MODULI STACKS FOR LINEAR CATEGORIES

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Abstract. We propose a simple definition of the moduli stack of objects in a locally proper Cauchy complete linear category and study its basic properties. Moreover, we slightly modify a construction by Calabrese-Groechenig in case of a finitely cocomplete category. **Preliminary version; do not distribute!**

Contents

1. The moduli stack of objects in a locally proper category

Let R be a commutative ring, and let $\mathcal{E}$ be an R-linear category. In examples, R will be a field k, and similarly, $\mathcal{E}$ will usually be a quasi-abelian category (cf. [9]). By a stack over R, we mean a stack in groupoids on the big fppf-site $\mathrm{Aff}_R = \mathrm{Aff}_R^{\mathrm{fppf}}$ of affine schemes over R.

Definition 1.1. Let $\mathcal{D}$ be an R-linear category. The tensor product $\mathcal{E} \otimes_R \mathcal{D}$ over R has the same objects as $\mathcal{E} \times \mathcal{D}$, but with morphisms defined by the tensor product,

$$\mathrm{Hom}_{\mathcal{E}\otimes_R\mathcal{D}}((M,N),(M',N')) = \mathrm{Hom}_{\mathcal{E}}(M,M') \otimes_R \mathrm{Hom}_{\mathcal{D}}(N,N'),$$

and composition given by the tensor product of the composition in $\mathcal{E}$ and $\mathcal{D}$.

Example 1.2. Let $\mathcal{E}$ be the category of abelian varieties over a field k. Then

$$\mathcal{E}_{\mathbb{Q}} = \mathcal{E} \otimes_{\mathbb{Z}} \mathbb{Q}$$

is the isogeny category of $\mathcal{E}$.

We denote by Grpd the (2-)category of groupoids, and by $\mathcal{D}^\times$ the maximal subgroupoid of a category $\mathcal{D}$. Note that $(-)^\times$ is right adjoint to Grpd $\hookrightarrow$ Cat, hence commutes with limits.

Definition 1.3. The naive moduli stack of objects in $\mathcal{E}$ is the stack over R defined by

$$\mathcal{N}_{\mathcal{E}} \colon \mathrm{Aff}_R \longrightarrow \mathrm{Grpd},\ S = \mathrm{Spec}(A) \longmapsto (\mathcal{E} \otimes_R A)^\times,$$

where on the right-hand side, A is the category with one object with endomorphism ring A.

Remark 1.4. If $\mathrm{Spec}(A') \twoheadrightarrow \mathrm{Spec}(A)$ is an fppf-cover in Aff_R, then the equivalence

$$\mathcal{E}_A := \mathcal{E} \otimes_R A \xrightarrow{\sim} \varprojlim{}^h \left(\mathcal{E}_{A'} \rightrightarrows \mathcal{E}_{A'\otimes_A A'} \ \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}\ \mathcal{E}_{A'\otimes_A A'\otimes_A A'}\right) \tag{1.1}$$

follows from faithfully flat descent. Thus $\mathcal{N}_{\mathcal{E}}$ is indeed a stack.

Definition 1.5. The moduli stack of objects in $\mathcal{E}$ is the stack over R defined by

$$\mathcal{M}_{\mathcal{E}} \colon \mathrm{Aff}_R \longrightarrow \mathrm{Grpd},\ S = \mathrm{Spec}(A) \longmapsto (\mathcal{E} \mathbin{\widehat{\otimes}_R} A)^\times,$$

where $- \mathbin{\widehat{\otimes}_R} -$ denotes the (R-linear) Cauchy completion of the tensor product.

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Remark 1.6. By its universal property, the Cauchy completion is a left adjoint. But it also commutes with finite 2-limits (cf. [2], §1.2), and thus (1.1) implies that $\mathcal{M}_{\mathcal{E}}$ is also a stack.

Note that $\mathcal{M}_{\mathcal{E}}$ is functorial in $\mathcal{E}$ with respect to arbitrary functors. In particular, applied to the unit $\mathcal{E} \to \widehat{\mathcal{E}}$, this induces an equivalence $\mathcal{M}_{\mathcal{E}} \xrightarrow{\sim} \mathcal{M}_{\widehat{\mathcal{E}}}$. Thus, while Definition 1.5 makes sense in general, we may assume $\mathcal{E}$ to be Cauchy complete henceforth.

Definition 1.7. Let S be a qcqs scheme, and qcoh_S the category of finitely presented quasi-coherent $\mathcal{O}_S$-modules. We denote by vect_S its full subcategory of locally free sheaves of finite rank. The dual of an object $F \in \mathrm{qcoh}_S$ is defined to be the quasi-coherent $\mathcal{O}_S$-module

$$F^\vee = \mathcal{H}\mathrm{om}_{\mathcal{O}_S}(F, \mathcal{O}_S).$$

The sheaf F is called reflexive, if the canonical map to its bidual is an isomorphism,

$$F \xrightarrow{\sim} F^{\vee\vee}.$$

The category refl_S is the full subcategory of reflexive $\mathcal{O}_S$-modules

$$\mathrm{vect}_S \subseteq \mathrm{refl}_S \subseteq \mathrm{qcoh}_S\,.$$

Similarly, if E is an R-algebra, we write Mod_E for the category of (right) E-modules, and

$$\mathrm{proj}_E \subseteq \mathrm{mod}_E$$

for its full subcategories of finitely presented flat, resp. finitely presented, modules.

Remark 1.8. The category proj_A is the Cauchy completion of A itself. This can be thought of as a two-step process, where completion under (finite) direct sums yields the category of finite free A-modules, the idempotent completion of which is proj_A. All of this occurs inside the category $\mathrm{Fun}_R(A^{\mathrm{op}}, \mathrm{Mod}_R) \cong \mathrm{Mod}_A$ of arbitrary A-modules.

Write $S = \mathrm{Spec}(A)$. Because of the above, it seems reasonable to think of $\mathcal{E} \mathbin{\widehat{\otimes}}_R A$ as parametrizing "flat S-families of objects in $\mathcal{E}$". For $s \in S$, there is an induced "fibre" functor

$$\mathcal{E} \mathbin{\widehat{\otimes}}_R A \longrightarrow \mathcal{E} \mathbin{\widehat{\otimes}}_R \kappa(s).$$

Definition 1.9. The underived moduli stack of an R-linear category $\mathcal{E}$ is defined by

$$\mathcal{U}_{\mathcal{E}} \colon \mathrm{Aff}_R \longrightarrow \mathrm{Grpd},\ \ S \longmapsto \mathrm{Fun}_R(\mathcal{E}^{\mathrm{op}}, \mathrm{vect}_S)^\times$$

as a functor of points.

Definition 1.10. We say that $\mathcal{E}$ is locally flat if for all $M, N \in \mathcal{E}$, the R-module $\mathrm{Hom}(M, N)$ is flat. The category $\mathcal{E}$ is called locally proper if $\mathrm{Hom}(M, N) \in \mathrm{proj}_R$ for all $M, N \in \mathcal{E}$.

Remark 1.11. The stack $\mathcal{U}_{\mathcal{E}}$ was introduced by Toën-Vaquié [11], who show that (unlike in our linear setting – hence our facetious terminology) the analogous definition in the dg-world actually provides a reasonable (higher) moduli stack of objects.

Let $S = \mathrm{Spec}(A) \in \mathrm{Aff}_R$. Similarly as for $\mathcal{N}_{\mathcal{E}}$, the assignment $\mathcal{U}_{\mathcal{E}}$ defines a stack over R, by faithfully flat descent for vect_S, now using the fact that $\mathrm{Fun}_R(\mathcal{E}^{\mathrm{op}}, -)$ is left exact.

If $\mathcal{E}$ is locally proper, then the Yoneda embedding induces a fully faithful functor

$$\mathcal{E} \otimes_R A \hookrightarrow \mathrm{Fun}_A(\mathcal{E}_A^{\mathrm{op}}, \mathrm{proj}_A) \cong \mathrm{Fun}_R(\mathcal{E}^{\mathrm{op}}, \mathrm{proj}_A).$$

This defines a representable morphism $\eta \colon \mathcal{N}_{\mathcal{E}} \to \mathcal{U}_{\mathcal{E}}$ of stacks over R, which factors as

$$\begin{array}{ccc} \mathcal{N}_{\mathcal{E}} & \xrightarrow{\ \eta\ } & \mathcal{U}_{\mathcal{E}} \\ & \searrow \quad \nearrow_{\widehat{\eta}} & \\ & \mathcal{M}_{\mathcal{E}} & \end{array} \tag{1.2}$$

since proj_A is Cauchy complete, and hence so is $\mathrm{Fun}_R(\mathcal{E}^{\mathrm{op}}, \mathrm{proj}_A)$.

Example 1.12. Let X be a proper scheme over R. Then the moduli stack of vector bundles

$$\mathrm{Bun}_X = \coprod_{d \in \mathbb{N}} \mathrm{Bun}_{X, \mathrm{GL}_d}$$

is an algebraic stack (which is always meant in the sense of [10]), locally of finite presentation over R. Indeed, let Coh_X be the stack assigning to $S \in \mathrm{Aff}_R$ the groupoid of S-flat finitely presented quasi-coherent sheaves on $X \times_R S$. Then by *loc.cit.*, Theorem 09DS, Coh_X has the desired properties (also cf. [6], Théorème 4.6.2.1; [1], §4.4). But $\mathrm{Bun}_X \subseteq \mathrm{Coh}_X$ is an open substack.

Now denote by $\mathcal{E} = \mathrm{vect}_X$. Then the morphism(1.2) factors as

$$\begin{array}{ccccc} \mathcal{N}_{\mathcal{E}} & \xrightarrow{\quad\eta\quad} & \mathcal{U}_{\mathcal{E}} & \ni & \mathrm{Hom}_{\mathcal{O}_{X\times S}}(\mathrm{pr}_X^*(-), E) \\ & {\scriptstyle \mathrm{pr}_X^*}\searrow \quad \nearrow{\scriptstyle \widetilde{\eta}} & & \nearrow & \\ & & \mathrm{Bun}_X & \ni & E, \qquad \text{on } S\text{-points.} \end{array} \tag{1.3}$$

It is clear that neither map is an equivalence. However, we obtain a natural equivalence

$$\mathcal{M}_{\mathcal{E}} \xrightarrow{\sim} \mathrm{Bun}_X .$$

In fact, the morphism pr_X^* in (1.3) factors through $\mathcal{N}_{\mathcal{E}} \hookrightarrow \mathcal{M}_{\mathcal{E}}$, and $\widetilde{\eta}$ maps into $\mathcal{M}_{\mathcal{E}}$.

If X is flat over R, we can consider the full subcategory $\mathrm{qcoh}_X^\flat \subseteq \mathrm{qcoh}_X$ of R-flat sheaves to get a monomorphism $\mathcal{M}_{\mathrm{qcoh}_X^\flat} \hookrightarrow \mathrm{Coh}_X$. At least if $R = k$ is a field, this is an equivalence

$$\mathcal{M}_{\mathrm{qcoh}_X} \xrightarrow{\sim} \mathrm{Coh}_X .$$

Example 1.13. Suppose that $R = k$ is a field, and let $\mathcal{E} = \mathrm{rep}_k(Q)$ be the category of finite dimensional representations over k of a finite (connected) quiver Q. For each $\underline{d} \in \mathbb{N}^{Q_0}$, set

$$\mathbb{V}_{Q,\underline{d}} = \mathrm{Spec}(\mathrm{Sym}(V_{Q,\underline{d}}^\vee)),$$

where

$$V_{Q,\underline{d}} = \bigoplus_{\alpha\in Q_1} \mathrm{Hom}_k(k^{d_{s(\alpha)}}, k^{d_{t(\alpha)}}).$$

Each $\mathbb{V}_{Q,\underline{d}}$ carries an action of the corresponding general linear group by change of basis,

$$g.(\varphi_\alpha)_{\alpha\in Q_1} = (g_{t(\alpha)}\varphi_\alpha g_{s(\alpha)}^{-1})_{\alpha\in Q_1} \text{ for } g \in \mathrm{GL}_{\underline{d}}(A),$$

where $\mathrm{GL}_{\underline{d}} = \prod_{i\in Q_0} \mathrm{GL}_{d_i}$ and $\varphi_\alpha \in \mathrm{Hom}_k(k^{d_{s(\alpha)}}, k^{d_{t(\alpha)}}) \otimes_k A$. Then the moduli stack

$$\mathcal{M}'_{\mathcal{E}} := \coprod_{\underline{d}\in\mathbb{N}^{Q_0}} [\mathbb{V}_{Q,\underline{d}}/ \mathrm{GL}_{\underline{d}}]$$

parametrizes representations of Q in vect_S (of constant rank on S), for $S = \mathrm{Spec}(A) \in \mathrm{Aff}_k$. Thus, the obvious map $\mathcal{N}_{\mathcal{E}} \to \mathcal{M}'_{\mathcal{E}}$ yields an equivalence of algebraic stacks locally of finite type

$$\mathcal{M}_{\mathcal{E}} \xrightarrow{\sim} \mathcal{M}'_{\mathcal{E}}.$$

On the other hand, the morphism $\eta' \colon \mathcal{M}'_{\mathcal{E}} \to \mathcal{U}_{\mathcal{E}}$ induced by (1.2) is only an equivalence if Q is discrete. Otherwise, we may assume $Q = A_2$. Let $F \colon \mathcal{E}^{\mathrm{op}} \to \mathrm{vect}_k$ be a (k-linear) functor, and let $E = (k \hookrightarrow k^2)$ denote the regular representation. If $P \in \mathcal{E}$ is a projective E-module, then $F(P) \cong \mathrm{Hom}_{\mathcal{E}}(P, F(E))$, since the restricted Yoneda embedding is an equivalence, as

$$\mathrm{Fun}(\mathrm{proj}_E^{\mathrm{op}}, \mathrm{vect}_k) \xrightarrow{\sim} \mathrm{Fun}(E^{\mathrm{op}}, \mathrm{vect}_k),\ f \mapsto f(E).$$

Now, if F is representable, it is left exact. By the standard projective resolution, it follows that it is indeed represented by $F(E)$. But consider the functor $F := \mathrm{Hom}_{\mathcal{E}}(H(-), E)$, where

$$H(V \xrightarrow{\varphi} W) = (0 \to \ker(\varphi)).$$

Then $F(E) = 0$ (indeed, $F|_{\mathrm{proj}_E} = 0$), but $F \neq 0$ otherwise, hence it is not representable.

Instead, if $\widehat{Q}$ is the category of finitely presented projective objects in the category of all representations of Q over k, then $\mathcal{U}_{\widehat{Q}} \cong \mathcal{M}_{\mathcal{E}}$ provides the expected answer.

Definition 1.14. The R-linear category $\mathcal{E}$ is of finite type if it is Morita equivalent to a finitely presented R-algebra. We say that $\mathcal{E}$ is locally of finite type if all of its endomorphism algebras are finitely presented as R-algebras.

Remark 1.15. Note that if $\mathcal{E}$ is of finite type, then the Yoneda embedding

$$\mathcal{E} \hookrightarrow \mathrm{Fun}_R(\mathcal{E}^{\mathrm{op}}, \mathrm{Mod}_R) \cong \mathrm{Fun}_R(E^{\mathrm{op}}, \mathrm{Mod}_R) = \mathrm{Mod}_E$$

maps into the finitely presented projective objects. Thus $\mathcal{E}$ is locally of finite type.

Theorem 1.16. *If $\mathcal{E}$ is of finite type, then $\mathcal{U}_\mathcal{E}$ is an algebraic stack, locally of finite presentation over* $\mathrm{Spec}(R)$. *Moreover, if $\mathcal{E}$ is locally proper, the same holds for $\mathcal{N}_\mathcal{E}$ and $\mathcal{M}_\mathcal{E}$.*

Proof. Any Morita equivalence between $\mathcal{E}$ and a finitely presented R-algebra E induces an equivalence between $\mathcal{U}_\mathcal{E}(S)$ and the groupoid of $(E \otimes_R \mathcal{O}_S)$-modules finitely presented and flat as $\mathcal{O}_S$-modules. It is shown in [11], Theorem 1.1, that the resulting forgetful morphism

$$\mathcal{U}_\mathcal{E} \longrightarrow \coprod_{d \in \mathbb{N}} \mathrm{BGL}_d$$

is representable of finite presentation, implying the result for $\mathcal{U}_\mathcal{E}$ by [10], Lemma 05UM.

Now let $\mathcal{E}$ be locally proper. For any $M \in \mathcal{E}$, set $E_M := \mathrm{End}_\mathcal{E}(M)$, and $\mathcal{E}_M = \mathrm{proj}_{E_M}$ the corresponding Cauchy completion. Then (1.2) induces an equivalence

$$\mathcal{M}_{\mathcal{E}_M} \xrightarrow{\sim} \mathcal{U}_{E_M} \tag{1.4}$$

of locally finitely presented algebraic stacks over R, by the above.

Since $\mathcal{E}$ is Cauchy complete, the tautological inclusion $E_M \hookrightarrow \mathcal{E}$ induces a representable map $\mathcal{M}_{\mathcal{E}_M} \to \mathcal{M}_\mathcal{E}$ by faithfulness, and therefore a representable morphism

$$\rho\colon \coprod_{M \in \pi_0(\mathcal{E}^\times)} \mathcal{M}_{\mathcal{E}_M} \longrightarrow \mathcal{M}_\mathcal{E}.$$

Moreover, ρ is surjective and smooth (which can be seen locally for each summand). Therefore, if we choose a smooth surjection from a scheme $X_M \to \mathcal{M}_{\mathcal{E}_M}$ for each $M \in \pi_0(\mathcal{E}^\times)$, then the composition

$$\coprod_{M \in \pi_0(\mathcal{E}^\times)} X_M \longrightarrow \coprod_{M \in \pi_0(\mathcal{E}^\times)} \mathcal{M}_{\mathcal{E}_M} \xrightarrow{\rho} \mathcal{M}_\mathcal{E}$$

will have the same properties. Thus, $\mathcal{M}_\mathcal{E}$ is also an algebraic stack (its diagonal is representable by faithfulness), and moreover locally of finite presentation since the source of ρ is. Alternatively, the below argument for $\mathcal{N}_\mathcal{E}$ also applies here.

Finally, since $\mathcal{N}_\mathcal{E} \hookrightarrow \mathcal{M}_\mathcal{E}$ is representable, $\mathcal{N}_\mathcal{E}$ is again an algebraic stack. Now let $(A_i)_{i \in I}$ be a filtered inductive system in $\mathrm{Aff}_R^{\mathrm{op}}$. Then we have the natural equivalence

$$\varinjlim(\mathcal{E} \otimes_R A_i) \xrightarrow{\sim} \mathcal{E} \otimes_R \varinjlim A_i,$$

since extension of scalars is right exact. Thus, we conclude that $\mathcal{N}_\mathcal{E}$ is also locally finitely presented over R, by [10], Proposition 0CMY. □

2. Reflexive and coherent families

While qcoh_S and refl_S are not abelian in general, they are finitely cocomplete, which we will make use of in this section. For qcoh_S, this is most easily seen by using that it is exactly the category of finitely presented objects in the (abelian) category of quasi-coherent sheaves on S. In fact, if $S = \mathrm{Spec}(A)$, then

$$\mathrm{qcoh}_S = \mathrm{mod}_A$$

is the completion of A under finite colimits (as an R-linear category). This makes it the natural candidate for our below considerations.

For refl_S on the other hand, we can make use of the following basic result.

Lemma 2.1 (cf. [5], Corollary 1.2)**.** *Let S be a qcqs scheme, $F \in \mathrm{qcoh}_S$. Then $F^\vee \in \mathrm{refl}_S$.*

Proof. Locally on S, we can find a flat cover $E_1 \to E_0 \to F \to 0$, with $E_i \in \mathrm{vect}_S$. Then

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^\vee & \longrightarrow & E_0^\vee & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow \sim & & \downarrow g & & \\ 0 & \longrightarrow & F^{\vee\vee\vee} & \longrightarrow & E_0^{\vee\vee\vee} & \longrightarrow & Q^{\vee\vee} & & \end{array}$$

is a commutative diagram of $\mathcal{O}_S$-modules with exact rows, where $Q := \mathrm{im}(E_0^\vee \to E_1^\vee)$. By the snake lemma, $\ker(f) = 0$ and $\mathrm{coker}(f) \cong \ker(g) = 0$, since $Q \hookrightarrow E_1^\vee$ factors over g. □

Corollary 2.2. *The category* refl_S *is finitely cocomplete.*

Proof. The functor $\mathrm{qcoh}_S \longrightarrow \mathrm{refl}_S,\ F \longmapsto F^{\vee\vee}$, is left adjoint to the inclusion, via

$$\mathrm{Hom}(F^{\vee\vee}, E) \xrightarrow{\sim} \mathrm{Hom}(F, E),\ f \longmapsto f \circ (F \to F^{\vee\vee}),\ g^{\vee\vee} \longleftarrow g.$$

Hence the bidual of the colimit taken in qcoh_S of a finite diagram defines a colimit in refl_S. □

Remark 2.3. Let k be a field, and consider the local scheme $S = \mathrm{Spec}(k[[x,y]]/(xy))$, with cyclic module

$$(x) \subseteq \mathcal{O}_S.$$

This is easily seen to be reflexive directly, but of course it cannot be free. In fact, (x) is the cokernel of $y\colon \mathcal{O}_S \to \mathcal{O}_S$ in refl_S, which does not have a cokernel in vect_S, because y would have to act trivially on it. Thus vect_S is not finitely cocomplete, hence absent from the following definition.

From now on, we assume that the category $\mathcal{E}$ is finitely cocomplete.

Definition 2.4. The moduli stack of coherent, resp. reflexive, families of objects in $\mathcal{E}$ is the stack $\mathcal{Q}_\mathcal{E}$, resp. $\mathcal{R}_\mathcal{E}$, over R, defined as the fppf-stackification of the functor

$$\begin{aligned} \mathrm{Aff}_R \longrightarrow \mathrm{Grpd},\ S \longmapsto (\mathcal{E} \boxtimes_R \mathrm{qcoh}_S)^\times, \\ \text{resp. } \mathrm{Aff}_R \longrightarrow \mathrm{Grpd},\ S \longmapsto (\mathcal{E} \boxtimes_R \mathrm{refl}_S)^\times, \end{aligned} \tag{2.1}$$

where $- \boxtimes_R -$ denotes Kelly's tensor product of finitely cocomplete R-linear categories.

Remark 2.5. Let $f\colon S' \to S$ be any morphism in Aff_R. We obtain a (right exact) functor

$$\mathcal{E} \boxtimes_R f^*\colon\ \mathcal{E} \boxtimes_R \mathrm{qcoh}_S \longrightarrow \mathcal{E} \boxtimes_R \mathrm{qcoh}_{S'}$$

from the universal property of the tensor product ([7], Theorem 7). Similarly, the functor

$$\mathrm{refl}_S \longrightarrow \mathrm{refl}_{S'},\ F \longmapsto (f^*F)^{\vee\vee},$$

is right exact by definition, whence we obtain $\mathcal{E} \boxtimes_R \mathrm{refl}_S \longrightarrow \mathcal{E} \boxtimes_R \mathrm{refl}_{S'}$.

In general, it is not clear whether the stackification in (2.1) is redundant. However, as a special case of [8], Theorem 5.1, if $\mathcal{E}$ carries a symmetric monoidal structure, right exact in both variables, then $- \boxtimes_R -$ is a 2-coproduct, thus commutes with finite 2-limits.

Therefore, in this case, we are reduced to (fpqc-)descent for quasi-coherent modules, which implies descent for qcoh_S since finite presentation descends. For refl_S, we use that the dual

$$f^*F^\vee = \mathcal{H}om_{\mathcal{O}_S}(F, \mathcal{O}_S) \otimes_{\mathcal{O}_S} \mathcal{O}_{S'} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_{S'}}(F \otimes_{\mathcal{O}_S} \mathcal{O}_{S'}, \mathcal{O}_{S'}) = (f^*F)^\vee \tag{2.2}$$

is stable under (faithfully) flat base change $S' \to S$.

Remark 2.6. If $R = k$ is a field, then $\mathcal{Q}_\mathcal{E}$ has the expected groupoid of k-points $\mathcal{E}^\times$, since

$$\mathcal{D} \xrightarrow{\sim} \mathrm{Rex}_k(\mathrm{qcoh}_k, \mathcal{D}) = \mathrm{Rex}_k(\mathrm{vect}_k, \mathcal{D}),\ D \longmapsto (k^n \mapsto D^{\oplus n}),$$

for every finitely cocomplete k-linear category $\mathcal{D}$. So indeed, $\mathcal{E}$ satisfies the universal property

$$\mathrm{Rex}_k(\mathcal{E}, \mathcal{D}) \xrightarrow{\sim} \mathrm{Rex}_k(\mathcal{E}, \mathrm{Rex}_k(\mathrm{qcoh}_k, \mathcal{D}))$$

of $\mathcal{E} \boxtimes_k \mathrm{qcoh}_k$, and $\mathcal{Q}_\mathcal{E}(k) = (\mathcal{E} \boxtimes_k \mathrm{qcoh}_k)^\times$ by the universal property of the stackification. This argument of course also shows that $\mathcal{R}_\mathcal{E}(k) \cong \mathcal{E}^\times$ as well.

Moreover, if $\mathcal{E}$ is an abelian category of finite length, and $L|k$ is a finite extension, then

$$\mathcal{E} \boxtimes_k \operatorname{qcoh}_L = \mathcal{E} \boxtimes_k \operatorname{vect}_L = \mathcal{E} \boxtimes_k \operatorname{refl}_L$$

coincides with Deligne's tensor product of abelian categories, which follows as a special case from [7], Proposition 22, together with *loc.cit.*, Theorem 18.

Lemma 2.7. *There are natural representable morphisms of stacks over R, which factor as*

$$\begin{array}{ccccc} \mathcal{Q}_{\mathcal{E}} & \xleftarrow{\varphi} & \mathcal{N}_{\mathcal{E}} & \xrightarrow{\rho} & \mathcal{R}_{\mathcal{E}} \\ & {\scriptstyle \widehat{\varphi}}\nwarrow & \downarrow & \nearrow{\scriptstyle \widehat{\rho}} & \\ & & \mathcal{M}_{\mathcal{E}} & & \end{array} \tag{2.3}$$

Proof. Let $S = \operatorname{Spec}(A) \in \operatorname{Aff}_R$, and let $\mathcal{D}$ be a finitely cocomplete R-linear category with symmetric monoidal structure, right exact in both variables, and $\operatorname{End}(\mathbb{1}) = A$. In particular, this applies to $\mathcal{D} \in \{\operatorname{qcoh}_S, \operatorname{refl}_S\}$. The tautological embedding $A \hookrightarrow \mathcal{D}$ induces a natural functor

$$\mathcal{E} \otimes_R A \hookrightarrow \mathcal{E} \otimes_R \mathcal{D} \longrightarrow \mathcal{E} \boxtimes_R \mathcal{D}.$$

Thus, we obtain maps φ and ρ as in (2.3) by composition with the respective adjunction morphism of the stackification (which is representable).

Let us show that $\mathcal{N}_{\mathcal{E}} \to \mathcal{M}_{\mathcal{E}}$ factors these. First, the tautological functor from above yields

$$\mathcal{E} \mathbin{\widehat{\otimes}}_R A \hookrightarrow \mathcal{E} \mathbin{\widehat{\otimes}}_R \mathcal{D}.$$

By construction, both $\mathcal{E} \mathbin{\widehat{\otimes}}_R \mathcal{D}$ and $\mathcal{E} \boxtimes_R \mathcal{D}$ are full subcategories of $\operatorname{Fun}_R(\mathcal{E}^{\mathrm{op}} \otimes_R \mathcal{D}^{\mathrm{op}}, \operatorname{Mod}_R)$. The former consists of finite direct sums of retracts of representable functors. In particular, these are left exact and finitely presented. But that means that they lie in $\mathcal{E} \boxtimes_R \mathcal{D}$.

The resulting morphisms $\widehat{\varphi} \colon \mathcal{M}_{\mathcal{E}} \to \mathcal{Q}_{\mathcal{E}}$ and $\widehat{\rho} \colon \mathcal{M}_{\mathcal{E}} \to \mathcal{R}_{\mathcal{E}}$ are representable by fully faithfulness, and yield a factorization as claimed.

A foritori, as compositions of representable morphisms, φ and ρ are representable. $\square$

Remark 2.8. It is explained in [7], Remark 9, when the functor $\mathcal{E} \otimes_R \mathcal{D} \longrightarrow \mathcal{E} \boxtimes_R \mathcal{D}$ is fully faithful. In particular, this applies when $\operatorname{Spec}(R)$ is reduced of dimension 0.

Example 2.9. Let $R = k$ be a field, and X be a smooth projective (geometrically connected) curve over k. Using Proposition 2.10 below, we get a similar factorization as in (1.3), with $\mathcal{R}_{\mathcal{E}}$ in place of $\mathcal{U}_{\mathcal{E}}$. Namely, the functor

$$\operatorname{vect}_X \otimes_k \operatorname{refl}_S \longrightarrow \operatorname{refl}_{X \times_k S}, \; E \otimes F \longmapsto \operatorname{pr}_X^* E \otimes_{\mathcal{O}_{X \times S}} \operatorname{pr}_S^* F, \tag{2.4}$$

is right exact in both arguments, hence yields a functor $\operatorname{vect}_X \boxtimes_k \operatorname{refl}_S \longrightarrow \operatorname{refl}_{X \times_k S}$, which is fully faithful, because (2.4) is. Hence the inclusion of $\operatorname{vect}_{X \times_k S}$ induces a map as claimed,

$$\mathcal{N}_{\operatorname{vect}_X} \hookrightarrow \mathcal{M}_{\operatorname{vect}_X} \xrightarrow{\sim} \operatorname{Bun}_X \xrightarrow{\pi} \mathcal{R}_{\operatorname{vect}_X}.$$

By [5], Corollary 1.4, if S is regular of dimension ≤ 1, then π is an equivalence on S-points. On the other hand, consider $S = \operatorname{Spec}(k[[x,y]]/(xy))$, as in Remark 2.3. Since X is smooth, in particular pr_S is flat, and thus $\operatorname{pr}_S^*(x) \in \operatorname{refl}_{X \times_k S}$ is not locally free.

Similarly, we have a factorization (where X can be any proper scheme over k)

$$\mathcal{N}_{\operatorname{qcoh}_X} \hookrightarrow \mathcal{M}_{\operatorname{qcoh}_X} \xrightarrow{\sim} \operatorname{Coh}_X \xrightarrow{\psi} \mathcal{Q}_{\operatorname{qcoh}_X}$$

by the proof of Proposition 2.10. Then ψ is an equivalence on S-points if S is reduced of dimension 0.

Proposition 2.10. *Let k be a field. Let X be a smooth projective (geometrically connected) curve over k, and $S \in \operatorname{Aff}_k$. Then the functor* (2.4) *induces an equivalence of categories*

$$\operatorname{vect}_X \boxtimes_k \operatorname{refl}_S \xrightarrow{\sim} \operatorname{refl}_{X \times_k S}. \tag{2.5}$$

Proof. Since S is affine over k, we can write S as $S = \varprojlim S_\lambda$, with each $S_\lambda \in \mathrm{Aff}_k$ noetherian over k, and hence $X \times_k S = \varprojlim (X \times_k S_\lambda)$. The category qcoh_S is exhausted by coherent sheaves on the S_λ, in the sense that there is an equivalence of categories

$$\varinjlim \mathrm{qcoh}_{X \times_k S_\lambda} \xrightarrow{\sim} \mathrm{qcoh}_{X \times_k S} \tag{2.6}$$

by [4], Théorème 8.5.2 (*ii*). Now, we can apply [8], Theorem 1.7, which implies that

$$\mathrm{qcoh}_X \boxtimes_k \mathrm{qcoh}_{S_\lambda} \xrightarrow{\sim} \mathrm{qcoh}_{X \times_k S_\lambda}$$

is an equivalence, for all λ. The universal property implies that this commutes with (2.6). Therefore, we may assume that S is noetherian, and for $Q \in \mathrm{refl}_{X \times_k S}$, there exist $E_i \in \mathrm{qcoh}_X$ as well as $F_i \in \mathrm{qcoh}_S$ with

$$Q \cong \varinjlim (\mathrm{pr}_X^* E_i \otimes \mathrm{pr}_S^* F_i) = \mathrm{pr}_X^* E \otimes \mathrm{pr}_S^* F, \text{ with } E = \varinjlim E_i \text{ and } F = \varinjlim F_i.$$

Note that a fortiori, the first $\varinjlim$ is taken in $\mathrm{refl}_{(-)}$, whereas a priori, the other two are not. However, we can assume $E \in \mathrm{vect}_X$ by replacing it by its torsion-free quotient E/E_{tors}, since by right exactness of the functors pr_X^* and $- \otimes \mathrm{pr}_S^* F$, the sequence

$$\mathrm{pr}_X^*(E_{\mathrm{tors}}) \otimes \mathrm{pr}_S^* F \xrightarrow{=0} Q \longrightarrow \mathrm{pr}_X^*(E/E_{\mathrm{tors}}) \otimes \mathrm{pr}_S^* F \longrightarrow 0$$

is exact, since Q is torsion-free. Then $\mathrm{pr}_S^* F \in \mathrm{refl}_{X \times_k S}$ as well, because it is locally a direct summand of Q. Finally, $\mathrm{pr}_S^* F \cong (\mathrm{pr}_S^* F)^{\vee\vee} \cong \mathrm{pr}_S^*(F^{\vee\vee})$, and $F^{\vee\vee} \in \mathrm{refl}_S$. Thus (2.5) is essentially surjective, and we had already seen that (2.4) is fully faithful. □

3. Moduli stacks for finitely cocomplete categories

Let $\mathcal{E}$ be a finitely cocomplete R-linear category.

Definition 3.1 (cf. [3], Definition 2.1)**.** Suppose $\mathcal{E}$ is cocomplete. The moduli stack of points of $\mathcal{E}$ is the fppf-stackification $\mathcal{P}_\mathcal{E}$ of the functor

$$\mathrm{Aff}_R \longrightarrow \mathrm{Grpd}, \; S \longmapsto \mathrm{Pt}_\mathcal{E}(S).$$

We define the moduli stack of objects of $\mathcal{E}$ as a substack of $\mathcal{Q}_\mathcal{E}$ by analogous conditions.

(...)

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