_K,$$

and indeed, the fibre functor of a ***quasi-Tannakian category*** over K is an exact K-linear isofibration.

(c) In the same vein, this works for the functor $\omega\colon \mathrm{Fil}^{\Lambda}_{\mathcal{B}} \to \mathcal{B}$, where Λ is a totally ordered abelian group.

Slope filtrations and Hall algebras

It is explained in [1] how to express the aforementioned notions of semistability in our categorical setting.

Definition 2: Let $\mathcal{E}^\times$ denote the maximal subgroupoid. Suppose $\mathcal{E}$ is equipped with two maps as follows.

- The ***rank function*** $\mathrm{rk}\colon \pi_0(\mathcal{E}^\times) \to \mathbb{N}$, additive on exact sequences, such that $\mathrm{rk}(E) = 0 \Leftrightarrow E = 0$.
- A ***degree function*** $\deg\colon K_0(\mathcal{E}) \to \Lambda$, with $\deg(E) \le \deg(E')$ for all $E \to E'$ with (co-)kernel $= 0$.

Then E is ***semistable*** if $\mu(N) \le \mu(E)$ for all $0 \ne N \le E$, where $\mu(E) = \frac{\deg E}{\mathrm{rk}\, E} \in \Lambda_{\mathbb{Q}}$ is the ***slope*** of E.

The full subcategory $\mathcal{E}^{\mathrm{ss}}_\lambda$ of semistable objects of $\mathcal{E}$ of slope $\in \{\lambda, \infty\}$ is inherently abelian.

Example 2: (a) For the category $\mathcal{E}$ of vector bundles on a (connected) smooth projective curve X over the field K, we have the usual notions of rank and degree. Note that $\mathcal{O}_X \twoheadrightarrow \mathcal{O}_X(1)$ has (co-)kernel $= 0$.

(b) Let Q be a (connected) quiver, and $\mathcal{E} = \mathrm{rep}_K(Q)$. Then $\mathrm{rk}(M) := \sum_{i\in Q_0} \dim_K(M_i)$ for $M \in \mathcal{E}$, and any choice of $\theta \in \Lambda^{Q_0}$ defines a degree function via $\deg_\theta\colon K_0(\mathcal{E}) \xrightarrow{\dim} \mathbb{Z}^{\oplus Q_0} \xrightarrow{\theta} \Lambda$.

(c) Let $\mathcal{E} = \mathrm{Fil}^{\Lambda}_{L|K}$ with rank function $\mathrm{rk}(V, F^\bullet) = \dim_K(V)$. The degree is weighted by the jumps of $F^\bullet$,

$$\deg_\bullet(V, F^\bullet) = \sum_{\lambda\in\Lambda} \lambda \cdot \dim_L(F^\lambda V / F^{\lambda-1} V) \in \Lambda.$$

On isoc_K, define $\deg_\sigma(V,\varphi) := -\,\mathrm{val}_p(\det\varphi)$. The fibre product in (1) is endowed with $\deg := \deg_\bullet + \deg_\sigma$.

Several further examples appear in the survey article [1], where the following is proved in this generality.

Proposition 1: *There is a unique filtration $F^\bullet\colon \Lambda^{\mathrm{op}}_{\mathbb{Q}} \times \mathcal{E} \to \mathcal{E}$, such that $0 \subsetneq F^{\lambda_1}E \subsetneq \ldots \subsetneq F^{\lambda_n}E = E$, for $E \in \mathcal{E}$, is uniquely determined by $F^{\lambda_i}E/F^{\lambda_{i-1}}E$ being semistable of decreasing slopes $\lambda_1 > \ldots > \lambda_n$.*

If $K = \mathbb{F}_q$ and $\mathcal{E}$ is finitary, we can express Proposition 1 as an equation in the ***Hall algebra*** of $\mathcal{E}$. This is the convolution algebra $\mathrm{H}(\mathcal{E}) = \mathbb{Q}[\pi_0(\mathcal{E}^\times)]$ of finitely supported $\mathbb{Q}$-valued functions on $\pi_0(\mathcal{E}^\times)$, that is,

$$(\varphi * \psi)(E) = \sum_{N\le E} \varphi(N)\psi(E/N), \quad \text{for } \varphi,\psi \in \mathbb{Q}[\pi_0(\mathcal{E}^\times)].$$

More precisely, if we complete $\mathrm{H}(\mathcal{E})$ with respect to its $K_0(\mathcal{E})$-grading, where $\mathbb{1}_E$ lies in degree $[E]$, then

$$\mathbb{1}_{\pi_0(\mathcal{E}^\times)} = \sum_{\lambda_1 > \ldots > \lambda_n} \mathbb{1}_{\pi_0(\mathcal{E}^{\mathrm{ss}\times}_{\lambda_1})} * \ldots * \mathbb{1}_{\pi_0(\mathcal{E}^{\mathrm{ss}\times}_{\lambda_n})} \in \widehat{\mathrm{H}}(\mathcal{E}). \tag{3}$$

If $\mathcal{E}$ is ***hereditary***, the Euler form $\chi(M,N) = \dim \mathrm{Hom}(M,N) - \dim \mathrm{Ext}^1(M,N)$ defines a twisted group ring $\mathbb{Q}^{(\chi)}[K_0(\mathcal{E})]$, with $[M][N] = q^{-\chi(N,M)}[N \oplus M]$. By [5], Lemma 6.1, there is an ***integration morphism***

$$\int_{\mathcal{E}}\colon \widehat{\mathrm{H}}(\mathcal{E}) \longrightarrow \mathbb{Q}^{(\chi)}[[K_0(\mathcal{E})]], \quad \mathbb{1}_E \longmapsto \tfrac{1}{\#\,\mathrm{Aut}(E)}[E]. \tag{4}$$

Integrating (3) yields a formula (5), counting points of the ***moduli stack of objects*** $\mathcal{M}_\alpha$ of class $\alpha \in K_0(\mathcal{E})$.

Equivariant motivic Hall algebras

Over an arbitrary field K, the idea is to replace the number of points $q = \#\mathbb{A}^1(\mathbb{F}_q)$ by the affine line itself. To this end, we understand it as an element $\mathbb{L} = [\mathbb{A}^1_K] \in K_0(\mathrm{Var}/K)$ of the Grothendieck ring of varieties.

Definition 3: Let $\mathcal{Z}$ be a stack in groupoids on the big fppf-site $\mathrm{Aff}^{\mathrm{fppf}}_K$ of affine schemes over K. Then the (relative) ***Grothendieck ring of stacks*** $K_0(\mathrm{Sta}/\mathcal{Z})$ is the free $\mathbb{Z}$-module on geometric equivalence classes of algebraic stacks over $\mathcal{Z}$, of finite type and with affine stabilizers over K, modulo the relations

$$[\mathfrak{X} \amalg \mathfrak{X}'] = [\mathfrak{X}] + [\mathfrak{X}'],$$
$$[\mathfrak{X}_1] = [\mathfrak{X}_2] \text{ for all locally trivial Zariski fibrations } \mathfrak{X}_i \to \mathfrak{X}_0 \text{ with equivalent fibres.}$$

This ensures that $K_0(\mathrm{Var}/K)[\mathbb{L}^{-1}, (\mathbb{L}^n - 1)^{-1} \mid n \in \mathbb{N}] \xrightarrow{\sim} K_0(\mathrm{Sta}/K)$. As above, there is a HN-recursion

$$[\mathcal{M}_\alpha] = \sum_{\substack{\alpha_1 > \ldots > \alpha_n \\ \alpha_1 + \ldots + \alpha_n = \alpha}} \mathbb{L}^{-\sum_{i<j}\chi(\alpha_j,\alpha_i)} [\mathcal{M}^{\mathrm{ss}}_{\alpha_1}] \cdots [\mathcal{M}^{\mathrm{ss}}_{\alpha_n}] \in K_0(\mathrm{Sta}/K). \tag{5}$$

This makes sense, assuming $\dim_K \mathrm{Hom}_{\mathcal{E}}(-,-) < \infty$, by the following finiteness result, which builds on [6].

Theorem 1: *Let $-\,\widehat{\otimes}_K -$ denote the K-linear Cauchy completion of the tensor product. The functor*

$$\mathcal{M}\colon \mathrm{Aff}^{\mathrm{fppf}}_K \longrightarrow \mathrm{Grpd}, \; \mathrm{Spec}(A) \longmapsto (\mathcal{E}\,\widehat{\otimes}_K A)^\times,$$

defines an algebraic stack, locally of finite type over K, called the moduli stack of objects of $\mathcal{E}$.

The ***motivic Hall algebra*** $\mathcal{H}(\mathcal{E}) := K_0(\mathrm{Sta}/\mathcal{M})$ of $\mathcal{E}$ is the convolution algebra along the correspondence

$$\mathcal{M} \times_K \mathcal{M} \xleftarrow{(\partial_2,\partial_0)} \mathcal{S}_2(\mathcal{E}) \xrightarrow{\partial_1} \mathcal{M}, \tag{6}$$

where $\mathcal{S}_2(\mathcal{E})$ is the moduli stack of short exact sequences in $\mathcal{E}$, which are mapped in (6) to their outer terms and their middle term, respectively. That is, multiplication in $\mathcal{H}(\mathcal{E})$ is defined as the composition

$$K_0(\mathrm{Sta}/\mathcal{M}) \otimes K_0(\mathrm{Sta}/\mathcal{M}) \xrightarrow{-\times_K-} K_0(\mathrm{Sta}/\mathcal{M}\times_K\mathcal{M}) \xrightarrow{(\partial_2,\partial_0)^*} K_0(\mathrm{Sta}/\mathcal{S}_2(\mathcal{E})) \xrightarrow{(\partial_1)_*} K_0(\mathrm{Sta}/\mathcal{M}).$$

Let $\mathcal{N}$ be the moduli stack of $\mathcal{B}$. By replacing $K_0(\mathrm{Sta}/-)$ by its $\mathcal{D}$-equivariant variant $K_0^{\mathcal{D}}(\mathrm{Sta}/-)$, we get

$$\mathcal{H}^{\omega,\nu}(\mathcal{E}) := K_0^{\mathcal{D}}(\mathrm{Sta}/\mathcal{S}_1^{\omega,\nu}(\mathcal{E})) \cong \bigoplus_{D\in\pi_0(\mathcal{D}^\times)} K_0^{\mathrm{Aut}(D)}(\mathrm{Sta}/\mathcal{S}_1^{\omega,\nu}(\mathcal{E})), \;\text{ where } \mathcal{S}_1^{\omega,\nu}(\mathcal{E}) = \mathcal{M} \times_{\mathcal{N}} \underline{\mathcal{D}}^\times,$$

the ***equivariant motivic Hall algebra*** of $\mathcal{E}$, with parabolic induction product between the summands.

Theorem 2: *There is a natural map of simplicial stacks $\mathcal{S}^{\omega,\nu}_\bullet(\mathcal{E}) \to \underline{K_0(S_\bullet(\mathcal{E}\times_{\mathcal{B}}\mathcal{D}))}$, whose pushforward*

$$\int_{\mathcal{E}}^{\omega,\nu}\colon \widehat{\mathcal{H}}^{\omega,\nu}(\mathcal{E}) \longrightarrow K_0^{\mathcal{D}}(\mathrm{Sta}/K)^{(\chi)}[[K_0(\mathcal{E}\times_{\mathcal{B}}\mathcal{D})]]$$

is an algebra morphism if $\mathcal{E}$ is hereditary. For $\mathcal{B} = \mathcal{D} = 0$, this recovers the motivic version of (4) in [4].

Further directions

If $\mathcal{E}$ carries a duality structure, there is a module over the Hall algebra of $\mathcal{E}$ on isometry classes of selfdual objects, due to M. Young. We have an analogue of Theorem 2 for the ***equivariant motivic Hall module***. In general, we replace $K_0^{\mathcal{D}}(\mathrm{Sta}/-)$ with a ring of analytic stacks (on affinoid spaces) over a *non-Archimedean* field. This again yields a Hall algebra, since Waldhausen's S-construction defines a ***2-Segal stack*** (cf. [3]).

Definition 4: Let $k \ge 0$. The n-cells $S^{\langle k\rangle}_n(\mathcal{E})$ of the ***higher Waldhausen S-construction*** are defined as the full subcategory of the category of diagrams $E\colon \mathrm{Fun}([k],[n]) \longrightarrow \mathcal{E}, (\beta\colon [k]\to[n]) \longmapsto E_\beta$, with

- *(degeneracies)* for every functor $\alpha\colon [k-1] \to [n]$, we have $E_{s^*_{k-1}\alpha} = \ldots = E_{s^*_0\alpha} = 0$, and
- *(faces)* for every $\gamma\colon [k+1] \to [n]$, the sequence $E_{d^*_{k+1}\gamma} \hookrightarrow E_{d^*_k\gamma} \longrightarrow \ldots \longrightarrow E_{d^*_1\gamma} \twoheadrightarrow E_{d^*_0\gamma}$ is exact.

Hesselholt and Madsen introduced $S^{\langle 2\rangle}_\bullet(\mathcal{E})$ in the context of real algebraic K-theory. We illustrate an element of its 4-skeleton, with image under the upper 3-Segal map $S^{\langle 2\rangle}_4(\mathcal{E}) \longrightarrow S^{\langle 2\rangle}_3(\mathcal{E}) \times_{S^{\langle 2\rangle}_2(\mathcal{E})} S^{\langle 2\rangle}_3(\mathcal{E})$ in red.

$$\begin{tikzcd}
 & & & E_{123} \arrow[dr, hook] & & & \\
 & & E_{023} \arrow[ur] \arrow[dr, hook, red] & & E_{124} \arrow[ddr] & & \\
 & & & E_{024} \arrow[ur, two heads, red] \arrow[dr, two heads, red] & & & \\
 & E_{013} \arrow[uur] \arrow[r, hook] & E_{014} \arrow[ur, red] \arrow[rr] & & E_{034} \arrow[r, two heads] \arrow[drr, two heads, red] & E_{134} \arrow[dr, two heads] & \\
E_{012} \arrow[ur, hook] \arrow[urr, hook, red] & & & & & & E_{234}
\end{tikzcd}$$

Theorem 3: *The simplicial category $S^{\langle k\rangle}_\bullet(\mathcal{E})$ is $2k$-Segal if $\mathcal{E}$ is abelian (and conversely, assuming $k \ge 2$).*

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