

HIGHER SEGAL STRUCTURES IN ALGEBRAIC K -THEORY

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ABSTRACT. We introduce higher dimensional analogues of simplicial constructions due to Segal and Waldhausen, respectively producing the direct sum and algebraic K -theory spectra of an exact category. We then investigate their fibrancy properties, based on the formalism of higher Segal spaces by Dyckerhoff-Kapranov.

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1. INTRODUCTION

Let \mathcal{E} be an exact category. In this article, we investigate certain simplicial categories naturally arising in the context of algebraic K -theory. They are obtained as generalizations of the following ubiquitous constructions, which produce a simplicial category from \mathcal{E} ;

- the Segal construction $S_{\oplus}(\mathcal{E})$,
- the Waldhausen construction $S(\mathcal{E})$.

From a topological perspective, their relevance lies in the fact that they provide deloopings of the direct sum and algebraic K -theory spectra of \mathcal{E} , respectively.

From an algebraic perspective, both constructions have fibrancy properties of structural importance. Namely, $S_{\oplus}(\mathcal{E})$ and $S(\mathcal{E})$ satisfy the 1-Segal and 2-Segal conditions, respectively, modelling the structure of an associative, resp. multi-valued associative, monoid.

This work centers around certain higher dimensional generalizations of the above;

- the k -dimensional Segal construction $S_{\oplus}^{(k)}(\mathcal{E})$,
- the k -dimensional Waldhausen construction $S^{(k)}(\mathcal{E})$.

For $k = 1$, we recover $S_{\oplus}(\mathcal{E})$ and $S(\mathcal{E})$, respectively. For $k = 2$, these constructions form the foundational basis of real algebraic K -theory as introduced by Hesselholt-Madsen [12], and studied further for example by Dotto [3]. In our context, \mathcal{E} is not endowed with a duality structure, and we consider these objects primarily as simplicial categories.

Similarly to the case $k = 1$, the higher dimensional constructions provide higher deloopings of algebraic K -theory and its split variant (Corollary 7.10). Their relevance from the algebraic perspective warrants further investigation, for which our main results lay the groundwork.

Theorem 1.1. *The higher Segal construction $S_{\oplus}^{(k)}(\mathcal{E})$ is lower $(2k - 1)$ -Segal. If \mathcal{E} is an abelian category, then the higher Waldhausen construction $S^{(k)}(\mathcal{E})$ is $2k$ -Segal.*

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For the precise statements, see Theorem 4.3 and Theorem 6.8, respectively.

Higher Segal objects were introduced in [6], with a focus on the 2-Segal conditions, in particular showing that they are responsible for the associativity of Hall and Hecke algebras. From a different perspective, unital 2-Segal spaces were defined and studied independently in [9] and its series of sequels.

Further work in this area includes a precise description of unital 2-Segal sets in terms of double categories in [1], and the introduction of relative Segal conditions in [20] and [22], which model the structure of modules over multi-valued associative monoids.

Let us briefly outline the structure of the paper. In §2, we summarize some basic theory of cyclic polytopes and their triangulations (from [15] and [23]), which is used in §3 to define and study relations between the higher Segal conditions, as in [7].

In §4, we define the simplicial category $S_{\oplus}^{(k)}(\mathcal{E})$ and prove the first part of Theorem 1.1.

Section §5 introduces the homological context for the second part of Theorem 1.1, which is finally proven in §6, after providing the definition of $S^{(k)}(\mathcal{E})$ and several examples.

In §7, we realize the higher Segal and Waldhausen constructions as totalizations of iterates of lower dimensional ones, from which we deduce delooping and additivity theorems.

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2. CYCLIC POLYTOPES

In this section, we recall results on polytopes relevant to the study of higher Segal objects.

Definition 2.1. Let $d \geq 0$, and consider the moment curve

$$\gamma_d: \mathbb{R} \longrightarrow \mathbb{R}^d, \quad t \longmapsto (t, t^2, t^3, \dots, t^d).$$

For a finite subset $N \subseteq \mathbb{R}$, the d -dimensional cyclic polytope on the vertices N is defined to be the convex hull of the set $\gamma_d(N) \subseteq \mathbb{R}^d$, and denoted by

$$C(N, d) = \text{conv}(\gamma_d(N)).$$

The combinatorial type of the polytope $C(N, d)$ only depends on the cardinality of N . We will usually consider N to be the set $[n] = \{0, 1, \dots, n\}$, where $n \geq 0$.

Cyclic polytopes are simplicial polytopes, i.e., their boundary forms a simplicial complex, which organizes into two components (with non-empty intersection), as follows.

Definition 2.2. A point x in the boundary of $C([n], d+1)$ is called a lower point, if

$$(x - \mathbb{R}_{>0}) \cap C([n], d+1) = \emptyset,$$

where the half-line of positive real numbers $\mathbb{R}_{>0} \subseteq \mathbb{R}^{d+1}$ is embedded into the last coordinate. Similarly, x is said to be an upper point, if

$$(x + \mathbb{R}_{>0}) \cap C([n], d+1) = \emptyset.$$

The lower and upper points of the boundary form simplicial subcomplexes of $C([n], d+1)$, which admit the following purely combinatorial description.

Definition 2.3. Let $I \subseteq [n]$. A gap of I is a vertex $j \in [n] \setminus I$ in the complement of I . A gap is said to be even, resp. odd, if the cardinality $\#\{i \in I \mid i > j\}$ is even, resp. odd. The subset I is called even, resp. odd, if all gaps of I are even, resp. odd.

Proposition 2.4 (Gale's evenness criterion; [23], Theorem 0.7). *Let $n \geq 0$, and let $I \subseteq [n]$ with $\#I = d+1$. Then I defines a d -simplex in the lower, resp. upper, boundary of $C([n], d+1)$ if and only if I is even, resp. odd.*

Forgetting the last coordinate of \mathbb{R}^{d+1} defines a projection map

$$p: C([n], d+1) \longrightarrow C([n], d).$$

For any $I \subseteq [n]$ with $\#I - 1 = r \leq d$, the projection p maps the geometric r -simplex

$$|\Delta^I| \subseteq C([n], d+1)$$

homeomorphically to an r -simplex in $C([n], d)$.

Definition 2.5. The lower triangulation \mathcal{T}_ℓ of the polytope $C([n], d)$ is the triangulation given by the projections under p of the simplices contained in the lower boundary of $C([n], d+1)$. Similarly, the upper triangulation \mathcal{T}_u of $C([n], d)$ is defined by the projections of the simplices contained in the upper boundary of the polytope $C([n], d+1)$.

Vice versa, any triangulation of $C([n], d)$ induces a piecewise linear section of p , whose image defines a simplicial subcomplex of $C([n], d+1)$. This interplay between the cyclic polytopes in different dimensions is what makes their combinatorics comparatively tractible.

Definition 2.6. Given a set $\mathcal{I} \subseteq 2^{[n]}$ of subsets of $[n]$, we denote by

$$\Delta^{\mathcal{I}} \subseteq \Delta^n \tag{2.1}$$

the simplicial subset of Δ^n whose m -simplices are given by those maps $[m] \rightarrow [n]$ which factor over some $I \in \mathcal{I}$.

From the above discussion, it follows that we have canonical homeomorphisms

$$|\Delta^{\mathcal{T}_\ell}| \cong C([n], d), \text{ and } |\Delta^{\mathcal{T}_u}| \cong C([n], d),$$

expressing the lower, resp. upper, triangulation of $C([n], d)$ geometrically.

Definition 2.7. Let $I, J \subseteq [n]$ be subsets of cardinality $d+1$, as well as $|\Delta^I|$ and $|\Delta^J|$ the geometric d -simplices in $C([n], d) \subseteq \mathbb{R}^d$ they define, respectively. Let $L(|\Delta^J|)$ denote the set of lower boundary points of $|\Delta^J| = C(J, d)$, and similarly, let $U(|\Delta^I|)$ be the set of upper boundary points of $|\Delta^I| = C(I, d)$. We write

$$|\Delta^I| \prec |\Delta^J| \iff |\Delta^I| \cap |\Delta^J| \subseteq U(|\Delta^I|) \cap L(|\Delta^J|).$$

If $|\Delta^I| \prec |\Delta^J|$, then we say that $|\Delta^I|$ lies below the simplex $|\Delta^J|$.

Proposition 2.8 ([15], Corollary 5.9). *The transitive closure of \prec defines a partial order on the set of nondegenerate d -simplices in Δ^n .*

Remark 2.9. Suppose that $\mathcal{T} \subseteq 2^{[n]}$ consists of subsets of $[n]$ of cardinality $d+1$ and defines a triangulation of the cyclic polytope $C([n], d)$. In particular,

$$|\Delta^{\mathcal{T}}| \cong C([n], d).$$

Let $I_0 \in \mathcal{T}$. Then either $L(|\Delta^{I_0}|)$ is contained in $L(|\Delta^{\mathcal{T}}|)$, or there is some $I_1 \in \mathcal{T}$ such that the simplex $|\Delta^{I_0}|$ lies below $|\Delta^{I_1}|$. Iterating this argument, we obtain a chain

$$|\Delta^{I_0}| \prec |\Delta^{I_1}| \prec |\Delta^{I_2}| \prec \dots$$

of subsimplices of $|\Delta^{\mathcal{T}}|$. The statement of Proposition 2.8 implies that this chain is acyclic and therefore has to terminate after finitely many steps. Thus, there exists $I_\infty \in \mathcal{T}$ with

$$L(|\Delta^{I_\infty}|) \subseteq L(|\Delta^{\mathcal{T}}|).$$

Similarly, there exists a set $I_{-\infty} \in \mathcal{T}$ such that $U(|\Delta^{I_{-\infty}}|) \subseteq U(|\Delta^{\mathcal{T}}|)$.

3. HIGHER SEGAL CONDITIONS

Let \mathcal{C} be an ∞ -category which admits finite limits. Following [7], we introduce a framework of fibrancy properties of simplicial objects in \mathcal{C} governed by the combinatorics from §2 of cyclic polytopes and their triangulations.

Definition 3.1. For $n \geq d \geq 0$, we introduce the lower subposet of $2^{[n]}$ as follows;

$$\mathcal{L}([n], d) = \{J \mid J \subseteq I \text{ for some even } I \subseteq [n] \text{ with } \#I = d + 1\}.$$

Analogously, we define

$$\mathcal{U}([n], d) \subseteq 2^{[n]}$$

as the poset of all subsets contained in an odd subset $I \subseteq [n]$ of cardinality $\#I = d + 1$.

By Proposition 2.4, the sets of subsimplices of $C([n], d)$ described by $\mathcal{L}([n], d)$ and $\mathcal{U}([n], d)$ define the lower and upper triangulations of the cyclic polytope, respectively.

Definition 3.2. Let $d \geq 0$. A simplicial object $X \in \mathcal{C}_\Delta$ is called

- lower d -Segal if, for every $n \geq d$, the natural map

$$X_n \longrightarrow \varprojlim_{I \in \mathcal{L}([n], d)} X_I$$

is an equivalence;

- upper d -Segal if, for every $n \geq d$, the natural map

$$X_n \longrightarrow \varinjlim_{I \in \mathcal{U}([n], d)} X_I$$

is an equivalence;

- fully d -Segal if X is both lower and upper d -Segal.

Example 3.3. Let $X \in \mathcal{C}_\Delta$ be a simplicial object.

- (1) Then X is lower (or upper) 0-Segal if and only if $X \cong X_0$ is equivalent to the constant simplicial object on its 0-cells.
- (2) The simplicial object X is lower 1-Segal if, for every $n \geq 1$, the map

$$X_n \longrightarrow X_{\{0,1\}} \times_{X_{\{1\}}} X_{\{1,2\}} \times_{X_{\{2\}}} \cdots \times_{X_{\{n-1\}}} X_{\{n-1,n\}}$$

is an equivalence. That is, X is a Segal object in the sense of Rezk [16].

For $X \in \text{Set}_\Delta$, this means that X is equivalent to the nerve of the category with objects X_0 , morphisms X_1 , and composition induced by the correspondence

$$X_{\{0,1\}} \times_{X_{\{1\}}} X_{\{1,2\}} \xleftarrow{\sim} X_2 \longrightarrow X_{\{0,2\}}.$$

Furthermore, $X \in \text{Set}_\Delta$ is fully 1-Segal if and only if this defines a discrete groupoid. In fact, in general, an object $X \in \mathcal{C}_\Delta$ is upper 1-Segal if, for every $n \geq 1$, we have

$$X_n \xrightarrow{\sim} X_{\{0,n\}}.$$

- (3) The simplicial object X is lower 2-Segal if, for every $n \geq 2$, the map

$$X_n \longrightarrow X_{\{0,n-1,n\}} \times_{X_{\{0,n-1\}}} X_{\{0,n-2,n-1\}} \times_{X_{\{0,n-2\}}} \cdots \times_{X_{\{0,2\}}} X_{\{0,1,2\}}$$

is an equivalence. Similarly, X is upper 2-Segal if, for every $n \geq 2$,

$$X_n \xrightarrow{\sim} X_{\{0,1,n\}} \times_{X_{\{1,n\}}} X_{\{1,2,n\}} \times_{X_{\{2,n\}}} \cdots \times_{X_{\{n-2,n\}}} X_{\{n-2,n-1,n\}}.$$

It follows that X is fully 2-Segal if and only if it is 2-Segal in the sense of [6]. This is most readily seen by reducing to (1-)Segal objects as in (2) by applying the respective path space criteria, Proposition 3.7 and [6], Theorem 6.3.2 – or by Proposition 3.5.

Remark 3.4. Let X be a simplicial object in an ∞ -category \mathcal{C} which admits limits. Then we can form the right Kan extension of X along the (opposite of the) Yoneda embedding

$$\mathcal{Y}^{\text{op}}: N(\Delta^{\text{op}}) \longrightarrow \text{Fun}(N(\Delta^{\text{op}}), \text{Top})^{\text{op}},$$

where Top denotes the ∞ -category of spaces. In particular, by means of this extension, we may evaluate X on any simplicial set. Then we can reformulate the higher Segal conditions as follows. The simplicial object X is lower d -Segal if and only if, for every $n \geq d$, the inclusion

$$\Delta^{\mathcal{L}([n], d)} \subseteq \Delta^n$$

is an X -equivalence, that is, maps to an equivalence in \mathcal{C} under X (cf. [6], Proposition 5.1.4). Similarly, X is upper d -Segal if and only if, for every $n \geq d$, the inclusion

$$\Delta^{\mathcal{U}([n], d)} \subseteq \Delta^n$$

is an X -equivalence.

Proposition 3.5. *Let $d \geq 0$, and let X be a simplicial object in an ∞ -category \mathcal{C} with limits. Then X is fully d -Segal if and only if, for every $n \geq d$, and every triangulation of the cyclic polytope $C([n], d)$ defined by the poset of simplices $\mathcal{T} \subseteq 2^{[n]}$, the natural map*

$$X_n \longrightarrow \varprojlim_{I \in \mathcal{T}} X_I$$

is an equivalence.

Proof. By [15], Corollary 5.12, any triangulation \mathcal{T} of $C([n], d)$ can be connected to the lower and upper triangulations \mathcal{T}_ℓ and \mathcal{T}_u via a sequence of elementary flips of the form

$$L(|\Delta^I|) \subseteq |\Delta^I| \supseteq U(|\Delta^I|).$$

This implies that we have a zig-zag of X -equivalences connecting $\Delta^{\mathcal{T}}$ to $\Delta^{\mathcal{T}_\ell}$ and $\Delta^{\mathcal{T}_u}$ in the category of simplicial sets over Δ^n . By 2-out-of-3, this implies that the inclusion $\Delta^{\mathcal{T}} \subseteq \Delta^n$ is again an X -equivalence, which is what was to be shown.

For the converse, there is nothing to prove. \square

Definition 3.6. Let X be a simplicial object in an ∞ -category \mathcal{C} . The left path space $P^\triangleleft X$ is the simplicial object in \mathcal{C} defined as the pullback of X along the endofunctor

$$\Delta \rightarrow \Delta, [n] \mapsto [0] \oplus [n].$$

Here, for two linearly ordered sets I and J , the ordinal sum $I \oplus J$ is the disjoint union $I \amalg J$ of sets, endowed with the linear order where $i \leq j$ for every pair of $i \in I$ and $j \in J$.

Similarly, the right path space $P^\triangleright X$ is given by the pullback of X along

$$\Delta \rightarrow \Delta, [n] \mapsto [n] \oplus [0].$$

In terms of these constructions, we have the following criteria for various Segal conditions.

Proposition 3.7 (Path space criterion). *Let $d \geq 0$, and let \mathcal{C} be an ∞ -category with finite limits. Let $X \in \mathcal{C}_\Delta$ be a simplicial object.*

- *Suppose d is even. Then*
 - (1) *X is lower d -Segal if and only if $P^\triangleleft X$ is lower $(d-1)$ -Segal,*
 - (2) *X is upper d -Segal if and only if $P^\triangleright X$ is lower $(d-1)$ -Segal.*
- *Suppose d is odd. Then the following conditions are equivalent.*
 - (i) *X is upper d -Segal.*
 - (ii) *$P^\triangleleft X$ is upper $(d-1)$ -Segal.*
 - (iii) *$P^\triangleright X$ is lower $(d-1)$ -Segal.*

Proof. We show that if d is even, then X is an upper d -Segal object if and only if $P^\triangleright X$ is lower $(d-1)$ -Segal. All other assertions follow by analogous arguments.

However, the claim is an immediate consequence of the following observation. Let $\ell([n], d)$, resp. $u([n], d)$, denote the set of maximal elements of the poset $\mathcal{L}([n], d)$, resp. $\mathcal{U}([n], d)$. Then the map $\ell([n-1], d-1) \rightarrow u([n], d)$, $I \mapsto I \oplus [0]$, is a bijection. Namely, a subset $I \subseteq [n]$ of cardinality $d+1$ is odd if and only if $n \in I$ and $I \setminus \{n\}$ is an even subset of $[n-1]$. \square

Example 3.8. Applying the bijections from the proof of Proposition 3.7 to Example 3.3 (2) twice says precisely that $u([n], 3) = \{\{0, 1, 2, n\}, \{0, 2, 3, n\}, \dots, \{0, n-2, n-1, n\}\}$.

Remark 3.9. There is no path space criterion for lower d -Segal objects if d is odd. While

$$\ell([5], 3) = \{\{0, 1, 2, 3\}, \{0, 1, 3, 4\}, \{0, 1, 4, 5\}, \{1, 2, 4, 5\}, \{2, 3, 4, 5\}, \{1, 2, 3, 4\}\},$$

the maps $c^{\heartsuit}: \ell([4], 2) \rightarrow \ell([5], 3)$, $I \mapsto [0] \oplus I$, and $c^{\spadesuit}: u([4], 2) \rightarrow \ell([5], 3)$, $I \mapsto I \oplus [0]$, are not even jointly surjective, and their images

$$\text{im}(c^{\heartsuit}) = \{\{0, 1, 2, 3\}, \{0, 1, 3, 4\}, \{0, 1, 4, 5\}\}, \quad \text{im}(c^{\spadesuit}) = \{\{0, 1, 4, 5\}, \{1, 2, 4, 5\}, \{2, 3, 4, 5\}\}$$

intersect. Rather, in general, the complement of $\text{im}(c^{\spadesuit})$ in $\ell([n], d)$ is given by $\ell([n-1], d)$. By induction, we obtain a disjoint decomposition of sets

$$\ell([n], d) = \coprod_{i=d}^n u([i-1], d-1) \oplus \{i\}. \quad (3.1)$$

Proposition 3.10. *Let X be a simplicial object in an ∞ -category \mathcal{C} which admits limits. Assume that X is lower or upper d -Segal. Then X is fully $(d+1)$ -Segal.*

Proof. We show the statement assuming that X is lower d -Segal. The proof for upper d -Segal objects is similar. Let $n \geq d+1$ and consider a collection \mathcal{T} defining a triangulation $|\Delta^{\mathcal{T}}|$ of the cyclic polytope $C([n], d+1)$. Recall that \mathcal{T}_ℓ defines the simplicial complex $L(|\Delta^{\mathcal{T}}|)$ of lower facets, and the projection $p: C([n], d+1) \rightarrow C([n], d)$ identifies $|\Delta^{\mathcal{T}_\ell}| \subseteq C([n], d+1)$ with the simplicial subcomplex defining the lower triangulation

$$p(|\Delta^{\mathcal{T}_\ell}|) \subseteq C([n], d).$$

Thus, we obtain a commutative diagram

$$\begin{array}{ccc} \Delta^{\mathcal{T}} & \xrightarrow{\iota} & \Delta^n \\ \kappa \uparrow & \nearrow \iota_\ell & \\ \Delta^{\mathcal{T}_\ell} & & \end{array} \quad (3.2)$$

of simplicial sets, in which by definition, $\iota \in \mathcal{S}_{d+1}$ and $\iota_\ell \in \mathcal{S}_d$. That is, \mathcal{S}_{d+1} denotes the set of $(d+1)$ -Segal coverings (cf. [6], §5.2), while \mathcal{S}_d denotes the set of lower d -Segal coverings, so that \mathcal{S}_d -local objects are precisely the lower d -Segal objects.

In order to deduce $(d+1)$ -Segal descent for X , we have to show that $\iota \in \overline{\mathcal{S}}_d$, by which we denote the set of \mathcal{S}_d -equivalences, by the tautological part of Proposition 3.5.

By the 2-out-of-3 property of $\overline{\mathcal{S}}_d$, it suffices to show that $\kappa \in \overline{\mathcal{S}}_d$. We will do so by showing that κ can be obtained as an iterated pushout along morphisms in \mathcal{S}_d .

By Remark 2.9, the triangulation $|\Delta^{\mathcal{T}}|$ of $C([n], d+1)$ contains a maximal $(d+1)$ -simplex of the form $|\Delta^I|$, defined by a singleton collection $\{I\} \subseteq \binom{[n]}{d+2}$. Let \mathcal{I}_ℓ be the set which defines the lower facets of $|\Delta^I|$, defining a triangulation $|\Delta^{\mathcal{I}_\ell}|$ of $C(I, d)$. Then the inclusion of simplicial sets

$$\kappa_\ell: \Delta^{\mathcal{I}_\ell} \longrightarrow \Delta^I$$

is contained in \mathcal{S}_d . Further, since $|\Delta^I|$ is a maximal simplex, we have a pushout diagram

$$\begin{array}{ccc} \Delta^{\mathcal{T}} & \longleftarrow & \Delta^I \\ \kappa^{(1)} \uparrow & \lrcorner & \uparrow \kappa_\ell \\ \Delta^{\mathcal{T}^{(1)}} & \longleftarrow & \Delta^{\mathcal{I}_\ell} \end{array}$$

of simplicial sets, where $\mathcal{T}^{(1)} = \mathcal{T} \setminus \{I\}$. Thus, the map $\kappa^{(1)}$ lies in $\overline{\mathcal{S}}_d$, and $|\Delta^{\mathcal{T}^{(1)}}|$ is an admissible simplicial subcomplex of $C([n], d+1)$ with one $(d+1)$ -simplex less than $|\Delta^{\mathcal{T}}|$.

Assume that the triangulation $|\Delta^{\mathcal{T}}|$ consists of exactly r simplices of dimension $(d+1)$. By iterating the argument just given, we obtain a chain of morphisms

$$\Delta^{\mathcal{T}_\ell} \hookrightarrow \Delta^{\mathcal{T}^{(r-1)}} \hookrightarrow \dots \hookrightarrow \Delta^{\mathcal{T}^{(1)}} \hookrightarrow \Delta^{\mathcal{T}}$$

in $\overline{\mathcal{S}}_d$ whose composite is the morphism κ from (3.2). Thus, it is also contained in $\overline{\mathcal{S}}_d$. \square

4. THE HIGHER SEGAL CONSTRUCTION

Let \mathcal{D} be a pointed category with finite products. In this section, we study a generalization of a construction due to Segal [18] which is similar to a construction proposed (for $k = 2$) by Hesselholt and Madsen [12]. The higher dimensional variants (for $k \geq 3$) are straightforward to define, but do not seem to have appeared in the literature as of yet.

Let Fin_* denote the category of finite pointed sets. For $T \in \text{Fin}_*$, we denote by $\mathcal{P}(T)$ its poset of pointed subsets, considered as a small category.

Definition 4.1. Let $T \in \text{Fin}_*$ be a finite pointed set. A \mathcal{D} -valued presheaf

$$\mathcal{F}: \mathcal{P}(T)^{\text{op}} \longrightarrow \mathcal{D}$$

on $\mathcal{P}(T)$ is called a sheaf if, for every pointed subset $U \subseteq T$, the canonical map

$$\mathcal{F}(U) \longrightarrow \prod_{u \in U \setminus \{*\}} \mathcal{F}(\{*, u\})$$

is an isomorphism. We denote by $\text{Sh}(T, \mathcal{D})$ the category of \mathcal{D} -valued sheaves on $\mathcal{P}(T)$.

Given a map $\rho: T \rightarrow T'$ in Fin_* , we define the pointed preimage functor

$$\rho^\times: \mathcal{P}(T') \longrightarrow \mathcal{P}(T), \quad U \longmapsto \rho^{-1}(U \setminus \{*\}) \amalg \{*\}.$$

Then the direct image functor $\mathcal{F} \mapsto \rho_* \mathcal{F} = \mathcal{F} \circ \rho^\times$ makes the assignment

$$\text{Sh}(-, \mathcal{D}): \text{Fin}_* \longrightarrow \text{Cat}$$

into a functor with values in the category of small categories.

Definition 4.2. Let $k \geq 1$. The k -dimensional Segal construction of \mathcal{D} is defined to be the simplicial category

$$S_{\oplus}^{(k)}(\mathcal{D}) = \text{Sh}(S^k, \mathcal{D}) \in \text{Cat}_{\Delta},$$

where $S^k = \Delta^k / \partial \Delta^k$ is considered as a simplicial object in Fin_* .

The goal of this section is to prove the following result, which is due to Segal [18], §2, for $k = 1$. Throughout, a lower, resp. upper, d -Segal category means a lower, resp. upper, d -Segal object in Cat , which is not to be confused with a Segal category in the sense of [4].

Theorem 4.3. *Let $k \geq 1$, and \mathcal{D} a pointed category with finite products. The k -dimensional Segal construction $S_{\oplus}^{(k)}(\mathcal{D})$ is a lower $(2k-1)$ -Segal category. In particular, it is fully $2k$ -Segal.*

Proof. The last part is an application of Proposition 3.10. Now let $n \geq 2k - 1$, and set

$$\mathcal{L} = \mathcal{L}([n], 2k - 1).$$

For $I, J \in \mathcal{L}$ with $I \supseteq J$, we denote by $\rho_{I,J}: S_I^k \rightarrow S_J^k$ the corresponding map of pointed sets, and further write $\rho_I = \rho_{[n],I}$ for brevity. We have to show that the canonical functor

$$\text{Sh}(S_n^k, \mathcal{D}) \longrightarrow \varprojlim_{I \in \mathcal{L}} \text{Sh}(S_I^k, \mathcal{D}) \quad (4.1)$$

is an equivalence of categories. Note that all the transition maps on the right-hand side are isofibrations; given $\Phi: (\rho_{I,J})_* \mathcal{F} \xrightarrow{\sim} \mathcal{G}'$, we extend $\mathcal{G}(\rho_{I,J}^\times V) := \mathcal{G}'(V)$ for $V \in \mathcal{P}(S_J^k)$ by

$$\mathcal{G}(U) := \prod_{\substack{\alpha \in U \\ \alpha|_J \neq *}} \mathcal{F}(\{*, \alpha\})$$

otherwise. This is functorial, since \mathcal{D} is pointed, and so Φ exhibits $\mathcal{G}(\rho_{I,J}^\times V)$ as the product

$$\mathcal{G}(\rho_{I,J}^\times V) = \prod_{\alpha|_J \in V \setminus \{*\}} \mathcal{F}(\{*, \alpha\}).$$

The lifting of Φ itself is then tautological. Now consider the functor

$$\mathcal{P}: \mathcal{L} \longrightarrow \text{Cat}, \quad I \longmapsto \mathcal{P}(S_I^k).$$

We form the following version of its Grothendieck construction

$$\pi: \mathfrak{X}_{\mathcal{P}} \longrightarrow \mathcal{L}^{\text{op}}.$$

The category $\mathfrak{X}_{\mathcal{P}}$ has objects (I, U) , where $I \in \mathcal{L}$ and $U \in \mathcal{P}(S_I^k)$, and there is a unique morphism $(I, U) \leq (J, V)$ if $I \supseteq J$ and $U \subseteq \rho_{I,J}^{\times} V$. The functor π is a cartesian fibration, where a morphism $(I, U) \leq (J, V)$ is cartesian if

$$U = \rho_{I,J}^{\times} V.$$

The category $\varprojlim_{I \in \mathcal{L}} \text{Sh}(S_I^k, \mathcal{D})$ can be identified with the full subcategory of $\text{Fun}(\mathfrak{X}_{\mathcal{P}}^{\text{op}}, \mathcal{D})$ spanned by those presheaves \mathcal{F} which satisfy the following conditions.

- (a) The presheaf \mathcal{F} maps cartesian morphisms to isomorphisms in \mathcal{D} .
- (b) For every $I \in \mathcal{L}$, the restriction of \mathcal{F} to the fibre $\pi^{-1}(I) = \mathcal{P}(S_I^k)$ is a sheaf.

A \mathcal{D} -valued presheaf on $\mathcal{P}(S_n^k)$ defines a presheaf on $\mathfrak{X}_{\mathcal{P}}$ via pullback along the functor

$$\varphi_0: \mathfrak{X}_{\mathcal{P}} \longrightarrow \mathcal{P}(S_n^k), (I, U) \longmapsto \rho_I^{\times} U.$$

The lower Segal functor (4.1) is then obtained by restricting this pullback functor along φ_0 to the category of sheaves on $\mathcal{P}(S_n^k)$.

Since the functor φ_0 maps cartesian morphisms to the identity map in $\mathcal{P}(S_n^k)$, it factors over a unique functor $\varphi: L\mathfrak{X}_{\mathcal{P}} \rightarrow \mathcal{P}(S_n^k)$, where $L\mathfrak{X}_{\mathcal{P}}$ denotes the localization of $\mathfrak{X}_{\mathcal{P}}$ along the set of cartesian morphisms. Note further that imposing condition (a) on a presheaf \mathcal{F} on the category $\mathfrak{X}_{\mathcal{P}}$ is equivalent to the requirement that \mathcal{F} factors through $L\mathfrak{X}_{\mathcal{P}}$.

We obtain an adjunction of presheaf categories as follows,

$$\varphi!: \mathcal{D}_{L\mathfrak{X}_{\mathcal{P}}} \longleftarrow \mathcal{D}_{\mathcal{P}(S_n^k)} : \varphi^* \quad (4.2)$$

where the functor φ^* maps the subcategory of sheaves to the subcategory $\varprojlim_{I \in \mathcal{L}} \text{Sh}(S_I^k, \mathcal{D})$. Finally, we introduce the sheafification functor

$$\sigma: \mathcal{D}_{\mathcal{P}(S_n^k)} \longrightarrow \text{Sh}(S_n^k, \mathcal{D})$$

as the left adjoint to the inclusion. A priori, this may require \mathcal{D} to have pushouts; however, this assumption is shown to be unnecessary in Remark 7.6. Now (4.2) induces an adjunction,

$$\sigma \circ \varphi!: \varprojlim_{I \in \mathcal{L}} \text{Sh}(S_I^k, \mathcal{D}) \longleftarrow \text{Sh}(S_n^k, \mathcal{D}) : \varphi^* \quad (4.3)$$

which we claim to be a pair of mutually inverse functors. In order to verify this, we show that the unit and counit are isomorphisms. For the former, it suffices to show that, for every sheaf $\mathcal{G} \in \text{Sh}(S_n^k, \mathcal{D})$ and every subset $\{*, \alpha\} \subseteq S_n^k$ of cardinality 2, the unit morphism

$$(\varphi! \varphi^* \mathcal{G})(\{*, \alpha\}) \longrightarrow \mathcal{G}(\{*, \alpha\})$$

is invertible. We have

$$(\varphi! \varphi^* \mathcal{G})(\{*, \alpha\}) \cong \varinjlim_{\substack{(I, U) \in L\mathfrak{X}_{\mathcal{P}}^{\text{op}} \\ \alpha \in \rho_I^{\times}(U)}} \mathcal{G}(\rho_I^{\times}(U)).$$

According to Lemma 4.4 (1) below, the indexing category $\varphi^{\text{op}}/\{*, \alpha\}$ of the colimit has a final object $(I_{\alpha}, \{*, \alpha|_{I_{\alpha}}\})$, with $\rho_{I_{\alpha}}^{\times}(\{*, \alpha|_{I_{\alpha}}\}) = \{*, \alpha\}$. This immediately implies the claim.

We proceed to prove that, for every object $\mathcal{F} \in \varprojlim_{I \in \mathcal{L}} \text{Sh}(S_I^k, \mathcal{D})$, the counit morphism

$$\mathcal{F} \longrightarrow \varphi^* \sigma \varphi! \mathcal{F}$$

is invertible. Similarly as above, it suffices to show for all $(J, \{*, \beta\}) \in L\mathfrak{X}_{\mathcal{P}}$ that the map

$$\mathcal{F}(J, \{*, \beta\}) \longrightarrow (\varphi^* \sigma \varphi! \mathcal{F})(J, \{*, \beta\}) \quad (4.4)$$

is an isomorphism in \mathcal{D} . Using Lemma 4.4 (1), we compute the right-hand side as

$$(\sigma \varphi! \mathcal{F})(\rho_J^{\times} \{*, \beta\}) \cong \prod_{\alpha \in \rho_J^{-1}(\beta)} (\varphi! \mathcal{F})(\{*, \alpha\}) \cong \prod_{\alpha \in \rho_J^{-1}(\beta)} \mathcal{F}(I_{\alpha}, \{*, \alpha|_{I_{\alpha}}\}).$$

Then Lemma 4.4 (2) implies in particular that the map (4.4) is indeed an isomorphism. \square

Lemma 4.4. *In the terminology introduced in the proof of Theorem 4.3, let (J, V) be an object of the category $L\mathfrak{X}_{\mathcal{P}}$. Then the following statements hold.*

- (1) *Let $\alpha \in \rho_J^{\times}(V) \setminus \{*\}$. There is a unique morphism*

$$(I_{\alpha}, \{*, \alpha|_{I_{\alpha}}\}) \longrightarrow (J, V)$$

in $L\mathfrak{X}_{\mathcal{P}}$, where

$$I_{\alpha} = \bigcup_{\alpha_i < \alpha_{i+1}} \{i, i+1\}.$$

- (2) *Let \mathcal{F} be an object of $\varprojlim_{I \in \mathcal{L}} \text{Sh}(S_I^k, \mathcal{D}) \subseteq \mathcal{D}_{L\mathfrak{X}_{\mathcal{P}}}$. There is an isomorphism*

$$\mathcal{F}(J, V) \xrightarrow{\sim} \prod_{\alpha \in \rho_J^{\times}(V) \setminus \{*\}} \mathcal{F}(I_{\alpha}, \{*, \alpha|_{I_{\alpha}}\}),$$

whose components are given by restriction along the unique morphisms from (1).

Proof. The even subsets of $[n]$ of cardinality $2k$ are precisely the disjoint unions of k subsets of the form $\{i, i+1\}$. Since $\alpha \neq *$, it follows that I_{α} is a (possibly non-disjoint) union of k such subsets. However, it is contained in the even subset of $[n]$ of cardinality $2k$ obtained by inductively filling for each $\alpha_{i-1} < \alpha_i < \alpha_{i+1}$ either the maximal gap $j < i$ of I_{α} or its minimal gap $j > i$.

The key observation is that the subsets $I \in \mathcal{L}$ which contain I_{α} are exactly those with

$$\rho_I^{\times}(\rho_I(\{*, \alpha\})) = \{*, \alpha\}.$$

This implies that for a morphism in $L\mathfrak{X}_{\mathcal{P}}$ of the form $(I_{\alpha}, \{*, \alpha|_{I_{\alpha}}\}) \leftarrow (I, U) \rightarrow (J', V')$, we always have $U = \{*, \alpha|_I\}$. Thus, the only condition on V' is that $\alpha|_{J'} \in V'$, and we can assume without loss of generality that $V = \{*, \alpha|_J\}$. In order to describe morphisms

$$\mu: (I_{\alpha}, \{*, \alpha|_{I_{\alpha}}\}) \longrightarrow (J, \{*, \alpha|_J\})$$

in $L\mathfrak{X}_{\mathcal{P}}$, we consider α as a sequence of k bars situated in a diagram of $[n]$, signifying the fact that $\alpha_j < \alpha_{j+1}$ by the bar $j|(j+1)$. An object $(I, \{*, \alpha|_I\})$ corresponds to marking the elements $i \in I \subseteq [n]$, and the zig-zag μ is a sequence of moves which shift the markings. Each move consists of adding and then removing certain markings (adhering to the constraints imposed by the definition of $L\mathfrak{X}_{\mathcal{P}}$).

The object $(I_{\alpha}, \{*, \alpha|_{I_{\alpha}}\})$ marks all elements adjacent to a bar; that is, we visualize it as a diagram of the following exemplary form,

$$- - \bullet | \bullet - \bullet | \bullet | \bullet - - \bullet | \dots | \bullet - - -$$

where ' \bullet ' indicates a marked element and ' $-$ ' an unmarked element of $[n]$. A single ' \bullet ' at a vertex $i \in [n]$ between two bars (i.e., ' $\bullet|$ ') corresponds precisely to the case $\alpha_{i-1} < \alpha_i < \alpha_{i+1}$ from above. Since $\alpha|_J \neq *$, this implies that $i \in J$; in fact, this condition states exactly that there is an element of J in every region cut out by the bars.

In order to see that μ is unique (if it exists), we first note that a ' \bullet ' can never cross a bar. Indeed, this would require a move

$$(I, \{*, \alpha|_I\}) \longleftarrow (H, \{*, \alpha|_H\}) \longrightarrow (\dots)$$

which adds a marking to some $i \in I_{\alpha}$. Then we can define $\beta \in \mathcal{P}(S_H^k)$ by $\beta|_{H \setminus \{i\}} \equiv \alpha|_{H \setminus \{i\}}$ and by replacing the jump $\alpha_{i-1} = \alpha_i < \alpha_{i+1}$ with $\beta_{i-1} < \beta_i = \beta_{i+1}$; but this contradicts the requirement that the left leg of the move be cartesian.

Then uniqueness follows from the fact that moves which are constrained within different sets of bars commute with one another, while the moves occurring between two particular bars all compose to the same shift of markings.

For the existence of μ , we observe that after adding markings for each ' $\bullet|$ ' as described above (filling the gaps of I_{α} ; where we can always choose the gap closest to an element of J), we can remove at least one marking adjacent to each bar (with the exception of the ' $\bullet|$ ', in which case the vertex lies in J already, as we have seen).

Then we can move each ' \bullet ' towards its intended position in J by repeatedly marking the adjacent vertex and removing the original; moreover, once a ' \bullet ' has reached its destination, we can duplicate it. This requires no further sets of the form $\{i, i+1\}$ to cover all markings, that is, we stay within \mathcal{L} in this process (as of course $J \in \mathcal{L}$ itself).

Finally, statement (2) follows from the above, since $\mathcal{F}(J, V) = \mathcal{F}(\rho_J^\times V)$, and similarly,

$$\mathcal{F}(I_\alpha, \{*, \alpha|_{I_\alpha}\}) = \mathcal{F}(\{*, \alpha\});$$

but condition (b) tells us that the restriction $\mathcal{F}|_{\mathcal{P}(S^k)}$ to the fibre $\pi^{-1}(J)$ is a sheaf. \square

5. STRINGENT CATEGORIES

Let \mathcal{E} be a proto-exact category ([6], Definition 2.4.2; for example, any exact category). In this self-contained section, we carry out some homological algebraic preparations for the proof of the main result in §6, which in particular requires no additive structure on \mathcal{E} .

Definition 5.1. A morphism $A \rightarrow B$ in \mathcal{E} is called admissible if it factors as the composition

$$\begin{array}{ccc} A & \xrightarrow{\quad} & B \\ & \searrow & \nearrow \\ & C & \end{array}$$

of an admissible epimorphism and an admissible monomorphism in \mathcal{E} .

Consider a sequence of admissible morphisms together with their corresponding (unique up to unique isomorphism) factorizations as above,

$$\begin{array}{ccccccc} A_k & \xrightarrow{\quad} & A_{k-1} & \xrightarrow{\quad} & \dots & \xrightarrow{\quad} & A_0 \\ & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow \\ & C_k & & C_{k-1} & & \dots & C_1 \end{array}$$

The sequence $A_k \rightarrow A_{k-1} \rightarrow \dots \rightarrow A_0$ will be called

- acyclic, if $C_{i+1} \rightarrow A_i \rightarrow C_i$ is a short exact sequence in \mathcal{E} for all $0 < i < k$.

An acyclic sequence in \mathcal{E} as above is called

- left exact, if $A_k \rightarrow A_{k-1}$ is an admissible monomorphism (equivalently, $A_k \xrightarrow{\sim} C_k$),
- right exact, if $A_1 \rightarrow A_0$ is an admissible epimorphism (i.e., $C_1 \xrightarrow{\sim} A_0$),
- exact, if it is both left exact and right exact.

Lemma 5.2. *Let \mathcal{E} be a pointed category. The following conditions are equivalent.*

- There exists a proto-exact structure on \mathcal{E} , in which a morphism is admissible only if it admits a kernel or a cokernel.*
- The class of all kernel-cokernel pairs defines a proto-exact structure on \mathcal{E} , and a morphism in \mathcal{E} is admissible if and only if it admits a kernel and a cokernel.*
- The pushout of a kernel by a cokernel exists in \mathcal{E} and is a kernel again, and the pullback of a cokernel by a kernel exists in \mathcal{E} and is a cokernel again. Furthermore, if a map f in \mathcal{E} admits a kernel and a cokernel, then it is strict, that is, it factors as the composition of a cokernel and a kernel. Equivalently, the natural map*

$$\text{coim}(f) \rightarrow \text{im}(f)$$

is an isomorphism.

Proof. Assume (i). The proto-exact structure on \mathcal{E} necessarily contains all kernel-cokernel pairs, since kernels and cokernels are admissible morphisms. Moreover, admissible morphisms admit kernels and cokernels, implying (ii). The converse is tautological.

Given (iii), we need to see that the class of all kernel-cokernel pairs defines a proto-exact structure on \mathcal{E} . Indeed, a composition of cokernels $B \twoheadrightarrow B' \twoheadrightarrow C$ admits a kernel

$$\begin{array}{ccccc} A & \twoheadrightarrow & A' & \twoheadrightarrow & 0 \\ \downarrow & \lrcorner & \downarrow & \square & \downarrow \\ B & \twoheadrightarrow & B' & \twoheadrightarrow & C \end{array}$$

and therefore is a cokernel in \mathcal{E} . Then we can conclude that (iii) \Leftrightarrow (ii) by definition.

Finally, the equivalence in (iii) is immediate from the fact that a factorization as in (iii) is unique (up to unique isomorphism). \square

Definition 5.3. A pointed category \mathcal{E} satisfying the equivalent conditions in Lemma 5.2 will be called a stringent category.

Remark 5.4. In [5], Definition 1.2, a proto-abelian category is defined as a pointed category on which the classes of all monomorphisms and epimorphisms define a proto-exact structure. For us, it will prove convenient to change this terminology slightly by additionally requiring the existence of all kernels and cokernels (rather than introducing another different term). This does not exclude any of the main examples of interest (like Example 5.5 (1) below).

Example 5.5. A pointed category which admits all kernels and cokernels is stringent if and only if it is proto-abelian. Similarly, a pre-abelian category is stringent if and only if it is abelian.

- (1) In particular, the category of (finite) \mathbb{F}_1 -vector spaces is stringent.
- (2) Consider the pointed category \mathcal{E} on a non-zero object V with $\text{End}(V) = \{0, 1, \varepsilon\}$, such that $\varepsilon^2 = 0$. It can easily be verified directly that ε admits neither a kernel nor a cokernel, and thus \mathcal{E} is stringent.

If F is a field, then the F -linear Cauchy completion of \mathcal{E} is an additive stringent category, namely the category of finite free $F[x]/(x^2)$ -modules.

Remark 5.6. An additive category \mathcal{E} is stringent if and only if the class of all kernel-cokernel pairs defines an exact structure on \mathcal{E} and a morphism in \mathcal{E} is admissible if and only if it admits a kernel and a cokernel. Similarly, the other conditions in Lemma 5.2 have evident additive analogues.

An additive stringent category \mathcal{E} is in particular weakly idempotent complete, in the sense of [2], Proposition 7.6. In fact, \mathcal{E} satisfies the stronger statement of [17], Proposition 1.1.8, whose proof applies verbatim. Indeed, note that if a composition of the form $A' \twoheadrightarrow A \xrightarrow{f} B$ is admissible, then its cokernel

$$\begin{array}{ccccc} A' & \twoheadrightarrow & A & \xrightarrow{f} & B \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \twoheadrightarrow & 0 & \twoheadrightarrow & C \end{array} \quad (5.1)$$

also defines a cokernel for f , which is therefore strict, and thus admits a kernel. In particular, the snake lemma for admissible morphisms holds in \mathcal{E} , as shown in [2], Corollary 8.13.

In fact, the snake lemma holds in any stringent category \mathcal{E} . The neat argument presented in *loc.cit.* does not quite apply here (as it ultimately relies on the additive structure of an exact category); however, the proof of [10], Proposition 4.3, does apply.

For this, we need to verify Heller's axioms for \mathcal{E} (cf. [2], Proposition B.1; with the obvious exception of additivity). Since cancellation follows directly from (5.1) by the coimage-image isomorphism, it only remains to prove the following result.

Proposition 5.7. *Let \mathcal{E} be a stringent category, and consider a diagram of the form*

$$\begin{array}{ccccc} A' & \twoheadrightarrow & B' & \twoheadrightarrow & C' \\ \downarrow & & \downarrow & & \downarrow \\ A & \twoheadrightarrow & B & \twoheadrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ A'' & \twoheadrightarrow & B'' & \twoheadrightarrow & C'' \end{array}$$

in \mathcal{E} , where all rows as well as all columns but the first are exact. Then $A' \twoheadrightarrow A \twoheadrightarrow A''$ is a short exact sequence as well.

Proof. First of all, by cancellation, $A' \twoheadrightarrow A$. Then we have the following cartesian squares.

$$\begin{array}{cccc} \begin{array}{ccc} A' & \twoheadrightarrow & B' \\ \downarrow & \lrcorner & \downarrow \\ 0 & \twoheadrightarrow & C' \\ \downarrow & \lrcorner & \downarrow \\ 0 & \twoheadrightarrow & C \end{array} & \begin{array}{ccc} A' & \twoheadrightarrow & B' \\ \downarrow & & \downarrow \\ A & \twoheadrightarrow & B \\ \downarrow & \lrcorner & \downarrow \\ 0 & \twoheadrightarrow & C \end{array} & \begin{array}{ccc} A' & \twoheadrightarrow & A \\ \downarrow & & \downarrow \\ B' & \twoheadrightarrow & B \\ \downarrow & \lrcorner & \downarrow \\ 0 & \twoheadrightarrow & B'' \end{array} & \begin{array}{ccc} A' & \twoheadrightarrow & A \\ \downarrow & & \downarrow \\ 0 & \twoheadrightarrow & A'' \\ \downarrow & \lrcorner & \downarrow \\ 0 & \twoheadrightarrow & B'' \end{array} \end{array}$$

The first diagram implies that the outer rectangle of the second is a pullback, hence so is its upper square, which agrees with the upper square of the third, implying that the outer rectangle of the fourth is a pullback, and thus its upper square. Therefore,

$$\operatorname{coker}(A' \twoheadrightarrow A) \xrightarrow{\sim} \operatorname{coker}(\ker(A \twoheadrightarrow A'')) \xrightarrow{\sim} \ker(\operatorname{coker}(A \twoheadrightarrow A'')) \twoheadrightarrow A''.$$

On the other hand, dually, we have the following cocartesian squares,

$$\begin{array}{ccc} B' & \twoheadrightarrow & C' \twoheadrightarrow C \\ \downarrow & \lrcorner & \downarrow \lrcorner \\ 0 & \twoheadrightarrow & 0 \twoheadrightarrow C'' \end{array} \quad \begin{array}{ccc} B' & \twoheadrightarrow & B \twoheadrightarrow C \\ \downarrow & \lrcorner & \downarrow \lrcorner \\ 0 & \twoheadrightarrow & B'' \twoheadrightarrow C'' \end{array}$$

which together imply that the right-hand square of the second diagram is again a pushout. But then the dual of Lemma 5.8 below tells us that $A \twoheadrightarrow A''$. \square

Lemma 5.8. *Let \mathcal{E} be a stringent category, and consider a pullback square of admissible morphisms in \mathcal{E} of the following form.*

$$\begin{array}{ccc} B_1 & \twoheadrightarrow & A_1 \\ \downarrow & \lrcorner & \downarrow \\ B_0 & \twoheadrightarrow & A_0 \end{array} \tag{5.2}$$

The induced map $C_1 \twoheadrightarrow C_0$ is an admissible monomorphism, where $C_i = \operatorname{coker}(B_i \twoheadrightarrow A_i)$.

Proof. We have the following composition of cartesian squares.

$$\begin{array}{ccc} B_1 & \twoheadrightarrow & A_1 \\ \downarrow & \lrcorner & \downarrow \\ B_0 & \twoheadrightarrow & A_0 \\ \downarrow & \lrcorner & \downarrow \\ 0 & \twoheadrightarrow & C_0 \end{array}$$

Thus, the composition $A_1 \rightarrow A_0 \twoheadrightarrow C_0$ is admissible with kernel B_1 , and therefore admits a factorization $A_1 \twoheadrightarrow C_1 \twoheadrightarrow C_0$, which fits uniquely into the diagram

$$\begin{array}{ccc} A_1 & \twoheadrightarrow & C_1 \\ \downarrow & & \downarrow \\ A_0 & \twoheadrightarrow & C_0 \end{array}$$

yielding the claim. \square

6. THE HIGHER WALDHAUSEN CONSTRUCTION

Let \mathcal{E} be a proto-exact category, with its subcategory of admissible monomorphisms, resp. admissible epimorphisms, denoted by $\mathcal{E}^\triangleleft$, resp. $\mathcal{E}^\triangleright$.

Let $k, n \geq 0$. We write $\text{Fun}([k], [n])$ for the category of functors between the standard ordinals $[k]$ and $[n]$, considered as small categories. Note that the objects of this category correspond bijectively to the set of k -simplices of the simplicial set Δ^n .

Definition 6.1. Let $k \geq 0$. For every $n \geq 0$, we define the category

$$S_n^{[k]}(\mathcal{E}) \subseteq \text{Fun}(\text{Fun}([k], [n]), \mathcal{E})$$

to be the full subcategory spanned by all diagrams A satisfying the following conditions.

(a) For every $(k-1)$ -simplex α in Δ^n , we have

$$A_{s_{k-1}^* \alpha} = \dots = A_{s_0^* \alpha} = 0.$$

(b) For every $(k+1)$ -simplex γ in Δ^n , the corresponding sequence

$$A_{d_{k+1}^* \gamma} \rightarrow A_{d_k^* \gamma} \rightarrow \dots \rightarrow A_{d_1^* \gamma} \rightarrow A_{d_0^* \gamma}$$

is acyclic.

We define $S_n^{\langle k \rangle}(\mathcal{E})$, resp. $S_n^{[k]}(\mathcal{E})$, as the full subcategory of $S_n^{[k]}(\mathcal{E})$ on all A such that

(b') For every $(k+1)$ -simplex γ in Δ^n , the following sequence is left, resp. right, exact.

$$A_{d_{k+1}^* \gamma} \rightarrow A_{d_k^* \gamma} \rightarrow \dots \rightarrow A_{d_1^* \gamma} \rightarrow A_{d_0^* \gamma}$$

Finally, we introduce $S_n^{\langle k \rangle}(\mathcal{E}) \subseteq S_n^{[k]}(\mathcal{E})$ as the full subcategory of diagrams A which satisfy

(b'') For every $(k+1)$ -simplex γ in Δ^n , the sequence

$$A_{d_{k+1}^* \gamma} \rightarrow A_{d_k^* \gamma} \rightarrow \dots \rightarrow A_{d_1^* \gamma} \rightarrow A_{d_0^* \gamma}$$

is exact.

By functoriality in $[n]$, we obtain simplicial categories

$$S^{\langle k \rangle}(\mathcal{E}), S^{\langle k \rangle}(\mathcal{E}), S^{[k]}(\mathcal{E}), S^{[k]}(\mathcal{E}) \in \text{Cat}_\Delta.$$

We call $S^{\langle k \rangle}(\mathcal{E})$ the k -dimensional Waldhausen construction of \mathcal{E} .

Remark 6.2. The $(k+1)$ -skeleton of $S^{\langle k \rangle}(\mathcal{E})$ has an immediate description. Namely,

$$S_0^{\langle k \rangle}(\mathcal{E}) = \dots = S_{k-1}^{\langle k \rangle}(\mathcal{E}) = 0, S_k^{\langle k \rangle}(\mathcal{E}) \cong \mathcal{E},$$

and $S_{k+1}^{\langle k \rangle}(\mathcal{E})$ is equivalent to the category of k -extensions in \mathcal{E} . The dimensionality of the Waldhausen construction refers to the k -skeleton of $|S^{\langle k \rangle}(\mathcal{E})^\times|$ being equivalent to $S^k \wedge |\mathcal{E}^\times|$.

Lemma 6.3. Let $k, n \geq 0$, let $0 \leq i \leq n$, and let \mathcal{E} be a proto-exact category. The face map

$$\partial_i: S_n^{\langle k \rangle}(\mathcal{E}) \rightarrow S_{n-1}^{\langle k \rangle}(\mathcal{E})$$

of the higher Waldhausen construction of \mathcal{E} is an isofibration.

Proof. Let $\Phi: \partial_i(A) \xrightarrow{\sim} B'$ be an isomorphism in $S_{n-1}^{(k)}(\mathcal{E})$. We construct a lift $B \in S_n^{(k)}(\mathcal{E})$ of B' as follows,

$$B: ([k] \xrightarrow{\beta} [n]) \mapsto \begin{cases} A_\beta & \text{if } \beta \notin \text{im}(d_i)_*, \\ B'_\alpha & \text{if } \beta = (d_i)_*\alpha. \end{cases}$$

The map in B for $\beta \leq \tilde{\beta}$ is given by the corresponding arrow in A , resp. B' , if $\beta, \tilde{\beta} \notin \text{im}(d_i)_*$, resp. both $\beta = (d_i)_*\alpha$ and $\tilde{\beta} = (d_i)_*\tilde{\alpha}$. Otherwise, we define

$$(B_\beta \rightarrow B_{\tilde{\beta}}) = \begin{cases} (A_\beta \rightarrow A_{\tilde{\beta}} \xrightarrow{\Phi} B'_\alpha) & \text{if } \beta \notin \text{im}(d_i)_* \text{ and } \tilde{\beta} = (d_i)_*\tilde{\alpha}, \\ (B'_\alpha \xleftarrow{\Phi} A_\beta \rightarrow A_{\tilde{\beta}}) & \text{if } \beta = (d_i)_*\alpha \text{ and } \tilde{\beta} \notin \text{im}(d_i)_*. \end{cases}$$

The lifting of Φ itself is then straightforward. \square

Remark 6.4. In particular, all the limits with transition maps given by compositions of ∂_i we consider throughout are computed by the respective 1-categorical limits.

This also applies to Theorem 7.7, whose proof in turn can be used to provide an alternative argument for Lemma 6.3 by reducing to the well-known case $k = 1$ (which can further be reduced to $k = 0$ via the equivalences of categories $S_{n-1}^{(0)}(\mathcal{E}) \cong S_n^{(1)}(\mathcal{E}) \cong S_{n-1}^{(0)}(\mathcal{E})$).

Example 6.5. Let $k \geq 0$, and let \mathcal{E} be a proto-exact category.

- (1) For $k = 0$, the degeneracy condition (a) is empty, and therefore,

$$S^{(0)}(\mathcal{E}) \cong N^\mathcal{E}(\mathcal{E}^\times) \cong \mathcal{E}$$

is the nerve of the maximal subgroupoid of \mathcal{E} , categorified by arbitrary morphisms in \mathcal{E} , which is equivalent to the constant object \mathcal{E} itself. Similarly, $S^{(0)}(\mathcal{E}) \cong N^\mathcal{E}(\mathcal{E}^\triangleleft)$, and dually, $S^{(0)}(\mathcal{E}) \cong N^\mathcal{E}(\mathcal{E}^\triangleright)$. Rather more subtly, $S^{(0)}(\mathcal{E}) \cong N^\mathcal{E}(\mathcal{E})$ if and only if \mathcal{E} is proto-abelian (in the sense of Remark 5.4), by [8], Proposition 3.1.

- (2) For $k = 1$, we recover a version of the original construction $S^{(1)}(\mathcal{E}) = S(\mathcal{E})$ from [21], whose n -cells are given by the category formed by strictly upper triangular diagrams with bicartesian squares, as follows.

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_{01} & \longrightarrow & A_{02} & \longrightarrow & \dots & \longrightarrow & A_{0n} \\ & & \downarrow & \square & \downarrow & \square & \downarrow & \square & \downarrow \\ & & 0 & \longrightarrow & A_{12} & \longrightarrow & \dots & \longrightarrow & A_{1n} \\ & & & & \downarrow & \square & \downarrow & \square & \downarrow \\ & & & & 0 & \longrightarrow & \dots & \longrightarrow & \vdots \\ & & & & & & \downarrow & \square & \downarrow \\ & & & & & & 0 & \longrightarrow & A_{(n-1)n} \\ & & & & & & & & \downarrow \\ & & & & & & & & 0 \end{array} \quad (6.1)$$

This is a refinement of Quillen's foundational construction $Q(\mathcal{E}) = N^\mathcal{E}(q(\mathcal{E}))$ in [14], where $q(\mathcal{E})$ denotes the category of correspondences in \mathcal{E} of the form $C \leftarrow B \rightarrow A$. The forgetful functor from the edgewise subdivision $eS(\mathcal{E}) \rightarrow Q(\mathcal{E})$ is an equivalence, that is, the whole diagram $A \in S_{2n+1}(\mathcal{E})$ of shape (6.1) is uniquely recovered from

$$\begin{array}{ccccccc} & & A_{(n-1)(n+1)} & & \dots & & A_{0(2n)} & & \\ & \swarrow & & \swarrow & & \swarrow & & \swarrow & \\ A_{n(n+1)} & & & & A_{(n-1)(n+2)} & & \dots & & A_{1(2n)} & & & & A_{0(2n+1)} \end{array} \quad (6.2)$$

by taking successive pullbacks and pushouts in \mathcal{E} .

- (3) For $k = 2$, the simplicial category $S^{(2)}(\mathcal{E})$ was introduced by Hesselholt-Madsen [12]. An element $A \in S_4^{(2)}(\mathcal{E})$ of its 4-cells is a diagram of the following form.

$$\begin{array}{ccccccc}
 A_{012} & \xrightarrow{\quad} & A_{013} & \xrightarrow{\quad} & A_{014} & & \\
 & & \downarrow & \lrcorner & \downarrow & & \\
 & & A_{023} & \xrightarrow{\quad} & A_{024} & & \\
 & & \searrow & & \downarrow & \searrow & \\
 & & & A_{123} & \xrightarrow{\quad} & A_{124} & \\
 & & & \downarrow & & \downarrow & \\
 & & & & A_{034} & \searrow & \\
 & & & & & \downarrow & \\
 & & & & & & A_{134} \\
 & & & & & & \searrow \\
 & & & & & & & A_{234}
 \end{array} \tag{6.3}$$

Note that the middle square is neither cartesian nor cocartesian. Rather, the diagram consists of bicartesian cubes (cf. Remark 6.6), as indicated in the following picture.

$$\begin{array}{ccccccc}
 A_{012} & \xrightarrow{\quad} & A_{013} & \xrightarrow{\quad} & A_{014} & & \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow & \searrow & \\
 0 & \xrightarrow{\quad} & 0 & \xrightarrow{\quad} & 0 & \xrightarrow{\quad} & 0 \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow & \searrow & \\
 0 & \xrightarrow{\quad} & A_{023} & \xrightarrow{\quad} & A_{024} & \xrightarrow{\quad} & 0 \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow & \searrow & \\
 0 & \xrightarrow{\quad} & 0 & \xrightarrow{\quad} & A_{123} & \xrightarrow{\quad} & A_{124} \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow & \searrow & \\
 0 & \xrightarrow{\quad} & 0 & \xrightarrow{\quad} & A_{034} & \xrightarrow{\quad} & 0 \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow & \searrow & \\
 0 & \xrightarrow{\quad} & 0 & \xrightarrow{\quad} & 0 & \xrightarrow{\quad} & A_{134} \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow & \searrow & \\
 0 & \xrightarrow{\quad} & 0 & \xrightarrow{\quad} & 0 & \xrightarrow{\quad} & A_{234}
 \end{array} \tag{6.4}$$

Remark 6.6. In general, $S^{(k)}(\mathcal{E})$ is composed of $(k+1)$ -dimensional bicartesian hypercubes. More precisely, let $A \in S_n^{[k]}(\mathcal{E})$. Then A lies in $S_n^{(k)}(\mathcal{E})$ if and only if

$$A_\beta = \lim_{\substack{\leftarrow \\ \beta < \beta' \\ \beta' - \beta \leq 1}} A_{\beta'} \tag{6.5}$$

for every k -simplex β in Δ^n with $\beta_{k-i} < n-i$ for $0 \leq i \leq k$. Then Lemma 6.7 below implies that the dual condition defines $S_n^{[k]}(\mathcal{E})$ inside $S_n^{(k)}(\mathcal{E})$, so that $A \in S_n^{[k]}(\mathcal{E})$ if and only if

$$\varinjlim_{\substack{\leftarrow \\ \beta' < \beta \\ \beta - \beta' \leq 1}} A_{\beta'} = A_\beta$$

for every k -simplex β in Δ^n with $i < \beta_i$ for all $0 \leq i \leq k$. Together, these yield the claim for

$$S_n^{(k)}(\mathcal{E}) = S_n^{[k]}(\mathcal{E}) \times_{S_n^{[k]}(\mathcal{E})} S_n^{[k]}(\mathcal{E}).$$

In order to see the first statement, note that the sequence of admissible morphisms $A_{d_\bullet^* \gamma}$ corresponding to some $(k+1)$ -simplex γ in Δ^n defines a hypercube $\text{conv}(A_{d_\bullet^* \gamma})$, formed by all A_β with $d_{k+1}^* \gamma \leq \beta \leq d_0^* \gamma$. If the maps (6.5) are isomorphisms, the minimal subhypercubes of $\text{conv}(A_{d_\bullet^* \gamma})$ are cartesian, and hence so is $\text{conv}(A_{d_\bullet^* \gamma})$ as their composition. Therefore,

$$A_{d_{k+1}^* \gamma} = \lim_{\substack{\leftarrow \\ d_{k+1}^* \gamma < \beta \leq d_0^* \gamma}} A_\beta = \ker(A_{d_k^* \gamma} \rightarrow A_{d_{k-1}^* \gamma}).$$

Conversely, let β be a k -simplex in Δ^n with $\beta_{k-i} < n-i$ for all $0 \leq i \leq k$. Then we can infer inductively that the hypercube Q_β on all $\beta' \geq \beta$ with $\beta' - \beta \leq 1$ is cartesian. First, assume that $|\beta| = \sum \beta_i$ is maximal. Thus, $\beta = (d_0^*)^{n-k} \Delta_n^n - 1 = d_{k+1}^* (d_0^*)^{n-k-1} \Delta_n^n$. But then Q_β is exactly given by the hypercube $\text{conv}(A_{d_{k+1}^* (d_0^*)^{n-k-1} \Delta_n^n})$.

In general, consider $\gamma = \beta \amalg \{n\}$. Then $d_{k+1}^* \gamma = \beta \leq \beta + 1 \leq d_0^* \gamma$, and hence $\text{conv}(A_{d_{k+1}^* \gamma})$ contains Q_β entirely. But its complement is covered by hypercubes $Q_{\tilde{\beta}}$ with $|\tilde{\beta}| > |\beta|$, which are cartesian by induction. Therefore, so are all of their compositions, and thus so is Q_β .

The following observation makes the inherent symmetry by duality precise.

Lemma 6.7. *The duality on Δ induces equivalences of simplicial categories*

$$\begin{aligned} S^{(k)}(\mathcal{E}) &\longrightarrow S^{(k)}(\mathcal{E}^{\text{op}}), \\ S^{[k]}(\mathcal{E}) &\longrightarrow S^{[k]}(\mathcal{E}^{\text{op}}), \\ S^{[k]}(\mathcal{E}) &\longrightarrow S^{[k]}(\mathcal{E}^{\text{op}}). \end{aligned}$$

Proof. This is immediate from the definitions. \square

We can now state our main result.

Theorem 6.8. *Let $k \geq 0$, and let \mathcal{E} be a stringent category. The k -dimensional Waldhausen construction $S^{(k)}(\mathcal{E})$ is a fully $2k$ -Segal category.*

For $k = 1$, this is one of the main results of [6], namely Proposition 2.4.8. In light of work in progress by Bergner, Osorno, Ozornova, Rovelli, and Scheimbauer, another proof is provided by Example 6.5 (2), where we have seen that $eS^{(1)}(\mathcal{E})$ is lower 1-Segal.

The case $k = 0$ is settled by Example 6.5 (1).

When $k = 2$, by Proposition 3.10, we can deduce Theorem 6.8 from Theorem 6.9 below. Its proof also serves to outline our strategy for the main theorem (up to the use of the double path space rather than both single path spaces).

An analogue of Theorem 6.8 in the context of stable ∞ -categories is a result of work in progress by Dyckerhoff and Jasso.

Theorem 6.9. *Let \mathcal{E} be a proto-abelian category. Then $S^{(2)}(\mathcal{E})$ is an upper 3-Segal category.*

Proof. By the path space criterion, it suffices to show that $P^\triangleleft P^\triangleright S^{(2)}(\mathcal{E})$ is lower 1-Segal. But Proposition 6.14 below shows that for all $n \geq 2$, the forgetful functor

$$P^\triangleleft P^\triangleright S_{n-2}^{(2)}(\mathcal{E}) \longrightarrow S_{n-2}^{[0]}(\mathcal{E}), \quad A \longmapsto (A_{01n} \rightarrow A_{02n} \rightarrow \cdots \rightarrow A_{0(n-1)n}),$$

is an equivalence of categories, identifying the double path space $P^\triangleleft P^\triangleright S^{(2)}(\mathcal{E}) \xrightarrow{\sim} N^\mathcal{E}(\mathcal{E})$ with the categorified nerve of \mathcal{E} , by Example 6.5 (1). \square

In [6], the 1-dimensional Waldhausen construction is shown to be fully 2-Segal for any proto-exact category. However, when $k \geq 2$, our assumption is necessary, which we illustrate in Example 6.10 below, at least in the additive case.

Example 6.10. Suppose that \mathcal{E} is an exact category which is not stringent. Then $S^{(2)}(\mathcal{E})$ is not 4-Segal. Indeed, let us see that the lower 3-Segal condition on the 4-cells of the left path space $P^\triangleleft S^{(2)}(\mathcal{E})$ is violated.

By [8], Proposition 3.1, there is a morphism $f: A \rightarrow B$ in \mathcal{E} which is not strict. However, by [2], Remark 8.2, it can be written as the composition of strict morphisms

$$A \xrightarrow{(1,f)} A \oplus B \xrightarrow{(0,1)} B.$$

Now suppose f admits a kernel C and consider the following possible element of $P^{\triangleleft}S_4^{(2)}(\mathcal{E})$.

$$\begin{array}{ccccccc}
 0 & \twoheadrightarrow & 0 & \twoheadrightarrow & C & \twoheadrightarrow & C & \twoheadrightarrow & A \\
 & & \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow (1,f) \\
 & & 0 & \twoheadrightarrow & A & \twoheadrightarrow & A & \twoheadrightarrow & A \oplus B \\
 & & & & \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow (0,1) \\
 & & & & 0 & \twoheadrightarrow & 0 & \twoheadrightarrow & B \\
 & & & & & & \downarrow \lrcorner & & \downarrow \\
 & & & & & & 0 & \twoheadrightarrow & B \\
 & & & & & & & & \downarrow \\
 & & & & & & & & 0
 \end{array} \tag{6.6}$$

Then the triple of diagrams

$$\begin{array}{ccc}
 \begin{array}{ccc} 0 & \twoheadrightarrow & C \\ \downarrow \lrcorner & & \downarrow \lrcorner \\ 0 & \twoheadrightarrow & A \\ & & \downarrow \lrcorner \\ & & 0 \end{array} & \begin{array}{ccc} C & \twoheadrightarrow & C \\ \downarrow \lrcorner & & \downarrow \lrcorner \\ A & \twoheadrightarrow & A \\ & & \downarrow \lrcorner \\ & & 0 \end{array} & \begin{array}{ccc} 0 & \twoheadrightarrow & C \\ \downarrow \lrcorner & & \downarrow \lrcorner \\ 0 & \twoheadrightarrow & A \\ & & \downarrow \lrcorner \\ & & 0 \end{array} \\
 \begin{array}{ccc} C & \twoheadrightarrow & A \\ \downarrow \lrcorner & & \downarrow \lrcorner \\ A & \twoheadrightarrow & A \oplus B \\ & & \downarrow \lrcorner \\ & & 0 \end{array} & \begin{array}{ccc} C & \twoheadrightarrow & A \\ \downarrow \lrcorner & & \downarrow \lrcorner \\ A & \twoheadrightarrow & A \oplus B \\ & & \downarrow \lrcorner \\ & & 0 \end{array} & \begin{array}{ccc} A & \twoheadrightarrow & A \\ \downarrow \lrcorner & & \downarrow \lrcorner \\ 0 & \twoheadrightarrow & 0 \\ & & \downarrow \lrcorner \\ & & 0 \end{array} \\
 \begin{array}{ccc} C & \twoheadrightarrow & A \\ \downarrow \lrcorner & & \downarrow \lrcorner \\ A & \twoheadrightarrow & A \oplus B \\ & & \downarrow \lrcorner \\ & & 0 \end{array} & \begin{array}{ccc} A & \twoheadrightarrow & A \\ \downarrow \lrcorner & & \downarrow \lrcorner \\ 0 & \twoheadrightarrow & 0 \\ & & \downarrow \lrcorner \\ & & 0 \end{array} & \begin{array}{ccc} A & \twoheadrightarrow & A \oplus B \\ \downarrow \lrcorner & & \downarrow \lrcorner \\ 0 & \twoheadrightarrow & 0 \\ & & \downarrow \lrcorner \\ & & 0 \end{array}
 \end{array}$$

defines an element in the right-hand side of the lower 3-Segal map for $P^{\triangleleft}S_4^{(2)}(\mathcal{E})$. However, it does not lie in its essential image, because the sequence

$$C \twoheadrightarrow A \xrightarrow{f} B$$

indexed by $\{0, 2, 4\}$ is not left exact (the map f not being strict), so $(6.6) \notin P^{\triangleleft}S_4^{(2)}(\mathcal{E})$.

Dually, since there exist non-strict morphisms admitting a cokernel in \mathcal{E} , the 4-cells of the right path space $P^{\triangleright}S^{(2)}(\mathcal{E})$ do not satisfy the upper 3-Segal condition (by Lemma 6.7).

The above arguments also imply that the assumption in Theorem 6.9 is necessary (if \mathcal{E} is additive). Namely, the lower 1-Segal condition for $P^{\triangleleft}P^{\triangleright}S^{(2)}(\mathcal{E}) \cong S^{[0]}(\mathcal{E})$ requires admissible morphisms in \mathcal{E} to be closed under composition, and thus \mathcal{E} must be abelian already.

Example 6.11. Let us illustrate the lowest 3-Segal conditions for $S^{(2)}(\mathcal{E})$, which is more conveniently done by depicting an element of its 4-cells as the following projection of (6.3).

$$\begin{array}{ccccccc}
 & & & & A_{123} & & \\
 & & & & \nearrow & & \searrow \\
 & & & & A_{023} & & A_{124} \\
 & & & & \searrow & & \nearrow \\
 & & & & A_{024} & & \\
 & & & & \nearrow & & \searrow \\
 & & & & A_{014} & & A_{034} \\
 & & & & \twoheadrightarrow & & \twoheadrightarrow \\
 & & & & A_{013} & & A_{134} \\
 & & & & \nearrow & & \searrow \\
 & & & & A_{012} & & A_{234}
 \end{array} \tag{6.7}$$

The red part marks the image of (6.7) in the right-hand side of the upper 3-Segal map

$$S_4^{(2)}(\mathcal{E}) \longrightarrow S_3^{(2)}(\mathcal{E}) \times_{S_2^{(2)}(\mathcal{E})} S_3^{(2)}(\mathcal{E}).$$

The upper 3-Segal condition says that the whole diagram (6.7) is uniquely recovered from the red subdiagram. Note that the complementary statement (for the lower 3-Segal map) is

false in general. In fact, it is equivalent to uniquely filling the frame of short exact sequences

$$\begin{array}{ccccc}
C_4 & \twoheadrightarrow & A_{023} & \twoheadrightarrow & A_{123} \\
\downarrow & & \downarrow & & \downarrow \\
C_3 & \twoheadrightarrow & A'_{024} & \twoheadrightarrow & A_{124} \\
\downarrow & & \downarrow & & \downarrow \\
C_2 & \twoheadrightarrow & C_1 & \twoheadrightarrow & C_0
\end{array} \tag{6.8}$$

where $C_i = \operatorname{coker}(A_{d_3^* d_i^* \Delta_4^4} \twoheadrightarrow A_{d_2^* d_i^* \Delta_4^4}) \cong \ker(A_{d_1^* d_i^* \Delta_4^4} \twoheadrightarrow A_{d_0^* d_i^* \Delta_4^4})$. However, there is an obstruction to this, which is parametrized by the quotient groupoid

$$[\operatorname{Ext}^1(C_0, C_4) / \operatorname{Hom}(C_0, C_4)],$$

as calculated in [5], Lemma 2.30 and Proposition 2.38, assuming that \mathcal{E} is abelian.

Remark 6.12. Suppose \mathcal{E} is additive. Then Theorem 6.9 does not generalize to the higher dimensional Waldhausen constructions, that is, $S^{(k)}(\mathcal{E})$ is not upper $(2k-1)$ -Segal for $k \neq 2$. Indeed, the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & A \xrightarrow{=} A \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & \longrightarrow & A & \xrightarrow{(1,0)} & A \oplus A \xrightarrow{(1,1)} A \\
& & & & \downarrow & & \downarrow \\
& & & & 0 & \longrightarrow & A \longrightarrow 0 \\
& & & & & & \downarrow \\
& & & & & & 0 \longrightarrow 0 \\
& & & & & & \downarrow \\
& & & & & & 0
\end{array} \tag{6.9}$$

is an element of the right-hand side of the lower 3-Segal map for $P^\triangleleft P^\triangleright S_4^{(3)}(\mathcal{E})$, but does not lie in its essential image.

Our next observation will prove essential for our inductive arguments.

Proposition 6.13 (Hyperplane lemma). *Let $1 \leq k \leq l < m \leq n$, and let \mathcal{E} be a stringent category. Then there is a natural functor*

$$\eta_{lm}^\triangleleft : S_n^{(k)}(\mathcal{E}) \longrightarrow S_l^{(k-1)}(\mathcal{E}), \quad A \longmapsto (\beta \mapsto \operatorname{coker}(A_{\beta \cup \{l\}} \twoheadrightarrow A_{\beta \cup \{m\}})).$$

Dually, there is a corresponding natural functor

$$\eta_{lm}^\triangleright : S_n^{(k)}(\mathcal{E}) \longrightarrow S_l^{(k-1)}(\mathcal{E}), \quad A \longmapsto (\beta \mapsto \ker(A_{\{n-m\} \cup \beta} \twoheadrightarrow A_{\{n-l\} \cup \beta})).$$

Moreover, both of these restrict to functors on the higher Waldhausen construction,

$$S_n^{(k)}(\mathcal{E}) \xrightleftharpoons[\eta_{lm}^\triangleright]{\eta_{lm}^\triangleleft} S_l^{(k-1)}(\mathcal{E}).$$

Proof. Let γ be a k -simplex in Δ^l , and $\gamma' = \gamma \amalg \{m\}$. If $l \in \gamma$, then the sequence $\eta_{lm}^\triangleleft(A)_{d_\bullet^* \gamma}$ is given by

$$\operatorname{coker}(A_{d_{k+1}^* \gamma'} \twoheadrightarrow A_{d_k^* \gamma'}) \twoheadrightarrow A_{d_{k-1}^* \gamma'} \longrightarrow A_{d_{k-2}^* \gamma'} \longrightarrow \dots \longrightarrow A_{d_0^* \gamma'}$$

which of course is indeed left exact. Furthermore, it is exact if and only if A lies in $S_n^{(k)}(\mathcal{E})$, by definition.

Now assume that $l \notin \gamma$. Then, for each vertex $0 < i < k$, let us write

$$\begin{array}{ccccc} (A_{d_{i+1}^* \gamma \cup \{l\}} \twoheadrightarrow) B_{i+1} & \twoheadrightarrow & A_{d_i^* \gamma \cup \{l\}} & \twoheadrightarrow & B_i (\twoheadrightarrow A_{d_{i-1}^* \gamma \cup \{l\}}) \\ & & \downarrow & & \downarrow \\ (A_{d_{i+1}^* \gamma \cup \{m\}} \twoheadrightarrow) C_{i+1} & \twoheadrightarrow & A_{d_i^* \gamma \cup \{m\}} & \twoheadrightarrow & C_i (\twoheadrightarrow A_{d_{i-1}^* \gamma \cup \{m\}}) \end{array}$$

for the corresponding short exact sequence. Taking cokernels yields a diagram as follows.

$$\begin{array}{ccccc} B_{i+1} & \twoheadrightarrow & C_{i+1} & \twoheadrightarrow & D_{i+1} \\ \downarrow & & \downarrow & & \downarrow \\ A_{d_i^* \gamma \cup \{l\}} & \twoheadrightarrow & A_{d_i^* \gamma \cup \{m\}} & \twoheadrightarrow & \eta_{lm}^\triangleleft(A)_{d_i^* \gamma} \\ \downarrow & & \downarrow & & \downarrow \\ B_i & \twoheadrightarrow & C_i & \twoheadrightarrow & D_i \end{array}$$

By the snake lemma, the right vertical sequence is short exact. Note that if $A \in S_n^{(k)}(\mathcal{E})$,

$$B_1 \xrightarrow{\sim} A_{d_0^* \gamma \cup \{l\}} \text{ and } C_1 \xrightarrow{\sim} A_{d_0^* \gamma \cup \{m\}}$$

which immediately implies also $D_1 \xrightarrow{\sim} \eta_{lm}^\triangleleft(A)_{d_0^* \gamma}$ by definition. It remains to prove that

$$\eta_{lm}^\triangleleft(A)_{d_k^* \gamma} \longrightarrow \eta_{lm}^\triangleleft(A)_{d_{k-1}^* \gamma}$$

is an admissible monomorphism. In order to see this, we may show that the diagram

$$\begin{array}{ccc} A_{d_k^* \gamma \cup \{l\}} & \twoheadrightarrow & A_{d_k^* \gamma \cup \{m\}} \\ \downarrow & & \downarrow \\ A_{d_{k-1}^* \gamma \cup \{l\}} & \twoheadrightarrow & A_{d_{k-1}^* \gamma \cup \{m\}} \end{array} \quad (6.10)$$

is pullback, by Lemma 5.8. In fact, we claim that it is the composition of pullback diagrams

$$\begin{array}{ccccccc} A_{d_k^* \gamma \cup \{l\}} & \twoheadrightarrow & A_{d_k^* \gamma \cup \{l+1\}} & \twoheadrightarrow & \dots & \twoheadrightarrow & A_{d_k^* \gamma \cup \{m-1\}} & \twoheadrightarrow & A_{d_k^* \gamma \cup \{m\}} \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ A_{d_{k-1}^* \gamma \cup \{l\}} & \twoheadrightarrow & A_{d_{k-1}^* \gamma \cup \{l+1\}} & \twoheadrightarrow & \dots & \twoheadrightarrow & A_{d_{k-1}^* \gamma \cup \{m-1\}} & \twoheadrightarrow & A_{d_{k-1}^* \gamma \cup \{m\}}. \end{array}$$

To prove this claim, for each $l \leq j < m$, we have the diagram

$$\begin{array}{ccc} A_{d_k^* \gamma \cup \{j\}} & \twoheadrightarrow & A_{d_k^* \gamma \cup \{j+1\}} \\ \downarrow & & \downarrow \\ A_{d_{k-1}^* \gamma \cup \{j\}} & \twoheadrightarrow & A_{d_{k-1}^* \gamma \cup \{j+1\}} \\ \downarrow & & \downarrow \\ 0 & \twoheadrightarrow & A_{d_{k-1}^* d_k^* \gamma \cup \{j, j+1\}} \end{array}$$

in which the lower and outer rectangles are pullback, and therefore, so is the upper. \square

Finally, the functor η_{lm}^\triangleright is given by the map η_{lm}^\triangleleft for \mathcal{E}^{op} , via Lemma 6.7. \square

We are now prepared to compute the path spaces of the higher Waldhausen construction. This constitutes the main step in the proof of our main theorem, as well as a generalization of [6], Lemma 2.4.9, which concerns the case $k = 1$.

Proposition 6.14. *Let $k \geq 1$, and assume that \mathcal{E} is a stringent category. Then there are equivalences of simplicial categories*

$$\begin{aligned} P^\triangleleft S^{(k)}(\mathcal{E}) &\longrightarrow S^{(k-1)}(\mathcal{E}), \quad A \longmapsto A_{[0] \oplus -}, \\ P^\triangleright S^{(k)}(\mathcal{E}) &\longrightarrow S^{[k-1]}(\mathcal{E}), \quad A \longmapsto A_{-\oplus [0]}, \end{aligned}$$

induced by the forgetful functors. For $k \geq 2$, there is an equivalence

$$P^{\triangleleft} P^{\triangleright} S^{(k)}(\mathcal{E}) \longrightarrow S^{[k-2]}(\mathcal{E}), \quad A \longmapsto A_{[0] \oplus - \oplus [0]}.$$

Proof. We prove the second statement first. For a diagram $A \in S_n^{[k-1]}(\mathcal{E})$, we construct its image in $P^{\triangleright} S_n^{(k)}(\mathcal{E}) = S_{n+1}^{(k)}(\mathcal{E})$ under the inverse functor as a right Kan extension. Namely, we extend by zero appropriately, and then into the k th dimension, as follows.

$$\begin{array}{ccc}
 \text{Fun}([k-1], [n]) & \xrightarrow{A} & \mathcal{E} \\
 \downarrow \iota & & \uparrow \lambda \\
 \text{Fun}([k-1], [n+1]) & \xrightarrow{A'} & \mathcal{E} \\
 \downarrow \text{incl} & \nearrow \widehat{A} & \\
 \text{Cyl}(\iota|_{\Delta_{k-1}^n}) & & \\
 \downarrow \lambda & & \\
 \text{Fun}([k], [n+1]) & &
 \end{array} \tag{6.11}$$

Here, we have set $\iota = (d_{n+1})_*$, and the category $\text{Cyl}(\iota|_{\Delta_{k-1}^n})$ is the cograph of its restriction to the skeleton. The functor λ is defined by $(s_{k-1})_*$ on Δ_{k-1}^n , and on $\text{Fun}([k-1], [n+1])$, it maps

$$\alpha \mapsto \alpha \cup \{n+1\}.$$

Explicitly, \widehat{A} is given by the diagram

$$\beta \longmapsto \varprojlim_{\beta \leq \lambda(\alpha)} A'_\alpha \cong \begin{cases} A_{\beta \setminus \{n+1\}} & \text{if } n+1 \in \beta, \\ \ker(A_{d_k^* \beta} \rightarrow A_{d_{k-1}^* \beta}) & \text{otherwise.} \end{cases}$$

Indeed, $d_k^* \beta$ is initial amongst those objects of the indexing category of the limit which come from $\text{Fun}([k-1], [n+1])$. If $n+1 \in \beta$, then this is the only contribution. Otherwise, there are additionally the objects of the form

$$[\alpha] \in \Delta_{k-1}^n \text{ with } d_{k-1}^* \beta \leq \alpha.$$

Therefore, in that case, the limit reduces to just the pullback

$$\widehat{A}_\beta \cong \varprojlim \left[\begin{array}{ccc} & A'_{d_k^* \beta} & \\ & \downarrow & \\ A'_{[d_{k-1}^* \beta]} & \longrightarrow & A'_{d_{k-1}^* \beta} \end{array} \right] \cong \varprojlim \left[\begin{array}{ccc} & A_{d_k^* \beta} & \\ & \downarrow & \\ 0 & \longrightarrow & A_{d_{k-1}^* \beta} \end{array} \right] = \ker(A_{d_k^* \beta} \rightarrow A_{d_{k-1}^* \beta}).$$

Now let γ be a $(k+1)$ -simplex in Δ^{n+1} . We claim that the corresponding sequence $\widehat{A}_{d_\bullet^* \gamma}$ is exact. If $n+1 \in \gamma$, then this is simply given by

$$\ker(A_{d_k^* d_{k+1}^* \gamma} \rightarrow A_{d_{k-1}^* d_{k+1}^* \gamma}) \twoheadrightarrow A_{d_k^* \gamma \setminus \{n+1\}} \longrightarrow \dots \longrightarrow A_{d_1^* \gamma \setminus \{n+1\}} \twoheadrightarrow A_{d_0^* \gamma \setminus \{n+1\}}$$

which is an exact sequence in \mathcal{E} by definition.

Otherwise, the relevant sequence is given by

$$\begin{array}{ccc}
 \ker(A_{d_k^* d_{k+1}^* \gamma} \rightarrow A_{d_{k-1}^* d_{k+1}^* \gamma}) & & \ker(A_{d_k^* d_{k+1}^* \gamma} \rightarrow A_{d_{k-1}^* d_{k+1}^* \gamma}) \\
 \downarrow & & \downarrow \\
 \ker(A_{d_k^* d_k^* \gamma} \rightarrow A_{d_{k-1}^* d_k^* \gamma}) & & \ker(A_{d_k^* d_{k+1}^* \gamma} \rightarrow A_{d_{k-1}^* d_k^* \gamma}) \\
 \downarrow & & \downarrow \\
 \ker(A_{d_k^* d_{k-1}^* \gamma} \rightarrow A_{d_{k-1}^* d_{k-1}^* \gamma}) & & \ker(A_{d_{k-1}^* d_{k+1}^* \gamma} \rightarrow A_{d_{k-1}^* d_k^* \gamma}) \\
 \downarrow & \text{or equivalently,} & \downarrow \\
 \ker(A_{d_k^* d_{k-2}^* \gamma} \rightarrow A_{d_{k-1}^* d_{k-2}^* \gamma}) & & \ker(A_{d_{k-2}^* d_{k+1}^* \gamma} \rightarrow A_{d_{k-2}^* d_k^* \gamma}) \\
 \downarrow & & \downarrow \\
 \vdots & & \vdots \\
 \downarrow & & \downarrow \\
 \ker(A_{d_k^* d_0^* \gamma} \rightarrow A_{d_{k-1}^* d_0^* \gamma}) & & \ker(A_{d_0^* d_{k+1}^* \gamma} \rightarrow A_{d_0^* d_k^* \gamma}).
 \end{array}$$

We prove exactness inductively. The first part of the sequence fits into a diagram of the form

$$\begin{array}{ccccc}
 \widehat{A}_{d_{k+1}^* \gamma} & \twoheadrightarrow & \widehat{A}_{d_k^* \gamma} & \twoheadrightarrow & A_{d_k^* d_{k+1}^* \gamma} \\
 \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
 0 & \twoheadrightarrow & \widehat{A}_{d_{k-1}^* \gamma} & \twoheadrightarrow & A_{d_{k-1}^* d_{k+1}^* \gamma} \\
 & & \downarrow & \lrcorner & \downarrow \\
 & & 0 & \twoheadrightarrow & A_{d_{k-1}^* d_k^* \gamma}.
 \end{array}$$

By definition, the bottom left square is pullback, so we can pull it back to the top and then to the left, since each outer rectangle is a pullback square by construction. Thus,

$$\widehat{A}_{d_{k+1}^* \gamma} \twoheadrightarrow \widehat{A}_{d_k^* \gamma} \twoheadrightarrow \widehat{A}_{d_{k-1}^* \gamma}$$

is a left exact sequence. Now, for each $0 < i < k$, let us write

$$\begin{array}{ccccc}
 (A_{d_{i+1}^* d_{k+1}^* \gamma} \twoheadrightarrow) B_{i+1} & \twoheadrightarrow & A_{d_i^* d_{k+1}^* \gamma} & \twoheadrightarrow & B_i (\twoheadrightarrow A_{d_{i-1}^* d_{k+1}^* \gamma}) \\
 \vdots & & \vdots & & \vdots \\
 (A_{d_{i+1}^* d_k^* \gamma} \twoheadrightarrow) C_{i+1} & \twoheadrightarrow & A_{d_i^* d_k^* \gamma} & \twoheadrightarrow & C_i (\twoheadrightarrow A_{d_{i-1}^* d_k^* \gamma})
 \end{array} \tag{6.12}$$

for the corresponding short exact sequences at the i th vertex of $d_{k+1}^* \gamma$ and $d_k^* \gamma$, respectively. First, we show that $B_k \rightarrow C_k$ is an admissible epimorphism. But we have

$$B_k = \text{coker}(\widehat{A}_{d_{k+1}^* \gamma} \twoheadrightarrow A_{d_k^* d_{k+1}^* \gamma}), \text{ and } C_k = \text{coker}(\widehat{A}_{d_k^* \gamma} \twoheadrightarrow A_{d_k^* d_k^* \gamma}).$$

Therefore, they fit into a diagram of the following form, which yields the claim.

$$\begin{array}{ccccc}
 \widehat{A}_{d_{k+1}^* \gamma} & \twoheadrightarrow & \widehat{A}_{d_k^* \gamma} & \twoheadrightarrow & A_{d_k^* d_{k+1}^* \gamma} \\
 \downarrow & & \downarrow & & \downarrow \\
 0 & \twoheadrightarrow & B'_k & \twoheadrightarrow & B_k \\
 & & \downarrow & & \downarrow \\
 & & 0 & \twoheadrightarrow & C_k
 \end{array}$$

In particular, $B'_k = \ker(B_k \rightarrow C_k)$. Next, we show that $B_{k-1} \rightarrow C_{k-1}$ admits a kernel B'_{k-1} in \mathcal{E} . In fact, consider the following diagram.

$$\begin{array}{ccccc}
B_k & \twoheadrightarrow & A_{d_{k-1}^* d_{k+1}^* \gamma} & \twoheadrightarrow & B_{k-1} \\
\downarrow & & \downarrow & & \downarrow (3) \\
C_k & \dashrightarrow (1) & E & \dashrightarrow (1') & D \\
\downarrow & & \downarrow & & \downarrow (2) \\
C_k & \twoheadrightarrow & A_{d_{k-1}^* d_k^* \gamma} & \twoheadrightarrow & C_{k-1} \\
\downarrow & & \downarrow & & \downarrow (2') \\
0 & \twoheadrightarrow & C'_{k-1} & \xrightarrow{=} & C'_{k-1}
\end{array}$$

We have (1) by Remark 5.6, and (1') is its cokernel. The snake lemma yields (2) and (2'), and (3) is obtained dually to (1).

Now the snake lemma implies that the top row of the following diagram is short exact.

$$\begin{array}{ccccc}
B'_k & \twoheadrightarrow & \widehat{A}_{d_{k-1}^* \gamma} & \twoheadrightarrow & B'_{k-1} \\
\downarrow & & \downarrow & & \downarrow \\
B_k & \twoheadrightarrow & A_{d_{k-1}^* d_{k+1}^* \gamma} & \twoheadrightarrow & B_{k-1} \\
\downarrow & & \downarrow & & \downarrow \\
C_k & \twoheadrightarrow & A_{d_{k-1}^* d_k^* \gamma} & \twoheadrightarrow & C_{k-1}
\end{array}$$

In particular, this settles the case $k = 2$. For $k \geq 3$, we can rewrite the diagram

$$\begin{array}{ccccc}
B_2 & \twoheadrightarrow & A_{d_1^* d_{k+1}^* \gamma} & \twoheadrightarrow & B_1 = A_{d_0^* d_{k+1}^* \gamma} \\
\downarrow & & \downarrow & & \downarrow \\
C_2 & \twoheadrightarrow & A_{d_1^* d_k^* \gamma} & \twoheadrightarrow & C_1 = A_{d_0^* d_k^* \gamma}
\end{array}$$

in terms of the hyperplane lemma (Proposition 6.13), namely as the upper part of the short exact sequence of acyclic sequences

$$\begin{array}{ccccc}
\eta_{(n-\gamma_1)(n-\gamma_0)}^{\triangleright}(A)_{d_{k-1}^* \alpha} & \twoheadrightarrow & A_{\{\gamma_0\} \cup d_{k-1}^* \alpha} & \twoheadrightarrow & A_{\{\gamma_1\} \cup d_{k-1}^* \alpha} \\
\downarrow & & \downarrow & & \downarrow \\
\eta_{(n-\gamma_1)(n-\gamma_0)}^{\triangleright}(A)_{d_{k-2}^* \alpha} & \twoheadrightarrow & A_{\{\gamma_0\} \cup d_{k-2}^* \alpha} & \twoheadrightarrow & A_{\{\gamma_1\} \cup d_{k-2}^* \alpha} \\
\downarrow & & \downarrow & & \downarrow \\
\eta_{(n-\gamma_1)(n-\gamma_0)}^{\triangleright}(A)_{d_{k-3}^* \alpha} & \twoheadrightarrow & A_{\{\gamma_0\} \cup d_{k-3}^* \alpha} & \twoheadrightarrow & A_{\{\gamma_1\} \cup d_{k-3}^* \alpha} \\
\vdots & & \vdots & & \vdots
\end{array} \tag{6.13}$$

where $\alpha = d_0^* d_1^* \gamma$. In particular, $B_2 \rightarrow C_2$ is an admissible morphism. Applying the snake lemma to the third morphism of short exact sequences in (6.13) tells us that the map

$$C'_2 = \text{coker}(B_2 \rightarrow C_2) \longrightarrow \text{coker}(A_{d_1^* d_{k+1}^* \gamma} \rightarrow A_{d_1^* d_k^* \gamma})$$

is an admissible monomorphism, and therefore, by applying it to the first, that $\widehat{A}_{d_1^* \gamma} \twoheadrightarrow \widehat{A}_{d_0^* \gamma}$.

Finally, we can iterate the argument, realizing (6.12) as the upper part of the diagram

$$\begin{array}{ccccc}
 \eta^{(i)}(A)_{d_{k-i}^* \alpha^{(i)}} \twoheadrightarrow A_{\{\gamma_0, \dots, \gamma_{i-1}\} \cup d_{k-i}^* \alpha^{(i)}} \twoheadrightarrow \eta^{(i-1)}(A)_{d_{k-i+1}^* \alpha^{(i-1)}} & & & & \\
 \downarrow & & \downarrow & & \downarrow \\
 \eta^{(i)}(A)_{d_{k-i-1}^* \alpha^{(i)}} \twoheadrightarrow A_{\{\gamma_0, \dots, \gamma_{i-1}\} \cup d_{k-i-1}^* \alpha^{(i)}} \twoheadrightarrow \eta^{(i-1)}(A)_{d_{k-i}^* \alpha^{(i-1)}} & & & & \\
 \downarrow & & \downarrow & & \downarrow \\
 \eta^{(i)}(A)_{d_{k-i-2}^* \alpha^{(i)}} \twoheadrightarrow A_{\{\gamma_0, \dots, \gamma_{i-1}\} \cup d_{k-i-2}^* \alpha^{(i)}} \twoheadrightarrow \eta^{(i-1)}(A)_{d_{k-i-1}^* \alpha^{(i-1)}} & & & & \\
 \vdots & & \vdots & & \vdots
 \end{array}$$

where $\eta^{(i)} = \eta_{(n-\gamma_i)(n-\gamma_{i-1})}^{\triangleright} \circ \dots \circ \eta_{(n-\gamma_1)(n-\gamma_0)}^{\triangleright}$ and $\alpha^{(i)} = d_0^* \dots d_i^* \gamma$. Then the sequence

$$B'_{i+1} \twoheadrightarrow \widehat{A}_{d_i^* \gamma} \longrightarrow B'_i$$

is the beginning of the corresponding long exact snake, where $B'_i = \ker(B_i \rightarrow C_i)$, and is therefore a short exact sequence, as above.

Finally, the equivalence $P^{\triangleleft} S^{(k)}(\mathcal{E}) \xrightarrow{\sim} S^{(k-1)}(\mathcal{E})$ follows via Lemma 6.7 from the one we have proven above. Furthermore, if $k \geq 2$, we obtain $P^{\triangleleft} P^{\triangleright} S^{(k)}(\mathcal{E}) \xrightarrow{\sim} S^{[k-2]}(\mathcal{E})$ as an immediate consequence of the two. Namely, let $A \in S_n^{[k-2]}(\mathcal{E})$. Then the left Kan extension analogous to (6.11) produces a diagram

$$\widehat{A} \in S_{n+1}^{[k-1]}(\mathcal{E}) = P^{\triangleleft} S_n^{[k-1]}(\mathcal{E}) \cong P^{\triangleleft} P^{\triangleright} S_n^{(k)}(\mathcal{E}),$$

as all arguments above apply verbatim to show that \widehat{A} consists of right exact sequences. \square

Remark 6.15. Proposition 6.14 can be seen as a higher analogue of the third isomorphism theorem, in that the equivalence of categories $S_{k+1}^{(k-1)}(\mathcal{E}) \xrightarrow{\sim} P^{\triangleleft} S_{k+1}^{(k)}(\mathcal{E}) = S_{k+2}^{(k)}(\mathcal{E})$ boils down to the following statement. Given a configuration of left exact sequences of the form

$$\begin{array}{ccccccc}
 A_k^{k+1} & \twoheadrightarrow & A_{k-1}^{k+1} & \twoheadrightarrow & \dots & \twoheadrightarrow & A_1^{k+1} & \twoheadrightarrow & A_0^{k+1} \\
 & & \searrow & & & & \searrow & & \downarrow \\
 & & A_{k-1}^k & \twoheadrightarrow & \dots & \twoheadrightarrow & A_1^k & \twoheadrightarrow & A_0^k \\
 & & & & & & \searrow & & \downarrow \\
 & & & & & & A_1^{k-1} & \twoheadrightarrow & A_0^{k-1} \\
 & & & & & & & & \downarrow \\
 & & & & & & & & \dots \\
 & & & & & & & & A_0^0
 \end{array}$$

where $A_i^j = A_{d_i^* d_j^* \Delta_{k+1}^{k+1}}$ in the previous notation, the induced maps between the cokernels

$$\text{coker}(A_1^{k+1} \rightarrow A_0^{k+1}) \longrightarrow \text{coker}(A_1^k \rightarrow A_0^k) \longrightarrow \dots \longrightarrow \text{coker}(A_1^0 \rightarrow A_0^0)$$

constitute an exact sequence in \mathcal{E} .

We are now prepared to prove our main result.

Proof of Theorem 6.8. Our goal is to show that $S^{(k-1)}(\mathcal{E})$ is a lower $(2k-1)$ -Segal category. Then, by Proposition 6.14 as well as Lemma 6.7, the path space criterion completes the proof.

Let $n \geq 2k$. We will prove by induction that the lower $(2k-1)$ -Segal map

$$S_n^{(k-1)}(\mathcal{E}) \longrightarrow \varprojlim_{I \in \mathcal{L}([n], 2k-1)} S_I^{(k-1)}(\mathcal{E}) \quad (6.14)$$

is an equivalence. Throughout, for $0 \leq i \leq n$, let δ_i refer to the i th face map of Δ_n^n , even when applied to any subsimplex of it. That is, δ_i^* removes the vertex i , and $\widehat{\delta}_i^*$ adjoins it.

First, consider the case $n = 2k$. By Lemma 6.16, the only k -simplex in Δ^{2k} not contained in an even subset of $[2k]$ of cardinality $2k$ already is

$$\varepsilon = \{0, 2, \dots, 2k\} = \delta_{2k-1}^* \delta_{2k-3}^* \cdots \delta_1^* \Delta_{2k}^{2k}.$$

But if A lies in the right-hand side of (6.14), then we can form the unique compositions

$$\begin{array}{ccccccc} A_{\delta_{2k}^* \varepsilon} & \dashrightarrow & A_{\delta_{2k-2}^* \varepsilon} & \dashrightarrow & \dots & \dashrightarrow & A_{\delta_0^* \varepsilon} \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ & & A_{\delta_{2k}^* \widehat{\delta}_{2k-1}^* \delta_{2k-2}^* \varepsilon} & & A_{\delta_{2k-2}^* \widehat{\delta}_{2k-3}^* \delta_{2k-4}^* \varepsilon} & \dots & A_{\delta_2^* \widehat{\delta}_1^* \delta_0^* \varepsilon} \end{array}$$

completing A to an element of $S_{2k}^{(k-1)}(\mathcal{E})$. It remains to be shown that the resulting sequence

$$A_{\delta_{2k}^* \varepsilon} \longrightarrow A_{\delta_{2k-2}^* \varepsilon} \longrightarrow \dots \longrightarrow A_{\delta_0^* \varepsilon}$$

is left exact. We proceed inductively. The case $k = 2$ is settled by Theorem 6.9. In general, since $A_{\delta_{2k}^* \varepsilon} \hookrightarrow A_{\delta_{2k-2}^* \varepsilon}$ is an admissible monomorphism (as a composition of such), it suffices to prove that

$$\text{coker}(A_{\delta_{2k}^* \varepsilon} \hookrightarrow A_{\delta_{2k-2}^* \varepsilon}) \longrightarrow A_{\delta_{2k-4}^* \varepsilon} \longrightarrow \dots \longrightarrow A_{\delta_0^* \varepsilon} \quad (6.15)$$

is a left exact sequence. For this, we use the hyperplane lemma. Namely, the functor

$$\eta_{(2k-2)2k}^\triangleleft : S_{2k}^{(k-1)}(\mathcal{E}) \longrightarrow S_{2k-2}^{(k-2)}(\mathcal{E})$$

constructed in Proposition 6.13 is compatible with the corresponding lower Segal maps on both sides, in that it induces a commutative diagram of the following form.

$$\begin{array}{ccc} S_{2k}^{(k-1)}(\mathcal{E}) & \longrightarrow & \varprojlim_{I \in \mathcal{L}([2k], 2k-1)} S_I^{(k-1)}(\mathcal{E}) \\ \downarrow \eta_{(2k-2)2k}^\triangleleft & & \downarrow \eta_{(2k-2)2k}^\triangleleft \\ S_{2k-2}^{(k-2)}(\mathcal{E}) & \longrightarrow & \varprojlim_{J \in \mathcal{L}([2k-2], 2k-3)} S_J^{(k-2)}(\mathcal{E}) \end{array}$$

Indeed, this is because we have $d_0^* d_0^* \varepsilon = \delta_{2k-3}^* \delta_{2k-5}^* \cdots \delta_1^* \Delta_{2k-2}^{2k-2}$. But then, by induction, the lower horizontal map is an equivalence, which by the above means precisely that (6.15) is a left exact sequence.

In order to prove the Segal conditions for the higher cells $S_n^{(k-1)}(\mathcal{E})$, we once again employ induction, now on the dimension n . If A lies in the right-hand side of the lower $(2k-1)$ -Segal map for $S_n^{(k-1)}(\mathcal{E})$, we first need to see that taking compositions completes A to a well-defined diagram of shape $\text{Fun}([k-1], [n])$ in \mathcal{E} .

By Lemma 6.16, we need only consider sequences indexed by simplices γ with all vertices separated by gaps. For gaps $i \in [n]$ of size 1, there is (as before) a unique composition,

$$\begin{array}{ccc} A_{\delta_{i+1}^* \gamma} & \dashrightarrow & A_{\delta_{i-1}^* \gamma} \\ & \searrow & \nearrow \\ & A_{\delta_{i+1}^* \widehat{\delta}_i^* \delta_{i-1}^* \gamma} & \end{array}$$

For a gap of γ of size $l+1$, say $\{i, \dots, i+l\} \subseteq [n]$, each $0 \leq j \leq l$ defines the composition

$$\begin{array}{ccc} A_{\delta_{i+l+1}^* \gamma} & \dashrightarrow & A_{\delta_{i-1}^* \gamma} \\ & \searrow & \nearrow \\ & A_{\delta_{i+l+1}^* \widehat{\delta}_{i+j}^* \delta_{i-1}^* \gamma} & \end{array}$$

By induction, all possible compositions can be reduced to one of these. On the other hand, they all agree, since for all $0 \leq j < j' \leq l$, the following diagram commutes.

$$\begin{array}{ccc} A_{\delta_{i+l+1}^* \gamma} & \longrightarrow & A_{\delta_{i+l+1}^* \widehat{\delta}_{i+j'}^* \delta_{i-1}^* \gamma} \\ \downarrow & \nearrow & \downarrow \\ A_{\delta_{i+l+1}^* \widehat{\delta}_{i+j}^* \delta_{i-1}^* \gamma} & \longrightarrow & A_{\delta_{i-1}^* \gamma} \end{array}$$

Finally, we apply induction to obtain the remaining exactness conditions for the completed diagram of A . Namely, the sequence indexed by γ is left exact, since $\partial_i(A)$ lies in the right-hand side of the lower $(2k-1)$ -Segal map of $S_{n-1}^{(k-1)}(\mathcal{E})$, for any gap i of γ . \square

Lemma 6.16. *Let $n \geq 2k$. Let γ be a k -subsimplex of Δ^n with a pair of adjacent simplices. Then γ is contained in an even subset $I \subseteq [n]$ of cardinality $\#I = 2k$.*

Proof. For $n = 2k$, this is clear. For the induction step, we can assume that $n \in \gamma$, otherwise the statement follows tautologically from the induction hypothesis. Let $0 < m < n$ be the maximal gap of γ . By induction, $(\gamma \cup \{m\}) \setminus \{n\}$ is contained in an even subset $I' \subseteq [n-1]$ with $\#I' = 2k$. But then γ is contained in $I = (I' \setminus \{m\}) \cup \{n\}$, which is even in $[n]$. \square

7. TOTALIZATION AND REALIZATIONS

In this section, we provide another perspective on the higher dimensional Segal and Waldhausen constructions by showing that they are totalizations of iterations of their respective classical variants. In particular, the delooping and additivity statements for the latter will imply corresponding results for the former.

Definition 7.1. Let \mathcal{C} be an ∞ -category with finite limits, and $X \in \mathcal{C}_{\Delta \times k}$ a k -fold simplicial object, $k \geq 0$. The totalization of X is the simplicial object $TX \in \mathcal{C}_{\Delta}$, where T is defined as the right adjoint of $(\oplus^{\text{op}})^* : \mathcal{C}_{\Delta} \rightarrow \mathcal{C}_{\Delta \times k}$ for the k -fold ordinal sum functor $\oplus : \Delta^{\times k} \rightarrow \Delta$.

First, we establish a relation between the totalization and the higher Segal conditions.

Proposition 7.2. *Let \mathcal{C} be an ∞ -category with finite limits, and let $X \in \mathcal{C}_{\Delta \times k}$ be a k -fold lower 1-Segal object, $k \geq 1$. Then $TX \in \mathcal{C}_{\Delta}$ is a lower $(2k-1)$ -Segal object.*

Proof. For $k = 1$, this is a tautology. Now assume the statement for some $k \geq 1$, and let X be a $(k+1)$ -fold lower 1-Segal object. We want to prove that the natural map

$$(TX)_n \longrightarrow \varprojlim_{H \in \mathcal{L}([n], 2k+1)} (TX)_H \quad (7.1)$$

is an equivalence, for every $n \geq 2k+1$. If $k = 2$, the totalization is computed by

$$(TX)_n \cong \text{eq} \left(\prod_{i=0}^n X_{\{0, \dots, i\}, \{i, \dots, n\}} \xrightarrow[\chi^{\text{op}}]{\psi^{\text{op}}} \prod_{I \oplus J = [n]} X_{I, J} \right),$$

where the components of ψ^\triangleleft are given by $\partial_{\bullet,0} \circ \text{pr}_i$ for $I = \{0, \dots, i\}$, and χ^\triangleright consists of the functors $\partial_{i,\bullet} \circ \text{pr}_i$ where $J = \{i, \dots, n\}$ (cf. [19], §3), for all $n \geq 0$. Equivalently, this can be expressed as

$$(TX)_n \cong \varprojlim_{\substack{I \cup J = [n] \\ I \leq J}} X_{I,J}.$$

Therefore, by induction, for every $k \geq 2$, the right-hand side of (7.1) is given by

$$\varprojlim_{H \in \mathcal{L}([n], 2k+1)} (TX)_H = \varprojlim_{H \in \mathcal{L}([n], 2k+1)} \varprojlim_{\substack{H_0 \cup \dots \cup H_k = H \\ H_0 \leq \dots \leq H_k}} X_{H_0, \dots, H_k}. \quad (7.2)$$

Now, we consider $X = X_{\bullet, \bullet} \in (\mathcal{C}_\Delta)_{\Delta \times k}$ as a k -fold lower 1-Segal object in \mathcal{C}_Δ . Hence, by induction, $TX_{m, \bullet}$ is lower $(2k-1)$ -Segal for all $m \geq 0$. Thus,

$$(TX)_n = (T([m] \mapsto TX_{m, \bullet}))_n = \varprojlim_{\substack{I_0 \cup I = [n] \\ I_0 \leq I}} (TX_{I_0, \bullet})_I \xrightarrow{\sim} \varprojlim_{\substack{I_0 \cup I = [n] \\ I_0 \leq I}} \varprojlim_{J \in \mathcal{L}(I, 2k-1)} (TX_{I_0, \bullet})_J.$$

Again spelling out the totalization, and using the Segal property in the first variable, we get

$$\varprojlim_{\substack{I_0 \cup I = [n] \\ I_0 \leq I}} \varprojlim_{J \in \mathcal{L}(I, 2k-1)} (TX_{I_0, \bullet})_J = \varprojlim_{\substack{I_0 \cup I = [n] \\ I_0 \leq I}} \varprojlim_{J \in \mathcal{L}(I, 2k-1)} \varprojlim_{\substack{J_1 \cup \dots \cup J_k = J \\ J_1 \leq \dots \leq J_k}} \varprojlim_{J_0 \in \mathcal{L}(I_0, 1)} X_{J_0, J_1, \dots, J_k}.$$

But altogether, this is equivalent to (7.2), which is immediate from the description of the maximal elements of $\mathcal{L}([n], 2k+1)$ as disjoint unions of $k+1$ sets of the form $\{i, i+1\} \subseteq [n]$. \square

Remark 7.3. Let $\mathcal{C} = N(\text{Set})$. Then the functor defined by Proposition 7.2,

$$T: \{k\text{-fold categories}\} \longrightarrow \{\text{lower } (2k-1)\text{-Segal sets}\} \quad (7.3)$$

is not an equivalence. Indeed, let $k = 2$. Then the essential image cannot contain $\text{ob}(S^{(1)}(\mathcal{E}))$ for every stringent category \mathcal{E} , as the corresponding double category would have precisely one object, which precludes the existence of a bijection on the higher cells in general.

In a positive direction, the main result of [1] shows that for $k = 2$, the functor

$$(\oplus^{\text{op}})^*: \{\text{fully 2-Segal sets}\} \longrightarrow \{\text{double categories}\}$$

induces an equivalence between the subcategory of unital 2-Segal sets on the left and stable double categories with the extra datum of an augmentation on the right-hand side. This also implies that (7.3) is not fully faithful (which is hardly surprising, looking at the counit of the adjunction in Definition 7.1).

It is an interesting question how to generalize the equivalence to arbitrary k . As a first step, it should be possible to combine the path space criterion with (3.1) to see that $(\oplus^{\text{op}})^*$ maps fully $(2k-2)$ -Segal objects to k -fold Segal objects.

The definition of stability directly extends to k -fold categories (in terms of $(k+1)$ -cubes), and so does the notion of augmentation; the analogue of unitality for fully $(2k-2)$ -Segal objects should involve degenerate triangulations of the appropriate cyclic polytopes.

Let \mathcal{D} be a pointed category with finite products, and let \mathcal{E} be a proto-exact category.

Definition 7.4. We write $S_{\oplus}^{(k)}(\mathcal{D}) \in \text{Cat}_{\Delta \times k}$ and $S^{(k)}(\mathcal{E}) \in \text{Cat}_{\Delta \times k}$ for the k -fold iterates of the 1-dimensional Segal and Waldhausen constructions, respectively.

Theorem 7.5. *Let $k \geq 1$. The k -dimensional Segal construction of a pointed category \mathcal{D} with finite products is naturally equivalent to the totalization of its k -fold Segal construction,*

$$S_{\oplus}^{(k)}(\mathcal{D}) \xrightarrow{\sim} TS_{\oplus}^{(k)}(\mathcal{D}).$$

Proof. For $k = 1$, there is nothing to show. By induction, it suffices to prove, for every $k > 1$,

$$S_{\oplus}^{(k)}(\mathcal{D}) \xrightarrow{\sim} T(S_{\oplus}^{(k-1)}(S_{\oplus}^{(1)}(\mathcal{D}))).$$

Therefore, we need to construct, for all $k, n > 0$, a natural equivalence of categories

$$\mathrm{Sh}(S_n^k, \mathcal{D}) \xrightarrow{\sim} \varprojlim_{\substack{I \cup J = [n] \\ I \leq J}} \mathrm{Sh}(S_I^{k-1}, \mathrm{Sh}(S_J^1, \mathcal{D})). \quad (7.4)$$

Note that as in the proof of Theorem 4.3, the right-hand side is computed by the 1-categorical limit. Now, we first consider $\mathrm{Sh}(S_I^{k-1}, \mathrm{Sh}(S_J^1, \mathcal{D}))$ as a full subcategory of

$$\begin{aligned} & \mathrm{Fun}(\mathcal{P}(S_I^{k-1})^{\mathrm{op}}, \mathrm{Fun}(\mathcal{P}(S_J^1)^{\mathrm{op}}, \mathcal{D})) \\ &= \mathrm{Fun}(\mathcal{P}(S_I^{k-1})^{\mathrm{op}} \times \mathcal{P}(S_J^1)^{\mathrm{op}}, \mathcal{D}) \\ &= \mathrm{Fun}(\mathcal{P}(S_I^{k-1} \amalg S_J^1)^{\mathrm{op}}, \mathcal{D}), \end{aligned}$$

where \amalg is the coproduct of pointed sets. Define the two maps $\lambda, \rho: S_n^k \rightarrow S_I^{k-1} \amalg S_J^1$ by

$$\lambda(\alpha) = \begin{cases} \alpha|_I & \text{if } \alpha(I) \subseteq [k-1], \\ * & \text{otherwise,} \end{cases} \quad \text{and } \rho(\alpha) = \begin{cases} \alpha|_J & \text{if } \alpha(J) \subseteq \{k-1, k\}, \\ * & \text{otherwise.} \end{cases}$$

Then we claim that an equivalence as in (7.4) descends from the induced functors

$$\mathrm{Sh}(S_n^k, \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{P}(S_I^{k-1} \amalg S_J^1)^{\mathrm{op}}, \mathcal{D}), \quad \mathcal{F} \longmapsto (W \mapsto \mathcal{F}(\lambda^\times W \cap \rho^\times W)).$$

Firstly, let us show that the essential image of this functor lies in $\mathrm{Sh}(S_I^{k-1}, \mathrm{Sh}(S_J^1, \mathcal{D}))$. This amounts to proving that for every $U \in \mathcal{P}(S_I^{k-1})$, the assignment $V \mapsto \mathcal{F}(\lambda^\times U \cap \rho^\times V)$ is a sheaf on $\mathcal{P}(S_J^1)$. But this reduces straight-forwardly to the sheaf property of \mathcal{F} , as

$$\mathcal{F}(\lambda^\times U \cap \rho^\times V) \xrightarrow{\sim} \prod_{\varepsilon \in V} \prod_{\alpha \in \lambda^\times U \cap \rho^\times \{*, \varepsilon\}} \mathcal{F}(\{*, \alpha\}) \xleftarrow{\sim} \prod_{\varepsilon \in V} \mathcal{F}(\lambda^\times U \cap \rho^\times \{*, \varepsilon\}).$$

The analogous argument shows that $U \mapsto \mathcal{F}(\lambda^\times U \cap \rho^\times V)$ is a sheaf for every $V \in \mathcal{P}(S_J^1)$, which yields the rest of the claim, since products of sheaves are computed point-wise.

Next, to see that these functors form into a map into the limit boils down to the transitivity of restriction, $(\alpha|_I)|_{I \setminus J} = \alpha|_{I \setminus J}$ for $I, J \subseteq [n]$ as before.

Finally, to construct the inverse, given $(\mathcal{F}_{I,J}) \in \varprojlim_{\substack{I \cup J = [n] \\ I \leq J}} \mathrm{Sh}(S_I^{k-1}, \mathrm{Sh}(S_J^1, \mathcal{D}))$, we extend

$$\{*, \alpha\} \longmapsto \mathcal{F}_{I(\alpha), J(\alpha)}(\{*, \alpha|_{I(\alpha)}\})(\{*, \alpha|_{J(\alpha)}\}), \quad \text{for } I(\alpha) = \alpha^{-1}[k-1], \quad J(\alpha) = \{i_\alpha\} \amalg \alpha^{-1}(k),$$

where $i_\alpha \in I(\alpha)$ is the maximal element, to a sheaf on $\mathcal{P}(S_n^k)$. In one direction, we use

$$\lambda^\times \{*, \alpha|_{I(\alpha)}\} \cap \rho^\times \{*, \alpha|_{J(\alpha)}\} = \{*, \alpha\}$$

to see that the two constructions are inverse to one another. Conversely, let $\mathcal{F} \in \mathrm{Sh}(S_n^k, \mathcal{D})$ denote the image of some $(\mathcal{F}_{I,J})$. For every $(U, V) \in \mathcal{P}(S_I^{k-1}) \times \mathcal{P}(S_J^1)$, $I \cap J \neq \emptyset$, we get

$$\mathcal{F}(\lambda^\times U \cap \rho^\times V) \xrightarrow{\sim} \mathcal{F}_{I,J}(U)(V)$$

via the following isomorphism,

$$\prod_{\alpha \in \lambda^\times U \cap \rho^\times V} \mathcal{F}_{I(\alpha), J(\alpha)}(\{*, \alpha|_{I(\alpha)}\})(\{*, \alpha|_{J(\alpha)}\}) \xrightarrow{\sim} \prod_{(\beta, \varepsilon) \in U \times V} \mathcal{F}_{I,J}(\{*, \beta\})(\{*, \varepsilon\}).$$

This map arises via the mutually inverse identifications $\alpha \mapsto (\lambda(\alpha), \rho(\alpha))$ and $(\beta, \varepsilon) \mapsto \beta \cup \varepsilon$ of the respective indexing sets, with its components given by the fact that

$$(\mathcal{F}_{I,J}) \in \varprojlim_{\substack{I \cup J = [n] \\ I \leq J}} \mathrm{Sh}(S_I^{k-1}, \mathrm{Sh}(S_J^1, \mathcal{D})),$$

which provides, for each $\alpha = \beta \cup \varepsilon$ as above, a chain of isomorphisms

$$\mathcal{F}_{I(\alpha), J(\alpha)}(\{*, \alpha|_{I(\alpha)}\})(\{*, \alpha|_{J(\alpha)}\}) \cong \mathcal{F}_{I', J'}(\{*, \alpha|_{I'}\})(\{*, \alpha|_{J'}\}) \cong \dots \cong \mathcal{F}_{I,J}(\{*, \beta\})(\{*, \varepsilon\}),$$

where $I' = I(\alpha) \setminus \{i_\alpha\}$, and $J' = \{i_\alpha - 1\} \amalg J(\alpha)$. \square

Remark 7.6. Together with Proposition 7.2 above, Theorem 7.5 yields an alternative proof of Theorem 4.3.

On the other hand, the analogous reasoning for the Waldhausen construction is false. That is, the totalization of a k -fold 2-Segal object is not in general fully $2k$ -Segal. A counterexample is provided precisely by Theorem 6.8 as for $k = 1$, its stringency assumption is unnecessary.

Theorem 7.7. *Let $k \geq 0$, and let \mathcal{E} be a proto-exact category. There is a natural equivalence of simplicial categories between the k -dimensional Waldhausen construction of \mathcal{E} and the totalization of its k -fold Waldhausen construction,*

$$S^{(k)}(\mathcal{E}) \xrightarrow{\sim} TS^{(k)}(\mathcal{E}).$$

Proof. For $k \leq 1$, this is tautological. By the same reasoning as in the proof of Theorem 7.5, it is sufficient to construct, for all $n, k \geq 1$, a natural equivalence of categories

$$S_n^{(k)}(\mathcal{E}) \xrightarrow{\sim} \varprojlim_{\substack{I \cup J = [n] \\ I \leq J}} S_I^{(k-1)}(S_J^{(1)}(\mathcal{E})).$$

To define the functor, we use that the right-hand side is a full subcategory of

$$\begin{aligned} & \varprojlim_{\substack{I \cup J = [n] \\ I \leq J}} \text{Fun}(\text{Fun}([k-1], I), \text{Fun}(\text{Fun}([1], J), \mathcal{E})) \\ &= \varprojlim_{\substack{I \cup J = [n] \\ I \leq J}} \text{Fun}(\text{Fun}([k-1], I) \times \text{Fun}([1], J), \mathcal{E}) \\ &= \text{Fun}(\varprojlim_{\substack{I \cup J = [n] \\ I \leq J}} \text{Fun}([k-1], I) \times \text{Fun}([1], J), \mathcal{E}). \end{aligned}$$

Consider the following full subcategory of the indexing category for its elements.

$$\{(\alpha, \varepsilon) \in \varprojlim_{\substack{I \cup J = [n] \\ I \leq J}} \text{Fun}([k-1], I) \times \text{Fun}([1], J) \mid \alpha_{k-1} = \varepsilon_0\}$$

This is equivalent to $\text{Fun}([k], [n])$ via $(\alpha, \varepsilon) \mapsto \alpha \cup \varepsilon$, with inverse induced by the functor

$$\text{Fun}([k], [n]) \longrightarrow \varprojlim_{\substack{I \cup J = [n] \\ I \leq J}} \text{Fun}([k-1], I) \times \text{Fun}([1], J), \quad \beta \longmapsto (\beta|_{[k-1]}, \beta|_{\{k-1, k\}}).$$

Now let $A \in S_n^{(k)}(\mathcal{E})$. Then its left Kan extension $A^!$ along the inclusion

$$\text{Fun}([k], [n]) \hookrightarrow \{(\alpha, \varepsilon) \in \varprojlim_{\substack{I \cup J = [n] \\ I \leq J}} \text{Fun}([k-1], I) \times \text{Fun}([1], J) \mid \alpha_{k-1} = \varepsilon_0 \text{ or } \varepsilon_0 = \varepsilon_1\}$$

amounts to an (iterated) extension by zero (as in the proof of Proposition 6.14 for $k = 1$).

Then we define the image $\widehat{A} \in \varprojlim_{\substack{I \cup J = [n] \\ I \leq J}} S_I^{(k-1)}(S_J^{(1)}(\mathcal{E}))$ of A as a further left Kan extension to

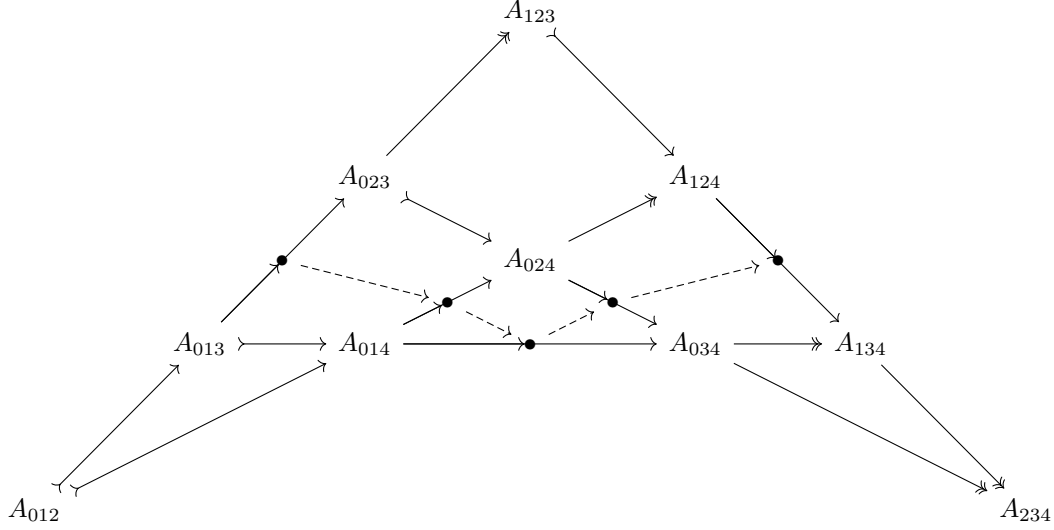
the whole indexing category $\varprojlim_{\substack{I \cup J = [n] \\ I \leq J}} \text{Fun}([k-1], I) \times \text{Fun}([1], J)$, which again is an iterated

version of the corresponding construction in the proof of Proposition 6.14. Recall that

$$\varprojlim_{\substack{I \cup J = [n] \\ I \leq J}} S_I^{(k-1)}(S_{J \setminus I}^{(0)}(\mathcal{E})) \cong \varprojlim_{\substack{I \cup J = [n] \\ I \leq J}} S_I^{(k-1)}(S_J^{(1)}(\mathcal{E}))$$

is an equivalence of categories, which is compatible with the transition maps in the limit, and which does not require the assumption that \mathcal{E} be stringent as in Proposition 6.14, by [6], Lemma 2.4.9. We proceed to show that this construction is well-defined, fully faithful, and essentially surjective, which completes the proof. \square

Example 7.8. As a special case of Theorem 7.7, we illustrate the equivalence of categories $S_4^{(2)}(\mathcal{E}) \cong S_{\{0,1\},\{1,2,3,4\}}^{(2)}(\mathcal{E}) \times_{S_{\{0,1\},\{2,3,4\}}^{(2)}(\mathcal{E})} S_{\{0,1,2\},\{2,3,4\}}^{(2)}(\mathcal{E}) \times_{S_{\{0,1,2\},\{3,4\}}^{(2)}(\mathcal{E})} S_{\{0,1,2,3\},\{3,4\}}^{(2)}(\mathcal{E})$ in the following diagram, where the glueing is represented by the two dashed sequences.



Let $k \geq 1$. As can be seen either directly or with the help of Theorem 7.7, the functor

$$\Phi: \text{ExCat} \longrightarrow \text{Top}_*, \quad \mathcal{E} \longmapsto |S^{(k-1)}(\mathcal{E})^\times|,$$

satisfies the hypotheses of [11], §1.3. This permits us to draw the following consequences on geometric realizations.

Corollary 7.9 (Additivity). *Let \mathcal{E} be an exact category, $k \geq 1$. There is a weak equivalence*

$$|S^{(k)}(S_2(\mathcal{E}))^\times| \xrightarrow{\sim} |S^{(k)}(\mathcal{E})^\times| \times |S^{(k)}(\mathcal{E})^\times|$$

induced by the functor $(\partial_2, \partial_0): S_2(\mathcal{E}) \rightarrow \mathcal{E} \times \mathcal{E}$. In fact, the simplicial space $|S^{(k)}(S_\bullet(\mathcal{E}))^\times|$ is a lower 1-Segal space.

Proof. By Theorem 7.7, this is precisely [11], Theorem 1.3.5 (2), with Φ as above. \square

Needless to say, the other versions of additivity in [11], Theorem 1.3.5, hold as well; this also allows us to deduce the following as in *loc.cit.* via the proof of [13], Proposition 3.6.2.

Corollary 7.10 (Delooping). *Let $k \geq 1$, and let $K(\mathcal{E})$ denote the algebraic K -theory space of the exact category \mathcal{E} . There is a natural homotopy equivalence*

$$\Omega^k |S^{(k)}(\mathcal{E})^\times| \xrightarrow{\sim} K(\mathcal{E}).$$

In conclusion, the algebraic K -theory spectrum of \mathcal{E} is given by the sequence of maps

$$|S^{(0)}(\mathcal{E})^\times| \longrightarrow \Omega |S^{(1)}(\mathcal{E})^\times| \longrightarrow \Omega^2 |S^{(2)}(\mathcal{E})^\times| \longrightarrow \dots,$$

where, for each $k \geq 1$, the morphism

$$\Omega^{k-1} |S^{(k-1)}(\mathcal{E})^\times| \longrightarrow \Omega^k |S^{(k)}(\mathcal{E})^\times|$$

is induced by the inclusions of the cells $S_{n-1}^{(k-1)}(\mathcal{E})$ into $S_n^{(k)}(\mathcal{E})$ as $S_1^{(1)}(S_{n-1}^{(k-1)}(\mathcal{E}))$, extended appropriately by zeroes, for all $n \geq 1$.

Remark 7.11. An alternative proof of these conclusions reduces directly from Theorem 7.7 to the case $k = 1$ ([21], Theorem 1.4.2 and Proposition 1.3.2, resp. Proposition 1.5.3). That

is, we use that the arrow $(*)$ in the following commutative diagram is an equivalence,

$$\begin{array}{ccccc} |S^{(k)}(\mathcal{E})^\times| & \xrightarrow{(*)} & |TS^{(k)}(\mathcal{E})^\times| & \xleftarrow{\sim} & |S^{(1)}(S^{(k-1)}(\mathcal{E}))^\times| \\ \sim \uparrow & & & & \sim \uparrow \\ |\mathrm{ob} S^{(k)}(\mathcal{E})| & \xrightarrow{\sim} & |\mathrm{ob}(TS^{(k)}(\mathcal{E}))| & \xleftarrow{\sim} & |\mathrm{ob} S^{(1)}(S^{(k-1)}(\mathcal{E}))|. \end{array}$$

The remaining equivalences result from Theorem 7.7, [21], Lemma 1.4.1 and its Corollary, as well as [19], Theorem 1. A generalization of the latter to bisimplicial spaces would of course render this detour redundant; moreover, we would also obtain the delooping

$$\Omega^k |S_{\oplus}^{(k)}(\mathcal{E})^\times| \xrightarrow{\sim} K_{\oplus}(\mathcal{E})$$

for the direct sum K -theory from Theorem 7.5 and the case $k = 1$ (cf. [18], Proposition 1.5). Via the above, we may instead construct such an equivalence using [21], Theorem 1.8.1.

REFERENCES

- [1] J. E. Bergner, A. M. Osorno, V. Ozornova, M. Rovelli, and C. I. Scheimbauer. 2-Segal sets and the Waldhausen construction. To appear in Proceedings of WIT II, Topology and its Applications, 2017.
- [2] T. Bühler. Exact categories. *Expositiones Mathematicae*, 28(1):1–69, 2010.
- [3] E. Dotto. *Stable real K-theory and real topological Hochschild homology*. PhD thesis, Copenhagen, 2012.
- [4] W.G. Dwyer, D.M. Kan, and J.H. Smith. Homotopy commutative diagrams and their realizations. *Journal of Pure and Applied Algebra*, 57(1):5–24, 1989.
- [5] T. Dyckerhoff. Higher categorical aspects of Hall Algebras. To appear in Birkhäuser, Advanced Courses in Mathematics, CRM, 2015.
- [6] T. Dyckerhoff and M. Kapranov. Higher Segal Spaces: Part I. To appear in Lecture Notes in Mathematics, Springer-Verlag, 2012.
- [7] T. Dyckerhoff and M. Kapranov. Higher Segal Spaces: Part II. In preparation, 2017.
- [8] P. Freyd. Representations in Abelian Categories. In S. Eilenberg, D.K. Harrison, S. MacLane, and H. Röhl, editors, *Proceedings of the Conference on Categorical Algebra*, volume 1, pages 95–120. Springer, 1966.
- [9] I. Gálvez-Carrillo, J. Kock, and A. Tonks. Decomposition spaces, incidence algebras and Möbius inversion I: basic theory. To appear in Advances in Mathematics, 2015.
- [10] A. Heller. Homological Algebra in Abelian Categories. *Annals of Mathematics*, 68(3):484–525, 1958.
- [11] L. Hesselholt and I. Madsen. On the K -theory of local fields. *Annals of Mathematics*, 158(1):1–113, 2003.
- [12] L. Hesselholt and I. Madsen. Real algebraic K -theory. Book project in progress, 2015.
- [13] R. McCarthy. The cyclic homology of an exact category. *Journal of Pure and Applied Algebra*, 93(3):251–296, 1994.
- [14] D. Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories*, volume 341 of *Lecture Notes in Mathematics*, pages 85–147. Springer-Verlag, 1973.
- [15] J. Rambau. Triangulations of cyclic polytopes and higher Bruhat orders. *Mathematika*, 44(1):162–194, 1997.
- [16] C. Rezk. A model for the homotopy theory of homotopy theory. *Transactions of the American Mathematical Society*, 353(3):973–1007, 2001.
- [17] J. P. Schneiders. Quasiabelian categories and sheaves. *Mémoires de la Société Mathématique de France (nouvelle série)*, 76(1):1–140, 1999.
- [18] G. Segal. Categories and cohomology theories. *Topology*, 13(1):293–312, 1974.
- [19] D. Stevenson. Décalage and Kan’s simplicial loop group functor. *Theory and Applications of Categories*, 26(28):768–787, 2012.
- [20] T. Walde. Hall monoidal categories and categorical modules. Preprint, 2016.
- [21] F. Waldhausen. Algebraic K -theory of spaces. In *Algebraic and Geometric Topology*, volume 1126 of *Lecture Notes in Mathematics*, pages 318–419. Springer-Verlag, 1985.
- [22] M. B. Young. Relative 2-Segal spaces. Preprint, 2016.
- [23] G. M. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, 1995.