HIGHER SEGAL STRUCTURES IN ALGEBRAIC K-THEORY

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Abstract. We introduce higher dimensional analogues of simplicial constructions due to Segal and Waldhausen, respectively producing the direct sum and algebraic K-theory spectra of an exact category. We interrelate them by totalizing, and investigate their fibrancy properties based on the formalism of higher Segal spaces of Dyckerhoff-Kapranov.

Contents

Introduction

Let $\mathcal{E}$ be an exact category. In this article, we investigate certain simplicial categories naturally arising in the context of algebraic K-theory. They are obtained as generalizations of the following ubiquitous constructions, which produce a simplicial category from $\mathcal{E}$;

- the Segal construction $S_{\oplus}(\mathcal{E})$,
- the Waldhausen construction $S(\mathcal{E})$.

From a topological perspective, their relevance lies in the fact that they provide deloopings of the direct sum and algebraic K-theory spectra of $\mathcal{E}$, respectively.

From an algebraic perspective, both constructions have fibrancy properties of structural importance. Namely, $S_{\oplus}(\mathcal{E})$ and $S(\mathcal{E})$ satisfy the Segal and 2-Segal conditions, respectively, modelling the structure of an associative, resp. multi-valued associative, monoid.

This work centers around certain higher dimensional generalizations of the above;

- the k-dimensional Segal construction $S_{\oplus}^{\langle k\rangle}(\mathcal{E})$,
- the k-dimensional Waldhausen construction $S^{\langle k\rangle}(\mathcal{E})$.

For $k = 1$, we recover $S_{\oplus}(\mathcal{E})$ and $S(\mathcal{E})$, respectively. For $k = 2$, these constructions form the foundational basis of real algebraic K-theory as introduced by Hesselholt-Madsen [14], and studied further for example by Dotto [5]. In our context, $\mathcal{E}$ is not endowed with a duality structure, and we consider these objects primarily as simplicial categories.

Similarly to the case $k = 1$, the higher dimensional constructions provide higher deloopings of algebraic K-theory and its split variant (Corollary 4.15). Their relevance from the algebraic perspective warrants further investigation, for which our main results lay the groundwork.

Theorem 0.1. *The higher Segal construction $S_{\oplus}^{\langle k\rangle}(\mathcal{E})$ is a lower $(2k-1)$-Segal category. The higher Waldhausen construction $S^{\langle k\rangle}(\mathcal{E})$ is a $2k$-Segal category.*

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Higher Segal objects were introduced in [8], with a focus on the 2-Segal conditions, in particular showing that they are responsible for the associativity of Hall and Hecke algebras. From a different perspective, unital 2-Segal spaces were defined and studied independently in [11] and its series of sequels.

Further work in this area includes a precise description of unital 2-Segal sets in terms of double categories in [2], and the introduction of relative Segal conditions in [22] and [24], which model the structure of modules over multi-valued associative monoids.

Let us briefly outline the structure of the paper. In §1, we summarize some basic theory of cyclic polytopes and their triangulations from [17] and [25], which is used in §2 to define and study relations between the higher Segal conditions from [9]. Novel results concern the interaction with diagonals and total simplicial objects of multisimplicial objects.

In §3, we define the simplicial category $S_{\oplus}^{\langle k \rangle}(\mathcal{E})$, realize it as the total simplicial object of the iterated 1-dimensional construction, and prove the first part of Theorem 0.1. Section 4 provides definitions of the higher S-construction and first examples; we show that $S^{\langle k \rangle}(\mathcal{E})$ is the total simplicial object of Waldhausen's original construction, from which we deduce delooping and additivity theorems, as well as the second part of Theorem 0.1.

Finally, §5 introduces the requisite homological context enabling us to establish further Segal properties for the higher S-constructions; it also includes some further examples.

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1. Cyclic polytopes

In this section, we recall results on polytopes relevant to the study of higher Segal objects.

Definition 1.1. Let $d \geq 0$, and consider the moment curve

$$\gamma_d \colon \mathbb{R} \longrightarrow \mathbb{R}^d,\ t \longmapsto (t, t^2, t^3, \ldots, t^d).$$

For a finite subset $N \subseteq \mathbb{R}$, the d-dimensional cyclic polytope on the vertices N is defined to be the convex hull of the set $\gamma_d(N) \subseteq \mathbb{R}^d$, and denoted by

$$C(N, d) = \operatorname{conv}(\gamma_d(N)).$$

The combinatorial type of the polytope $C(N, d)$ only depends on the cardinality of N. We will usually consider N to be the set $[n] = \{0, 1, \ldots, n\}$, where $n \geq 0$.

Cyclic polytopes are simplicial polytopes, i.e., their boundary forms a simplicial complex, which organizes into two components (with non-empty intersection), as follows.

Definition 1.2. A point x in the boundary of $C([n], d+1)$ is called a lower point, if

$$(x - \mathbb{R}_{>0}) \cap C([n], d+1) = \emptyset,$$

where the half-line of positive real numbers $\mathbb{R}_{>0} \subseteq \mathbb{R}^{d+1}$ is embedded into the last coordinate. Similarly, x is said to be an upper point, if

$$(x + \mathbb{R}_{>0}) \cap C([n], d+1) = \emptyset.$$

The lower and upper points of the boundary form simplicial subcomplexes of $C([n], d+1)$, which admit the following purely combinatorial description.

Definition 1.3. Let $I \subseteq [n]$. A gap of I is a vertex $j \in [n] \smallsetminus I$ in the complement of I. A gap is said to be even, resp. odd, if the cardinality $\#\{i \in I \mid i > j\}$ is even, resp. odd. The subset I is called even, resp. odd, if all gaps of I are even, resp. odd.

Proposition 1.4 (Gale's evenness criterion; [25], Theorem 0.7)**.** *Let $n \geq 0$, and let $I \subseteq [n]$ with $\#I = d+1$. Then I defines a d-simplex in the lower, resp. upper, boundary of $C([n], d+1)$ if and only if I is even, resp. odd.*

Forgetting the last coordinate of $\mathbb{R}^{d+1}$ defines a projection map

$$p\colon C([n], d+1) \longrightarrow C([n], d).$$

For any $I \subseteq [n]$ with $\#I - 1 = r \leq d$, the projection p maps the geometric r-simplex

$$|\Delta^I| \subseteq C([n], d+1)$$

homeomorphically to an r-simplex in $C([n], d)$.

Definition 1.5. The lower triangulation $\mathcal{T}_\ell$ of the polytope $C([n], d)$ is the triangulation given by the projections under p of the simplices contained in the lower boundary of $C([n], d+1)$. Similarly, the upper triangulation $\mathcal{T}_u$ of $C([n], d)$ is defined by the projections of the simplices contained in the upper boundary of the polytope $C([n], d+1)$.

Vice versa, any triangulation of $C([n], d)$ induces a piecewise linear section of p, whose image defines a simplicial subcomplex of $C([n], d+1)$. This interplay between the cyclic polytopes in different dimensions is what makes their combinatorics comparatively tractible.

Definition 1.6. Given a set $\mathcal{I} \subseteq 2^{[n]}$ of subsets of $[n]$, we denote by

$$\Delta^{\mathcal{I}} \subseteq \Delta^n \tag{1.1}$$

the simplicial subset of Δ^n whose m-simplices are given by those maps $[m] \to [n]$ which factor over some $I \in \mathcal{I}$.

From the above discussion, it follows that we have canonical homeomorphisms

$$|\Delta^{\mathcal{T}_\ell}| \xrightarrow{\simeq} C([n], d), \text{ and } |\Delta^{\mathcal{T}_u}| \xrightarrow{\simeq} C([n], d),$$

expressing the lower, resp. upper, triangulation of $C([n], d)$ geometrically.

Definition 1.7. Let $I, J \subseteq [n]$ be subsets of cardinality $d+1$, as well as $|\Delta^I|$ and $|\Delta^J|$ the geometric d-simplices in $C([n], d) \subseteq \mathbb{R}^d$ they define, respectively. Let $L(|\Delta^J|)$ denote the set of lower boundary points of $|\Delta^J| = C(J, d)$, and similarly, let $U(|\Delta^I|)$ be the set of upper boundary points of $|\Delta^I| = C(I, d)$. We write

$$|\Delta^I| \prec |\Delta^J| \Longleftrightarrow |\Delta^I| \cap |\Delta^J| \subseteq U(|\Delta^I|) \cap L(|\Delta^J|).$$

If $|\Delta^I| \prec |\Delta^J|$, then we say that $|\Delta^I|$ lies below the simplex $|\Delta^J|$.

Proposition 1.8 ([17], Corollary 5.9)**.** *The transitive closure of $\prec$ defines a partial order on the set of nondegenerate d-simplices in Δ^n.*

Remark 1.9. Suppose that $\mathcal{T} \subseteq 2^{[n]}$ consists of subsets of $[n]$ of cardinality $d+1$ and defines a triangulation of the cyclic polytope $C([n], d)$. In particular,

$$|\Delta^{\mathcal{T}}| \cong C([n], d).$$

Let $I_0 \in \mathcal{T}$. Then either $L(|\Delta^{I_0}|)$ is contained in $L(|\Delta^{\mathcal{T}}|)$, or there is some $I_1 \in \mathcal{T}$ such that the simplex $|\Delta^{I_0}|$ lies below $|\Delta^{I_1}|$. Iterating this argument, we obtain a chain

$$|\Delta^{I_0}| \prec |\Delta^{I_1}| \prec |\Delta^{I_2}| \prec \ldots$$

of subsimplices of $|\Delta^{\mathcal{T}}|$. The statement of Proposition 1.8 implies that this chain is acyclic and therefore has to terminate after finitely many steps. Thus, there exists $I_\infty \in \mathcal{T}$ with

$$L(|\Delta^{I_\infty}|) \subseteq L(|\Delta^{\mathcal{T}}|).$$

Similarly, there exists a set $I_{-\infty} \in \mathcal{T}$ such that $U(|\Delta^{I_{-\infty}}|) \subseteq U(|\Delta^{\mathcal{T}}|)$.

2. Higher Segal conditions

Let $\mathcal{C}$ be an ∞-category which admits finite limits. Following [9], we introduce a framework of fibrancy properties of simplicial objects in $\mathcal{C}$ governed by the combinatorics from §1 of cyclic polytopes and their triangulations.

Definition 2.1. For $n \geq d \geq 0$, we introduce the lower subposet of $2^{[n]}$ as follows;

$$\mathcal{L}([n], d) = \{J \mid J \subseteq I \text{ for some even } I \subseteq [n] \text{ with } \#I = d+1\}.$$

Analogously, we define

$$\mathcal{U}([n], d) \subseteq 2^{[n]}$$

as the poset of all subsets contained in an odd subset $I \subseteq [n]$ of cardinality $\#I = d+1$.

By Proposition 1.4, the sets of subsimplices of $C([n], d)$ described by $\mathcal{L}([n], d)$ and $\mathcal{U}([n], d)$ define the lower and upper triangulations of the cyclic polytope, respectively.

Definition 2.2. Let $d \geq 0$. A simplicial object $X \in \mathcal{C}_\Delta$ is called

- lower d-Segal if, for every $n \geq d$, the natural map
$$X_n \longrightarrow \varprojlim_{I \in \mathcal{L}([n],d)} X_I$$
is an equivalence;
- upper d-Segal if, for every $n \geq d$, the natural map
$$X_n \longrightarrow \varprojlim_{I \in \mathcal{U}([n],d)} X_I$$
is an equivalence;
- d-Segal if X is both lower and upper d-Segal.

Example 2.3. Let $X \in \mathcal{C}_\Delta$ be a simplicial object.

(1) Then X is lower (or upper) 0-Segal if and only if $X \xrightarrow{\simeq} X_0$ is equivalent to the constant simplicial object on its 0-cells.

(2) The simplicial object X is lower 1-Segal if, for every $n \geq 1$, the map
$$X_n \longrightarrow X_{\{0,1\}} \times_{X_{\{1\}}} X_{\{1,2\}} \times_{X_{\{2\}}} \cdots \times_{X_{\{n-1\}}} X_{\{n-1,n\}}$$
is an equivalence. That is, X is a Segal object in the sense of Rezk [18].

For $X \in \mathrm{Set}_\Delta$, this means that X is equivalent to the nerve of the category with objects X_0, morphisms X_1, and composition induced by the correspondence
$$X_{\{0,1\}} \times_{X_{\{1\}}} X_{\{1,2\}} \xleftarrow{\sim} X_2 \longrightarrow X_{\{0,2\}}.$$
Furthermore, $X \in \mathrm{Set}_\Delta$ is 1-Segal if and only if this defines a discrete groupoid. In fact, in general, an object $X \in \mathcal{C}_\Delta$ is upper 1-Segal if, for every $n \geq 1$, we have
$$X_n \xrightarrow{\simeq} X_{\{0,n\}}.$$

(3) The simplicial object X is lower 2-Segal if, for every $n \geq 2$, the map
$$X_n \longrightarrow X_{\{0,n-1,n\}} \times_{X_{\{0,n-1\}}} X_{\{0,n-2,n-1\}} \times_{X_{\{0,n-2\}}} \cdots \times_{X_{\{0,2\}}} X_{\{0,1,2\}}$$
is an equivalence. Similarly, X is upper 2-Segal if, for every $n \geq 2$, we have
$$X_n \xrightarrow{\simeq} X_{\{0,1,n\}} \times_{X_{\{1,n\}}} X_{\{1,2,n\}} \times_{X_{\{2,n\}}} \cdots \times_{X_{\{n-2,n\}}} X_{\{n-2,n-1,n\}}.$$
It follows that X is 2-Segal if and only if it is 2-Segal in the sense of [8]. This is most readily seen by reducing to Segal objects as in (2) by applying the respective path space criteria, Proposition 2.7 and [8], Theorem 6.3.2 – or by Proposition 2.5.

Remark 2.4. Let X be a simplicial object in an ∞-category $\mathcal{C}$ which admits limits. Then we can form the right Kan extension of X along the (opposite of the) Yoneda embedding

$$\mathcal{Y}^{\mathrm{op}}\colon N(\Delta^{\mathrm{op}}) \longrightarrow \mathrm{Fun}(N(\Delta^{\mathrm{op}}), \mathrm{Top})^{\mathrm{op}},$$

where Top denotes the ∞-category of spaces. In particular, by means of this extension, we may evaluate X on any simplicial set. Then we can reformulate the higher Segal conditions as follows. Let $\Sigma_d = \Sigma_{\ell|d} \cup \Sigma_{u|d}$ denote the collection of d-Segal coverings, where

$$\Sigma_{\ell|d} = \{\Delta^{\mathcal{L}([n],d)} \hookrightarrow \Delta^n \mid n \geq d\}, \text{ and } \Sigma_{u|d} = \{\Delta^{\mathcal{U}([n],d)} \hookrightarrow \Delta^n \mid n \geq d\},$$

see [8], §5.2. Then X is d-Segal if and only if it is Σ_d-local (cf. *loc.cit.*, Proposition 5.1.4). In this case, we say that the elements of Σ_d are X-equivalences (meaning, X maps them to equivalences in $\mathcal{C}$).

Similarly, X is lower, resp. upper, d-Segal if and only if it is $\Sigma_{\ell|d}$-local, resp. $\Sigma_{u|d}$-local.

Proposition 2.5. *Let $d \geq 0$, and let X be a simplicial object in an ∞-category $\mathcal{C}$ with limits. Then X is d-Segal if and only if, for every $n \geq d$, and every triangulation of the cyclic polytope $C([n], d)$ defined by the poset of simplices $\mathcal{T} \subseteq 2^{[n]}$, the natural map*

$$X_n \longrightarrow \varprojlim_{I \in \mathcal{T}} X_I$$

is an equivalence.

Proof. By [17], Corollary 5.12, any triangulation $\mathcal{T}$ of $C([n], d)$ can be connected to the lower and upper triangulations $\mathcal{T}_\ell$ and $\mathcal{T}_u$ via sequences of elementary flips of the form

$$L(|\Delta^I|) \subseteq |\Delta^I| \supseteq U(|\Delta^I|).$$

This implies that we have a zig-zag of X-equivalences of the form

$$\Delta^{\mathcal{L}([n],d)} \longrightarrow \ldots \longleftarrow \Delta^{\mathcal{T}} \longrightarrow \ldots \longleftarrow \Delta^{\mathcal{U}([n],d)}$$

in the category of simplicial sets over Δ^n. This implies that the inclusion $\Delta^{\mathcal{T}} \subseteq \Delta^n$ is again an X-equivalence by 2-out-of-3, which was to be shown.

For the converse, there is nothing to prove. $\square$

Definition 2.6. Let X be a simplicial object in an ∞-category $\mathcal{C}$. The left path space $P^{\triangleleft}X$ is the simplicial object in $\mathcal{C}$ defined as the pullback of X along the endofunctor

$$c^{\triangleleft}\colon \Delta \to \Delta,\ [n] \mapsto [0] \oplus [n].$$

Here, for two linearly ordered sets I and J, the ordinal sum $I \oplus J$ is the disjoint union $I \amalg J$ of sets, endowed with the linear order where $i \leq j$ for every pair of $i \in I$ and $j \in J$.

Similarly, the right path space $P^{\triangleright}X$ is given by the pullback of X along

$$c^{\triangleright}\colon \Delta \to \Delta,\ [n] \mapsto [n] \oplus [0].$$

Proposition 2.7 (Path space criterion)**.** *Let $d \geq 0$, and let $\mathcal{C}$ be an ∞-category with finite limits. Let $X \in \mathcal{C}_\Delta$ be a simplicial object.*

- *Suppose d is even. Then*
  1. *X is lower d-Segal if and only if $P^{\triangleleft}X$ is lower $(d-1)$-Segal,*
  2. *X is upper d-Segal if and only if $P^{\triangleright}X$ is lower $(d-1)$-Segal.*
- *Suppose d is odd. Then the following conditions are equivalent.*
  - (*i*) *X is upper d-Segal.*
  - (*ii*) *$P^{\triangleleft}X$ is upper $(d-1)$-Segal.*
  - (*iii*) *$P^{\triangleright}X$ is lower $(d-1)$-Segal.*

Proof. We show that if d is even, then X is an upper d-Segal object if and only if $P^{\triangleright}X$ is lower $(d-1)$-Segal. All other assertions follow by analogous arguments.

Let $\ell([n], d)$ denote the set of maximal elements of the poset $\mathcal{L}([n], d)$, so that

$$\Delta^{\mathcal{L}([n],d)} = \bigcup_{I \in \ell([n],d)} \Delta^I$$

is the minimal presentation, and similarly for $u([n],d) \subseteq \mathcal{U}([n],d)$.

Then the claim is an immediate consequence of the following observation. A subset $I \subseteq [n]$ of cardinality $d+1$ is odd if and only if $n \in I$ and $I \smallsetminus \{n\}$ is an even subset of $[n-1]$. Thus the map $c^{\triangleright} \colon \ell([n-1], d-1) \to u([n],d)$, $I \mapsto I \oplus [0]$, is a bijection. $\square$

Remark 2.8. There is no path space criterion for lower d-Segal objects if d is odd. While

$$\ell([5],3) = \{\{0,1,2,3\},\{0,1,3,4\},\{0,1,4,5\},\{1,2,4,5\},\{2,3,4,5\},\{1,2,3,4\}\},$$

the maps $c^{\triangleleft} \colon \ell([4],2) \to \ell([5],3)$, $I \mapsto [0] \oplus I$, and $c^{\triangleright} \colon u([4],2) \to \ell([5],3)$, $I \mapsto I \oplus [0]$, are not even jointly surjective, and their images

$$\operatorname{im}(c^{\triangleleft}) = \{\{0,1,2,3\},\{0,1,3,4\},\{0,1,4,5\}\},\ \operatorname{im}(c^{\triangleright}) = \{\{0,1,4,5\},\{1,2,4,5\},\{2,3,4,5\}\}$$

intersect. Rather, the complement of $\operatorname{im}(c^{\triangleright})$ in $\ell([n],d)$ is given by $\ell([n-1],d)$, for all $d \in \mathbb{N}$ and $n \geq d$. By induction, we obtain a disjoint decomposition of sets

$$\ell([n],d) = \coprod_{j=d}^{n} u([j-1], d-1) \oplus \{j\}. \tag{2.1}$$

In fact, this statement, resp. its dual, also follow by grouping the elements of $\ell([n],d)$ by their maximal, resp. minimal, vertex, and the following explicit description. If d is odd, then

$$\ell([n],d) = \left\{ \coprod_{i \in J} \{i, i+1\} \;\middle|\; J \subseteq [n] \text{ with } \#J = \frac{d+1}{2} \text{ and } |i-j| > 1 \text{ for all } i,j \in I \right\}, \tag{2.2}$$

for all $n \geq d$. All other $\ell([n],d)$ and $u([n],d)$ are described from this by applying the proof of Proposition 2.7.

Proposition 2.9. *Let X be a simplicial object in an ∞-category $\mathcal{C}$ which admits limits. Assume that X is lower or upper d-Segal. Then X is $(d+1)$-Segal.*

Proof. We show the statement assuming that X is lower d-Segal. The proof for upper d-Segal objects is similar. Let $n \geq d+1$ and consider a collection $\mathcal{T}$ defining a triangulation $|\Delta^{\mathcal{T}}|$ of the cyclic polytope $C([n], d+1)$. Recall that $\mathcal{T}_\ell$ defines the simplicial complex $L(|\Delta^{\mathcal{T}}|)$ of lower facets, and the projection $p \colon C([n], d+1) \to C([n], d)$ identifies $|\Delta^{\mathcal{T}_\ell}| \subseteq C([n], d+1)$ with the simplicial subcomplex defining the lower triangulation

$$p(|\Delta^{\mathcal{T}_\ell}|) \subseteq C([n], d).$$

Thus, we obtain a commutative diagram

$$\begin{tikzcd} \Delta^{\mathcal{T}} \arrow[r, "\iota"] & \Delta^n \\ \Delta^{\mathcal{T}_\ell} \arrow[u, "\kappa"] \arrow[ur, "\iota_\ell"'] & \end{tikzcd} \tag{2.3}$$

of simplicial sets, in which by definition, $\iota \in \Sigma_{d+1}$ and $\iota_\ell \in \Sigma_{\ell|d}$. In order to deduce $(d+1)$-Segal descent for X, we have to show that $\iota \in \overline{\Sigma}_{\ell|d}$, the collection of $\Sigma_{\ell|d}$-equivalences, by the tautological part of Proposition 2.5.

By the 2-out-of-3 property of $\overline{\Sigma}_{\ell|d}$, it suffices to show that $\kappa \in \overline{\Sigma}_{\ell|d}$. We will do so by showing that κ can be obtained as an iterated pushout along morphisms in $\Sigma_{\ell|d}$.

By Remark 1.9, the triangulation $|\Delta^{\mathcal{T}}|$ of $C([n], d+1)$ contains a maximal $(d+1)$-simplex of the form $|\Delta^I|$, defined by a singleton collection $\{I\} \subseteq \binom{[n]}{d+2}$. Let $\mathcal{I}_\ell$ be the set which defines the lower facets of $|\Delta^I|$, defining a triangulation $|\Delta^{\mathcal{I}_\ell}|$ of $C(I, d)$. Then the inclusion of simplicial sets

$$\kappa_\ell \colon \Delta^{\mathcal{I}_\ell} \longrightarrow \Delta^I$$

is contained in $\Sigma_{\ell|d}$. Further, since $|\Delta^I|$ is a maximal simplex, we have a pushout diagram

$$\begin{tikzcd} \Delta^{\mathcal{T}} \arrow[dr, phantom, "\lrcorner"] & \Delta^I \arrow[l] \\ \Delta^{\mathcal{T}^{(1)}} \arrow[u, "\kappa^{(1)}"] & \Delta^{\mathcal{I}_\ell} \arrow[l] \arrow[u, "\kappa_\ell"'] \end{tikzcd}$$

of simplicial sets, where $\mathcal{T}^{(1)} = \mathcal{T} \setminus \{I\}$. Thus, the map $\kappa^{(1)}$ lies in $\overline{\Sigma}_{\ell|d}$, and $|\Delta^{\mathcal{T}^{(1)}}|$ is an admissible simplicial subcomplex of $C([n], d+1)$ with one $(d+1)$-simplex less than $|\Delta^{\mathcal{T}}|$.

Assume that the triangulation $|\Delta^{\mathcal{T}}|$ consists of exactly r simplices of dimension $(d+1)$. By iterating the argument just given, we obtain a chain of morphisms

$$\Delta^{\mathcal{T}_\ell} \hookrightarrow \Delta^{\mathcal{T}^{(r-1)}} \hookrightarrow \ldots \hookrightarrow \Delta^{\mathcal{T}^{(1)}} \hookrightarrow \Delta^{\mathcal{T}}$$

in $\overline{\Sigma}_{\ell|d}$ whose composite is the morphism κ from (2.3). Thus, it is also contained in $\overline{\Sigma}_{\ell|d}$. $\square$

Definition 2.10. Let $\mathcal{C}$ be an ∞-category with finite limits, and $Y \in \mathcal{C}_{\Delta^{\times k}}$ a k-fold simplicial object, for $k \geq 0$. We will denote by $DY \in \mathcal{C}_\Delta$ the diagonal of Y, which is defined to be the pullback of Y under the diagonal embedding $\Delta \hookrightarrow \Delta^{\times k}$.

Remark 2.11. Let $\mathcal{C}$ be an ∞-category with finite limits, and let $Y \in \mathcal{C}_{\Delta^{\times k}}$ be a k-fold simplicial object, and let $r \geq 0$. We conjecture the following compatibilities.

(1) If Y is k-fold lower $(2r-1)$-Segal, $r \geq 1$, then DY is lower $(2kr-1)$-Segal.
(2) If Y is k-fold lower, resp. upper, $2r$-Segal, then DY is upper, resp. lower, $2kr$-Segal.
(3) If Y is k-fold upper $(2r+1)$-Segal, then DY is upper $(2kr+1)$-Segal.

Note that (1) implies the other two statements, by repeated applications of the path space criterion; indeed, $P^{\triangleleft} DY \simeq DP^{\triangleleft}_{(k)} Y$ and $P^{\triangleright} DY \simeq DP^{\triangleright}_{(k)} Y$, where the notation $P^{\triangleleft}_{(k)}$ and $P^{\triangleright}_{(k)}$ means application of the corresponding path space functor in each variable.

Definition 2.12. Let $V\colon \mathcal{C}_\Delta \to \mathcal{C}_{\Delta^{\times k}}$ be the total décalage functor, which is the pullback along the k-fold ordinal sum $\oplus\colon \Delta^{\times k} \to \Delta$. The total simplicial object of Y is denoted by

$$TY \in \mathcal{C}_\Delta,$$

where T is defined as the right adjoint of V.

Remark 2.13. Let $Y \in \mathcal{C}_{\Delta^{\times k}}$ be a k-fold simplicial object. If $k=2$, then for all $n \geq 0$, by cofinality, the n-cells of the total simplicial object of Y are computed by

$$(TY)_n \xrightarrow{\simeq} \mathrm{eq}\Big(\prod_{i=0}^{n} Y_{\{0,\ldots,i\},\{i,\ldots,n\}} \underset{\chi^{\triangleright}}{\overset{\psi^{\triangleleft}}{\rightrightarrows}} \prod_{I \oplus J = [n]} Y_{I,J} \Big),$$

where the components of $\psi^{\triangleleft}$ are given by $\partial_{\bullet,0} \circ \mathrm{pr}_i$ for $I = \{0,\ldots,i\}$, and $\chi^{\triangleright}$ consists of the functors $\partial_{i,\bullet} \circ \mathrm{pr}_i$ for $J = \{i,\ldots,n\}$, cf. [1], §III. Equivalently, this can be expressed as

$$(TY)_n \xrightarrow{\simeq} \varprojlim_{\substack{I_1 \cup \ldots \cup I_k = [n] \\ I_1 \leq \ldots \leq I_k}} Y_{I_1,\ldots,I_k} \tag{2.4}$$

for arbitrary $k \geq 1$, by induction.

Definition 2.14. Let $\underline{d} = (d_i) \in \mathbb{N}^k$, and let $Y \in \mathcal{C}_{\Delta^{\times k}}$ be a k-fold simplicial object. We say that Y is $\underline{d}$-Segal if it is d_i-Segal in the ith simplicial direction, for all $1 \leq i \leq k$.

Proposition 2.15. *Let $\mathcal{C}$ be an ∞-category with finite limits, $\underline{r} \in \mathbb{N}^k$, and let $Y \in \mathcal{C}_{\Delta^{\times k}}$ be a lower $(2\underline{r}-1)$-Segal object. Then $TY \in \mathcal{C}_\Delta$ is lower $(2r-1)$-Segal, where $r = \sum r_i$.*

Proof. For $k=1$, this is a tautology. Now assume the statement for some $k \geq 1$, and consider a lower $(2\underline{r}-1)$-Segal object $Y = Y_{\bullet,\bullet} \in (\mathcal{C}_\Delta)_{\Delta^{\times k}}$ in $\mathcal{C}_\Delta$ which is lower $(2s-1)$-Segal in the remaining coordinate. We want to prove that, for all $n \geq 2(r+s)-1$, the inclusion

$$\bigcup_{H \in \ell} \bigcup_{\substack{H = I \vee J \\ I \leq J}} \Delta^I \times \Delta^J \hookrightarrow \bigcup_{i=0}^{n} \Delta^{\{0,\ldots,i\}} \times \Delta^{\{i,\ldots,n\}} \tag{2.5}$$

is a Y-equivalence, where $\ell = \ell([n], 2(r+s)-1)$, and $I \vee J$ signifies the non-disjoint union. By eliminating redundant summands, we can write the left-hand side as

$$\bigcup_{H\in\ell}\Bigg(\bigcup_{\substack{H=I\vee J,\ I\leq J\\ 2r\leq \#I\leq 2r+1}} \Delta^I \times \Delta^J \cup \bigcup_{i=0}^{2r-1} \Delta^{[i]} \times \Delta^{H\smallsetminus[i-1]} \cup \bigcup_{j=n-2s}^{n-1} \Delta^{H\cap[j+1]} \times \Delta^{[n]\smallsetminus[j]}\Bigg).$$

Now, by induction, $TY_{m,\bullet}$ is lower $(2r-1)$-Segal, for all $m \geq 0$.

Combining this, as well as the lower $(2s-1)$-Segal property in the other variable, with Proposition 2.9, and using (2.2), the statement follows. □

Remark 2.16. Let $\mathcal{C} = \mathrm{Set}$. Then the functor defined by Proposition 2.15,

$$T\colon \{k\text{-fold categories}\} \longrightarrow \{\text{lower } (2k-1)\text{-Segal sets}\} \tag{2.6}$$

is not an equivalence. In fact, for $k = 2$, the main result of [2] shows that the functor

$$V\colon \{2\text{-Segal sets}\} \longrightarrow \{\text{double categories}\}$$

induces an equivalence between the subcategory of unital 2-Segal sets on the left- and stable double categories together with the extra datum of an augmentation on the right-hand side. In particular, this implies that (2.6) cannot be fully faithful. Moreover, we provide a class of counter-examples to its essential surjectivity in Example 5.17.

It is an interesting problem to generalize the above equivalence to arbitrary k. The definition of stability directly extends to k-fold categories (in terms of $(k+1)$-cubes), and so does the notion of augmentation; the analogue of unitality for higher Segal objects should involve degenerate triangulations of the appropriate cyclic polytopes.

Our considerations in §4 suggest a candidate for the inverse functor; on the other hand, it appears that for V to produce k-fold Segal objects, it still needs to be restricted to 2-Segal objects.

Our next result is the analogue of Remark 2.11 (2) for the total simplicial object; however, the proof is not as straight-forward, for the following reason.

Lemma 2.17. *Let $\mathcal{C}$ be an ∞-category with finite limits, and let $Y \in \mathcal{C}_{\Delta^{\times k}}$ be a k-fold simplicial object, $k \geq 1$. Then $P^{\triangleleft}TY \simeq TP^{\triangleleft}Y$ and $P^{\triangleright}TY \simeq TP^{\triangleright}Y$, where $P^{\triangleleft}$, resp. $P^{\triangleright}$, is applied in the first, resp. last, simplicial direction.*

Proof. This is an immediate consequence of the following base change squares,

$$\begin{tikzcd} \Delta^{\times k} \arrow[r, "c^{\triangleleft}\times \mathrm{id}"] \arrow[d, "\oplus"'] & \Delta^{\times k} \arrow[d, "\oplus"] \\ \Delta \arrow[r, "c^{\triangleleft}"] & \Delta \end{tikzcd} \qquad \begin{tikzcd} \Delta^{\times k} \arrow[r, "\mathrm{id}\times c^{\triangleright}"] \arrow[d, "\oplus"'] & \Delta^{\times k} \arrow[d, "\oplus"] \\ \Delta \arrow[r, "c^{\triangleright}"] & \Delta \end{tikzcd}$$

where $c^{\triangleleft}$ and $c^{\triangleright}$ are the maps from Definition 2.6. □

Proposition 2.18. *Let $\mathcal{C}$ be an ∞-category with finite limits. Let $k \geq 1$, and let $Y \in \mathcal{C}_{\Delta^{\times k}}$ be a $2\underline{r}$-Segal object, $\underline{r} \in \mathbb{N}^k$. Then $TY \in \mathcal{C}_{\Delta}$ is a $2r$-Segal object, where $r = \sum r_i$.*

Proof. For $k = 1$, there is nothing to show. In order to verify the upper $2k$-Segal condition by induction, we use the path space criterion as well as (2.1) to obtain

$$u([n], 2k) = \ell([n-1], 2k-1) \oplus \{n\} = \coprod_{j=2k-1}^{n-1} u([j-1], 2k-2) \oplus \{j, n\}. \tag{2.7}$$

This suffices, since for the lower $2k$-Segal condition, we can use the dual decomposition,

$$\ell([n], 2k) = \coprod_{i=1}^{n-2k+1} \{0, i\} \oplus \ell([n] \smallsetminus [i], 2k-2).$$

Now let $k = 2$. We aim to exhibit the n-cells of the total simplicial object as the limit

$$(TY)_n \xrightarrow{\simeq} \varprojlim_{H \in \mathcal{U}([n],4)} \varprojlim_{\substack{I \cup J = H \\ I \leq J}} Y_{I,J}.$$

By (2.7), the $H \in u([n], 4)$ are precisely $H = \{i-1, i, j-1, j, n\}$ with $0 < i < j-1 < n-1$. The corresponding factor in the limit is of the following form,

$$\begin{aligned} &Y_{(i-1)i,i(j-1)jn} \times_{Y_{(i-1)i,(j-1)jn}} Y_{(i-1)i(j-1),(j-1)jn} \times_{Y_{(i-1)i(j-1),jn}} Y_{(i-1)i(j-1)j,jn} \\ \simeq\ &Y_{(i-1)i,i(j-1)n} \times_{Y_{(i-1)i,(j-1)n}} Y_{(i-1)i(j-1),(j-1)jn} \times_{Y_{(i-1)(j-1),jn}} Y_{(i-1)(j-1)j,jn} \end{aligned} \tag{2.8}$$

by the upper 2-Segal property in the last and the lower 2-Segal property in the first coordinate, respectively. But the factors of the form $Y_{(i-1)i,i(j-1)n}$ and $Y_{(i-1)(j-1)j,jn}$ cancel precisely with the non-maximal elements of $\mathcal{U}([n], 4)$, with the exception of the extremal cases $Y_{01,12n}$ and $Y_{0(n-2)(n-1),(n-1)n}$, respectively.

On the other hand, we can describe the n-cells of the total simplicial object as follows,

$$\begin{aligned} (TY)_n \simeq\ &Y_{01,1\ldots n} \times_{Y_{01,2\ldots n}} Y_{012,2\ldots n} \times_{Y_{012,3\ldots n}} \cdots \times_{Y_{0\ldots(n-3),(n-2)(n-1)n}} \\ &Y_{0\ldots(n-2),(n-2)(n-1)n} \times_{Y_{0\ldots(n-2),(n-1)n}} Y_{0\ldots(n-1),(n-1)n} \\ \simeq\ &Y_{01,12n} \times_{Y_{01,2n}} Y_{012,23n} \times_{Y_{012,3n}} \cdots \times_{Y_{0\ldots(n-3),(n-2)(n-1)n}} \\ &Y_{0\ldots(n-2),(n-2)(n-1)n} \times_{Y_{0(n-2),(n-1)n}} Y_{0(n-2)(n-1),(n-1)n} \end{aligned} \tag{2.9}$$

by applying the upper 2-Segal property in the first variable wherever possible, as well as the lower 2-Segal property in the last variable to the final factor. But the factor $Y_{0\ldots(j-1),(j-1)jn}$ is precisely the limit of the (2.8) after cancellation for all $0 < i < j-1$. $\square$

3. The higher Segal construction

Let $\mathcal{D}$ be a pointed category with finite products. In this section, we study a generalization of a construction due to Segal [20] which is similar to a construction proposed (for $k = 2$) by Hesselholt and Madsen [14]. The higher dimensional variants (for $k \geq 3$) are straightforward to define, but do not seem to have appeared in the literature as of yet.

Let Fin_* denote the category of finite pointed sets. For $T \in \mathrm{Fin}_*$, we denote by $\mathcal{P}(T)$ its poset of pointed subsets, considered as a small category.

Definition 3.1. Let $T \in \mathrm{Fin}_*$ be a finite pointed set. A $\mathcal{D}$-valued presheaf

$$\mathcal{F}\colon \mathcal{P}(T)^{\mathrm{op}} \longrightarrow \mathcal{D}$$

on $\mathcal{P}(T)$ is called a sheaf if, for every pointed subset $U \subseteq T$, the canonical map

$$\mathcal{F}(U) \longrightarrow \prod_{u \in U \smallsetminus \{*\}} \mathcal{F}(\{*, u\})$$

is an isomorphism. We denote by $\mathrm{Sh}(T, \mathcal{D})$ the category of $\mathcal{D}$-valued sheaves on $\mathcal{P}(T)$.

Given a map $\rho\colon T \to T'$ in Fin_*, we define the pointed preimage functor

$$\rho^{\times}\colon \mathcal{P}(T') \longrightarrow \mathcal{P}(T),\ U \longmapsto \rho^{-1}(U \smallsetminus \{*\}) \amalg \{*\}.$$

Then the direct image functor $\mathcal{F} \mapsto \rho_*\mathcal{F} = \mathcal{F} \circ \rho^{\times}$ makes the assignment

$$\mathrm{Sh}(-, \mathcal{D})\colon \mathrm{Fin}_* \longrightarrow \mathrm{Cat}$$

into a functor with values in the category of small categories.

Definition 3.2. Let $k \geq 1$. The k-dimensional Segal construction of $\mathcal{D}$ is defined to be the simplicial category

$$S_{\oplus}^{\langle k \rangle}(\mathcal{D}) = \mathrm{Sh}(S^k, \mathcal{D}) \in \mathrm{Cat}_{\Delta},$$

where $S^k = \Delta^k / \partial\Delta^k$ is considered as a simplicial object in Fin_*.

Lemma 3.3. *Let $0 \leq i \leq n$. The face map $\partial_i\colon \mathrm{Sh}(S^k_n, \mathcal{D}) \to \mathrm{Sh}(S^k_{n-1}, \mathcal{D})$ is an isofibration.*

Proof. Let $\rho_i\colon S^k_n \to S^k_I$ be the corresponding map in Fin_*, where $I = [n] \smallsetminus \{i\}$. Given

$$\Phi\colon (\rho_i)_*\mathcal{F} = \partial_i(\mathcal{F}) \xrightarrow{\sim} \mathcal{G}',$$

we extend $\mathcal{G}(\rho_i^\times V) := \mathcal{G}'(V)$ for $V \in \mathcal{P}(S^k_I)$ by

$$\mathcal{G}(U) := \prod_{\substack{\alpha\in U\\ \alpha|_I \not\equiv *}} \mathcal{F}(\{*,\alpha\})$$

otherwise. This is functorial, since $\mathcal{D}$ is pointed, and so Φ exhibits $\mathcal{G}(\rho_i^\times V)$ as the product

$$\mathcal{G}(\rho_i^\times V) = \prod_{\alpha|_I \in V \smallsetminus \{*\}} \mathcal{F}(\{*,\alpha\}).$$

The lifting of Φ itself is then tautological. □

The goal of this section is to prove the following result, which is due to Segal [20], §2, for $k = 1$. Throughout, a lower, resp. upper, d-Segal category means a lower, resp. upper, d-Segal object in Cat, which is not to be confused with a Segal category in the sense of [6].

Theorem 3.4. *Let $k \geq 1$, and $\mathcal{D}$ a pointed category with finite products. The k-dimensional Segal construction $S^{\langle k\rangle}_\oplus(\mathcal{D})$ is a lower $(2k-1)$-Segal category. In particular, it is $2k$-Segal.*

Proof. The last part is an application of Proposition 2.9. Now let $n \geq 2k-1$, and set

$$\mathcal{L} = \mathcal{L}([n], 2k-1).$$

For $I, J \in \mathcal{L}$ with $I \supseteq J$, we denote by $\rho_{I,J}\colon S^k_I \to S^k_J$ the corresponding map of pointed sets, and further write $\rho_I = \rho_{[n],I}$ for brevity. We have to show that the canonical functor

$$\mathrm{Sh}(S^k_n, \mathcal{D}) \longrightarrow \varprojlim_{I\in\mathcal{L}} \mathrm{Sh}(S^k_I, \mathcal{D}) \tag{3.1}$$

is an equivalence of categories. Note that by Lemma 3.3, all transition maps on the right-hand side are isofibrations, so that the limit is 1-categorical. Now consider the functor

$$\mathcal{P}\colon \mathcal{L} \longrightarrow \mathrm{Cat},\ I \longmapsto \mathcal{P}(S^k_I).$$

We form the following version of its Grothendieck construction

$$\pi\colon \mathfrak{X}_\mathcal{P} \longrightarrow \mathcal{L}^{\mathrm{op}}.$$

The category $\mathfrak{X}_\mathcal{P}$ has objects (I, U), where $I \in \mathcal{L}$ and $U \in \mathcal{P}(S^k_I)$, and there is a unique morphism $(I, U) \leq (J, V)$ if $I \supseteq J$ and $U \subseteq \rho^\times_{I,J} V$. The functor π is a cartesian fibration, where a morphism $(I, U) \leq (J, V)$ is cartesian if

$$U = \rho^\times_{I,J} V.$$

The category $\varprojlim_{I\in\mathcal{L}} \mathrm{Sh}(S^k_I, \mathcal{D})$ can be identified with the full subcategory of $\mathrm{Fun}(\mathfrak{X}^{\mathrm{op}}_\mathcal{P}, \mathcal{D})$ spanned by those presheaves $\mathcal{F}$ which satisfy the following conditions.

(a) The presheaf $\mathcal{F}$ maps cartesian morphisms to isomorphisms in $\mathcal{D}$.
(b) For every $I \in \mathcal{L}$, the restriction of $\mathcal{F}$ to the fibre $\pi^{-1}(I) = \mathcal{P}(S^k_I)$ is a sheaf.

A $\mathcal{D}$-valued presheaf on $\mathcal{P}(S^k_n)$ defines a presheaf on $\mathfrak{X}_\mathcal{P}$ via pullback along the functor

$$\varphi_0\colon \mathfrak{X}_\mathcal{P} \longrightarrow \mathcal{P}(S^k_n),\ (I, U) \longmapsto \rho^\times_I U.$$

The lower Segal functor (3.1) is then obtained by restricting this pullback functor along φ_0 to the category of sheaves on $\mathcal{P}(S^k_n)$.

Since the functor φ_0 maps cartesian morphisms to the identity map in $\mathcal{P}(S^k_n)$, it factors over a unique functor $\varphi\colon L\mathfrak{X}_\mathcal{P} \to \mathcal{P}(S^k_n)$, where $L\mathfrak{X}_\mathcal{P}$ denotes the localization of $\mathfrak{X}_\mathcal{P}$ along the set of cartesian morphisms. Note further that imposing condition (a) on a presheaf $\mathcal{F}$ on the category $\mathfrak{X}_\mathcal{P}$ is equivalent to the requirement that $\mathcal{F}$ factors through $L\mathfrak{X}_\mathcal{P}$.

We obtain an adjunction of presheaf categories as follows,

$$\varphi_!\colon \mathcal{D}_{L\mathfrak{X}_\mathcal{P}} \longleftrightarrow \mathcal{D}_{\mathcal{P}(S^k_n)} :\varphi^* \tag{3.2}$$

where the functor φ^* maps the subcategory of sheaves to the subcategory $\varprojlim_{I\in\mathcal{L}} \mathrm{Sh}(S^k_I,\mathcal{D})$. Finally, we introduce the sheafification functor

$$\sigma\colon \mathcal{D}_{\mathcal{P}(S^k_n)} \longrightarrow \mathrm{Sh}(S^k_n,\mathcal{D})$$

as the left adjoint of the inclusion. Here, we need to require the existence of pushouts in $\mathcal{D}$; this assumption is shown to be unnecessary below. Now (3.2) induces an adjunction,

$$\sigma\circ\varphi_!\colon \varprojlim_{I\in\mathcal{L}} \mathrm{Sh}(S^k_I,\mathcal{D}) \longleftrightarrow \mathrm{Sh}(S^k_n,\mathcal{D}) :\varphi^* \tag{3.3}$$

which we claim to be a pair of mutually inverse functors. In order to verify this, we show that the unit and counit are isomorphisms. For the former, it suffices to show that, for every sheaf $\mathcal{G}\in\mathrm{Sh}(S^k_n,\mathcal{D})$ and every subset $\{*,\alpha\}\subseteq S^k_n$ of cardinality 2, the unit morphism

$$(\varphi_!\varphi^*\mathcal{G})(\{*,\alpha\}) \longrightarrow \mathcal{G}(\{*,\alpha\})$$

is invertible. We have

$$(\varphi_!\varphi^*\mathcal{G})(\{*,\alpha\}) \cong \varinjlim_{\substack{(I,U)\in L\mathfrak{X}^{\mathrm{op}}_{\mathcal{P}}\\ \alpha\in\rho^\times_I(U)}} \mathcal{G}(\rho^\times_I(U)).$$

According to Lemma 3.5 (1) below, the indexing category $\varphi^{\mathrm{op}}/\{*,\alpha\}$ of the colimit has a final object $(I_\alpha,\{*,\alpha|_{I_\alpha}\})$, with $\rho^\times_{I_\alpha}(\{*,\alpha|_{I_\alpha}\}) = \{*,\alpha\}$. This immediately implies the claim.

We proceed to prove that, for every object $\mathcal{F}\in\varprojlim_{I\in\mathcal{L}}\mathrm{Sh}(S^k_I,\mathcal{D})$, the counit morphism

$$\mathcal{F}\longrightarrow \varphi^*\sigma\varphi_!\mathcal{F}$$

is invertible. Similarly as above, it suffices to show for all $(J,\{*,\beta\})\in L\mathfrak{X}_{\mathcal{P}}$ that the map

$$\mathcal{F}(J,\{*,\beta\}) \longrightarrow (\varphi^*\sigma\varphi_!\mathcal{F})(J,\{*,\beta\}) \tag{3.4}$$

is an isomorphism in $\mathcal{D}$. Using Lemma 3.5 (1), we compute the right-hand side as

$$(\sigma\varphi_!\mathcal{F})(\rho^\times_J\{*,\beta\}) \cong \prod_{\alpha\in\rho_J^{-1}(\beta)} (\varphi_!\mathcal{F})(\{*,\alpha\}) \cong \prod_{\alpha\in\rho_J^{-1}(\beta)} \mathcal{F}(I_\alpha,\{*,\alpha|_{I_\alpha}\}).$$

Then Lemma 3.5 (2) implies in particular that the map (3.4) is indeed an isomorphism. □

Lemma 3.5. *In the terminology introduced in the proof of Theorem 3.4, let (J,V) be an object of the category $L\mathfrak{X}_{\mathcal{P}}$. Then the following statements hold.*

(1) *Let $\alpha\in\rho^\times_J(V)\smallsetminus\{*\}$. There is a unique morphism*

$$(I_\alpha,\{*,\alpha|_{I_\alpha}\}) \longrightarrow (J,V)$$

in $L\mathfrak{X}_{\mathcal{P}}$, where

$$I_\alpha = \bigcup_{\alpha_i<\alpha_{i+1}} \{i,i+1\}.$$

(2) *Let $\mathcal{F}$ be an object of $\varprojlim_{I\in\mathcal{L}}\mathrm{Sh}(S^k_I,\mathcal{D})\subseteq\mathcal{D}_{L\mathfrak{X}_{\mathcal{P}}}$. There is an isomorphism*

$$\mathcal{F}(J,V) \xrightarrow{\ \simeq\ } \prod_{\alpha\in\rho^\times_J(V)\smallsetminus\{*\}} \mathcal{F}(I_\alpha,\{*,\alpha|_{I_\alpha}\}),$$

whose components are given by restriction along the unique morphisms from (1).

Proof. The even subsets of $[n]$ of cardinality $2k$ are precisely the disjoint unions of k subsets of the form $\{i,i+1\}$. Since $\alpha\not\equiv *$, it follows that I_α is a (possibly non-disjoint) union of k such subsets. However, it is contained in the even subset of $[n]$ of cardinality $2k$ obtained by inductively filling for each $\alpha_{i-1}<\alpha_i<\alpha_{i+1}$ either the maximal gap $j<i$ of I_α or its minimal gap $j>i$.

The key observation is that the subsets $I\in\mathcal{L}$ which contain I_α are exactly those with

$$\rho^\times_I(\rho_I(\{*,\alpha\})) = \{*,\alpha\}.$$

This implies that for a morphism in $L\mathfrak{X}_{\mathcal{P}}$ of the form $(I_\alpha, \{*, \alpha|_{I_\alpha}\}) \leftarrow (I, U) \rightarrow (J', V')$, we always have $U = \{*, \alpha|_I\}$. Thus, the only condition on V' is that $\alpha|_{J'} \in V'$, and we can assume without loss of generality that $V = \{*, \alpha|_J\}$. In order to describe morphisms

$$\mu \colon (I_\alpha, \{*, \alpha|_{I_\alpha}\}) \longrightarrow (J, \{*, \alpha|_J\})$$

in $L\mathfrak{X}_{\mathcal{P}}$, we consider α as a sequence of k bars situated in a diagram of $[n]$, signifying the fact that $\alpha_j < \alpha_{j+1}$ by the bar $j|(j+1)$. An object $(I, \{*, \alpha|_I\})$ corresponds to marking the elements $i \in I \subseteq [n]$, and the zig-zag μ is a sequence of moves which shift the markings. Each move consists of adding and then removing certain markings (adhering to the constraints imposed by the definition of $L\mathfrak{X}_{\mathcal{P}}$).

The object $(I_\alpha, \{*, \alpha|_{I_\alpha}\})$ marks all elements adjacent to a bar; that is, we visualize it as a diagram of the following exemplary form,

$$- - \bullet| \bullet - \bullet | \bullet | \bullet - - \bullet| \ \ldots \ | \bullet - - -$$

where '$\bullet$' indicates a marked element and '$-$' an unmarked element of $[n]$. A single '$\bullet$' at a vertex $i \in [n]$ between two bars (i.e., '$|\bullet|$') corresponds precisely to the case $\alpha_{i-1} < \alpha_i < \alpha_{i+1}$ from above. Since $\alpha|_J \not\equiv *$, this implies that $i \in J$; in fact, this condition states exactly that there is an element of J in every region cut out by the bars.

In order to see that μ is unique (if it exists), we first note that a '$\bullet$' can never cross a bar. Indeed, this would require some move

$$(I, \{*, \alpha|_I\}) \longleftarrow (H, \{*, \alpha|_H\}) \longrightarrow X$$

which adds a marking to some $i \in I_\alpha$. Then we can define $\beta \in \mathcal{P}(S^k_H)$ by $\beta|_{H \smallsetminus \{i\}} \equiv \alpha|_{H \smallsetminus \{i\}}$ and by replacing the jump $\alpha_{i-1} = \alpha_i < \alpha_{i+1}$ with $\beta_{i-1} < \beta_i = \beta_{i+1}$; but this contradicts the requirement that the left leg of the move be cartesian.

Then uniqueness follows from the fact that moves which are constrained within different sets of bars commute with one another, while the moves occuring between two particular bars all compose to the same shift of markings.

For the existence of μ, we observe that after adding markings for each '$| \bullet |$' as described above (filling the gaps of I_α; where we can always choose the gap closest to an element of J), we can remove at least one marking adjacent to each bar (with the exception of the '$| \bullet |$', in which case the vertex lies in J already, as we have seen).

Then we can move each '$\bullet$' towards its intended position in J by repeatedly marking the adjacent vertex and removing the original; moreover, once a '$\bullet$' has reached its destination, we can duplicate it. This requires no further sets of the form $\{i, i+1\}$ to cover all markings, that is, we stay within $\mathcal{L}$ in this process (as of course $J \in \mathcal{L}$ itself).

Finally, statement (2) follows from the above, since $\mathcal{F}(J, V) = \mathcal{F}(\rho^\times_J V)$, and similarly,

$$\mathcal{F}(I_\alpha, \{*, \alpha|_{I_\alpha}\}) = \mathcal{F}(\{*, \alpha\});$$

but condition (b) tells us that the restriction $\mathcal{F}|_{\mathcal{P}(S^k_J)}$ to the fibre $\pi^{-1}(J)$ is a sheaf. □

Note that the n-cells of the 1-dimensional Segal construction constitute a pointed category with finite products again. We write $S^{(k)}_\oplus(\mathcal{D}) \in \mathrm{Cat}_{\Delta^{\times k}}$ for the k-fold iterate of $S^{\langle 1\rangle}_\oplus(\mathcal{D})$.

The following result not only provides another perspective on the higher Segal construction, but together with Proposition 2.15, it yields an alternative proof of Theorem 3.4.

Theorem 3.6. *Let $k \geq 1$. The k-dimensional Segal construction of a pointed category $\mathcal{D}$ with finite products is naturally equivalent to the total simplicial object of its k-fold Segal construction,*

$$S^{\langle k\rangle}_\oplus(\mathcal{D}) \xrightarrow{\simeq} TS^{(k)}_\oplus(\mathcal{D}).$$

Proof. For $k = 1$, there is nothing to show. By induction, it suffices to prove, for every $k > 1$,

$$S^{\langle k\rangle}_\oplus(\mathcal{D}) \xrightarrow{\simeq} T(S^{\langle k-1\rangle}_\oplus(S^{\langle 1\rangle}_\oplus(\mathcal{D}))).$$

Therefore, we need to construct, for all $k, n > 0$, a natural equivalence of categories

$$\mathrm{Sh}(S_n^k, \mathcal{D}) \xrightarrow{\simeq} \varprojlim_{\substack{I \cup J = [n] \\ I \leq J}} \mathrm{Sh}(S_I^{k-1}, \mathrm{Sh}(S_J^1, \mathcal{D})). \tag{3.5}$$

Note that by Lemma 3.3, the right-hand side is computed by the 1-categorical limit. Now, we first consider $\mathrm{Sh}(S_I^{k-1}, \mathrm{Sh}(S_J^1, \mathcal{D}))$ as a full subcategory of

$$\begin{aligned} &\mathrm{Fun}(\mathcal{P}(S_I^{k-1})^{\mathrm{op}}, \mathrm{Fun}(\mathcal{P}(S_J^1)^{\mathrm{op}}, \mathcal{D})) \\ \simeq\ &\mathrm{Fun}(\mathcal{P}(S_I^{k-1})^{\mathrm{op}} \times \mathcal{P}(S_J^1)^{\mathrm{op}}, \mathcal{D})) \\ \simeq\ &\mathrm{Fun}(\mathcal{P}(S_I^{k-1} \amalg S_J^1)^{\mathrm{op}}, \mathcal{D})), \end{aligned}$$

where $\amalg$ is the coproduct of pointed sets. Define the two maps $\lambda, \rho \colon S_n^k \to S_I^{k-1} \amalg S_J^1$ by

$$\lambda(\alpha) = \begin{cases} \alpha|_I & \text{if } \alpha(I) \subseteq [k-1], \\ * & \text{otherwise,} \end{cases} \quad \text{and } \rho(\alpha) = \begin{cases} \alpha|_J & \text{if } \alpha(J) \subseteq \{k-1, k\}, \\ * & \text{otherwise.} \end{cases}$$

Then we claim that an equivalence as in (3.5) descends from the induced functors

$$\mathrm{Sh}(S_n^k, \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{P}(S_I^{k-1} \amalg S_J^1)^{\mathrm{op}}, \mathcal{D})),\ \mathcal{F} \longmapsto (W \mapsto \mathcal{F}(\lambda^\times W \cap \rho^\times W)).$$

Firstly, let us show that the essential image of this functor lies in $\mathrm{Sh}(S_I^{k-1}, \mathrm{Sh}(S_J^1, \mathcal{D}))$. This amounts to proving that for every $U \in \mathcal{P}(S_I^{k-1})$, the assignment $V \mapsto \mathcal{F}(\lambda^\times U \cap \rho^\times V)$ is a sheaf on $\mathcal{P}(S_J^1)$. But this reduces straight-forwardly to the sheaf property of $\mathcal{F}$, as

$$\mathcal{F}(\lambda^\times U \cap \rho^\times V) \xrightarrow{\sim} \prod_{\varepsilon \in V} \prod_{\alpha \in \lambda^\times U \cap \rho^\times \{*, \varepsilon\}} \mathcal{F}(\{*, \alpha\}) \xleftarrow{\sim} \prod_{\varepsilon \in V} \mathcal{F}(\lambda^\times U \cap \rho^\times \{*, \varepsilon\}).$$

The analogous argument shows that $U \mapsto \mathcal{F}(\lambda^\times U \cap \rho^\times V)$ is a sheaf for every $V \in \mathcal{P}(S_J^1)$, which yields the rest of the claim, since products of sheaves are computed point-wise.

Next, to see that these functors form into a map into the limit boils down to the transitivity of restriction, $(\alpha|_I)|_{I \smallsetminus J} = \alpha|_{I \smallsetminus J}$ for $I, J \subseteq [n]$ as before.

Finally, to construct the inverse, given $(\mathcal{F}_{I,J}) \in \varprojlim_{\substack{I \cup J = [n] \\ I \leq J}} \mathrm{Sh}(S_I^{k-1}, \mathrm{Sh}(S_J^1, \mathcal{D}))$, we extend

$$\{*, \alpha\} \longmapsto \mathcal{F}_{I(\alpha), J(\alpha)}(\{*, \alpha|_{I(\alpha)}\})(\{*, \alpha|_{J(\alpha)}\}),\ \text{for } I(\alpha) = \alpha^{-1}[k-1],\ J(\alpha) = \{i_\alpha\} \amalg \alpha^{-1}(k),$$

where $i_\alpha \in I(\alpha)$ is the maximal element, to a sheaf on $\mathcal{P}(S_n^k)$. In one direction, we use

$$\lambda^\times \{*, \alpha|_{I(\alpha)}\} \cap \rho^\times \{*, \alpha|_{J(\alpha)}\} = \{*, \alpha\}$$

to see that the two constructions are inverse to one another. Conversely, let $\mathcal{F} \in \mathrm{Sh}(S_n^k, \mathcal{D})$ denote the image of some $(\mathcal{F}_{I,J})$. For every $(U, V) \in \mathcal{P}(S_I^{k-1}) \times \mathcal{P}(S_J^1)$, $I \cap J \neq \emptyset$, we get

$$\mathcal{F}(\lambda^\times U \cap \rho^\times V) \xrightarrow{\sim} \mathcal{F}_{I,J}(U)(V)$$

via the following isomorphism,

$$\prod_{\alpha \in \lambda^\times U \cap \rho^\times V} \mathcal{F}_{I(\alpha), J(\alpha)}(\{*, \alpha|_{I(\alpha)}\})(\{*, \alpha|_{J(\alpha)}\}) \xrightarrow{\sim} \prod_{(\beta, \varepsilon) \in U \times V} \mathcal{F}_{I,J}(\{*, \beta\})(\{*, \varepsilon\}).$$

This map arises via the mutually inverse identifications $\alpha \mapsto (\lambda(\alpha), \rho(\alpha))$ and $(\beta, \varepsilon) \mapsto \beta \cup \varepsilon$ of the respective indexing sets, with its components given by the fact that

$$(\mathcal{F}_{I,J}) \in \varprojlim_{\substack{I \cup J = [n] \\ I \leq J}} \mathrm{Sh}(S_I^{k-1}, \mathrm{Sh}(S_J^1, \mathcal{D})),$$

which provides, for each $\alpha = \beta \cup \varepsilon$ as above, a chain of isomorphisms

$$\mathcal{F}_{I(\alpha), J(\alpha)}(\{*, \alpha|_{I(\alpha)}\})(\{*, \alpha|_{J(\alpha)}\}) \cong \mathcal{F}_{I', J'}(\{*, \alpha|_{I'}\})(\{*, \alpha|_{J'}\}) \cong \ldots \cong \mathcal{F}_{I,J}(\{*, \beta\})(\{*, \varepsilon\}),$$

where $I' = I(\alpha) \smallsetminus \{i_\alpha\}$, and $J' = \{i_\alpha - 1\} \amalg J(\alpha)$. $\square$

Corollary 3.7 (Delooping). *Let $k \geq 1$. There is a natural homotopy equivalence*

$$\Omega^k |S_{\oplus}^{\langle k \rangle}(\mathcal{D})^{\times}| \xrightarrow{\simeq} K^{\oplus}(\mathcal{D}), \tag{3.6}$$

where $K^{\oplus}(\mathcal{D})$ denotes the direct sum K-theory space of $\mathcal{D}$.

Proof. By Theorem 3.6 as well as Lemma 3.8 below, this reduces to the case $k = 1$, which is a special case of [20], Proposition 1.5. $\square$

Lemma 3.8. *Let $Y \in \mathrm{Top}_{\Delta^{\times k}}$ be a k-fold simplicial space. Then the natural map $DY \to TY$ induces an equivalence*

$$|Y| \simeq |DY| \xrightarrow{\simeq} |TY|.$$

Proof. Let us consider $Y_{\bullet,\bullet} \in (\mathrm{Set}_{\Delta})_{\Delta^{\times k}}$. When $k = 2$, we obtain the claim from

$$|DY| \simeq |D([m] \mapsto DY_{m,\bullet})| \xleftarrow{\simeq} |T([m] \mapsto DY_{m,\bullet})| \xrightarrow{\simeq} |T([m] \mapsto TY_{m,\bullet})| \simeq |TY|,$$

which holds by [4], Theorem 1.1 and §7 (also cf. [21], Theorem 1). By iterating this argument, the statement follows for all k. $\square$

4. The higher Waldhausen construction

Let $\mathcal{E}$ be a proto-exact category ([8], Definition 2.4.2; for example, any exact category). We denote its wide subcategories of admissible monomorphisms, resp. admissible epimorphisms, by $\mathcal{E}^{\triangleleft}$, resp. $\mathcal{E}^{\triangleright}$.

Definition 4.1. A morphism $A \to B$ in $\mathcal{E}$ is called admissible if it factors as the composition

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, two heads] & & B \\ & C \arrow[ur, tail] & \end{tikzcd}$$

of an admissible epimorphism and an admissible monomorphism in $\mathcal{E}$.

Consider a sequence of admissible morphisms together with their corresponding (unique up to unique isomorphism) factorizations as above,

$$\begin{tikzcd} A_k \arrow[rr] \arrow[dr, two heads] & & A_{k-1} \arrow[rr] \arrow[dr, two heads] & & \dots \arrow[rr] \arrow[dr, two heads] & & A_0. \\ & C_k \arrow[ur, tail] & & C_{k-1} \arrow[ur, tail] & \dots & C_1 \arrow[ur, tail] & \end{tikzcd}$$

The sequence $A_k \to A_{k-1} \to \ldots \to A_0$ will be called

- acyclic, if $C_{i+1} \rightarrowtail A_i \twoheadrightarrow C_i$ is a short exact sequence in $\mathcal{E}$ for all $0 < i < k$.

An acyclic sequence in $\mathcal{E}$ as above is called

- left exact, if $A_k \to A_{k-1}$ is an admissible monomorphism (equivalently, $A_k \xrightarrow{\sim} C_k$),
- right exact, if $A_1 \to A_0$ is an admissible epimorphism (i.e., $C_1 \xrightarrow{\sim} A_0$),
- exact, if it is both left exact and right exact.

Let $k, n \geq 0$. We write $\mathrm{Fun}([k], [n])$ for the category of functors between the standard ordinals $[k]$ and $[n]$, considered as small categories. Note that the objects of this category correspond bijectively to the set of k-simplices of the simplicial set Δ^n.

Definition 4.2. Let $k \geq 0$. For every $n \geq 0$, we define the category

$$S_n^{[k]}(\mathcal{E}) \subseteq \mathrm{Fun}(\mathrm{Fun}([k], [n]), \mathcal{E})$$

to be the full subcategory spanned by all diagrams A satisfying the following conditions.

(a) For every $(k-1)$-simplex α in Δ^n, we have

$$A_{s_{k-1}^* \alpha} = \ldots = A_{s_0^* \alpha} = 0.$$

(b) For every $(k+1)$-simplex γ in Δ^n, the corresponding sequence

$$A_{d^*_{k+1}\gamma} \longrightarrow A_{d^*_k\gamma} \longrightarrow \dots \longrightarrow A_{d^*_1\gamma} \longrightarrow A_{d^*_0\gamma}$$

is acyclic.

We define $S^{\langle k]}_n(\mathcal{E})$, resp. $S^{[k\rangle}_n(\mathcal{E})$, as the full subcategory of $S^{[k]}_n(\mathcal{E})$ on all A such that

(b') For every $(k+1)$-simplex γ in Δ^n, the following sequence is left, resp. right, exact.

$$A_{d^*_{k+1}\gamma} \longrightarrow A_{d^*_k\gamma} \longrightarrow \dots \longrightarrow A_{d^*_1\gamma} \longrightarrow A_{d^*_0\gamma}$$

Finally, we introduce $S^{\langle k\rangle}_n(\mathcal{E}) \subseteq S^{[k]}_n(\mathcal{E})$ as the full subcategory of diagrams A which satisfy

(b'') For every $(k+1)$-simplex γ in Δ^n, the sequence

$$A_{d^*_{k+1}\gamma} \longrightarrow A_{d^*_k\gamma} \longrightarrow \dots \longrightarrow A_{d^*_1\gamma} \longrightarrow A_{d^*_0\gamma}$$

is exact.

By functoriality in $[n]$, we obtain simplicial categories

$$S^{\langle k\rangle}(\mathcal{E}),\ S^{\langle k]}(\mathcal{E}),\ S^{[k\rangle}(\mathcal{E}),\ S^{[k]}(\mathcal{E}) \in \mathrm{Cat}_\Delta\,.$$

We call $S^{\langle k\rangle}(\mathcal{E})$ the k-dimensional Waldhausen construction of $\mathcal{E}$.

Remark 4.3. The $(k+1)$-skeleton of $S^{\langle k\rangle}(\mathcal{E})$ has an immediate description. Namely,

$$S^{\langle k\rangle}_0(\mathcal{E}) \simeq \dots \simeq S^{\langle k\rangle}_{k-1}(\mathcal{E}) \simeq 0,\ S^{\langle k\rangle}_k(\mathcal{E}) \simeq \mathcal{E},$$

while $S^{\langle k\rangle}_{k+1}(\mathcal{E})$ is equivalent to the category of k-extensions in $\mathcal{E}$. The dimensionality of the Waldhausen construction also refers to the fact that the k-skeleton of $|S^{\langle k\rangle}(\mathcal{E})^\times|$ is equivalent to the k-fold suspension $S^k \wedge |\mathcal{E}^\times|$, and not the dimension of the diagrams it classifies.

Example 4.4. Let $k \geq 0$, and let $\mathcal{E}$ be a proto-exact category.

(1) For $k = 0$, the degeneracy condition (a) is empty, and therefore,

$$S^{\langle 0\rangle}(\mathcal{E}) \simeq N^{\mathcal{E}}(\mathcal{E}^\times) \simeq \mathcal{E}$$

is the nerve of the maximal subgroupoid of $\mathcal{E}$, categorified by arbitrary morphisms in $\mathcal{E}$, which is equivalent to the constant object $\mathcal{E}$ itself. Similarly, $S^{\langle 0]}(\mathcal{E}) \simeq N^{\mathcal{E}}(\mathcal{E}^{\triangleleft})$, and dually, $S^{[0\rangle}(\mathcal{E}) \simeq N^{\mathcal{E}}(\mathcal{E}^{\triangleright})$. Rather more subtly, $S^{[0]}(\mathcal{E}) \xrightarrow{\simeq} N^{\mathcal{E}}(\mathcal{E})$ if and only if $\mathcal{E}$ is proto-abelian (in the sense of Remark 5.3), by [10], Proposition 3.1.

(2) For $k = 1$, we recover a version of the original construction $S^{\langle 1\rangle}(\mathcal{E}) \simeq S(\mathcal{E})$ from [23], whose n-cells are given by the category formed by strictly upper triangular diagrams with bicartesian squares, as follows.

$$\begin{tikzcd}
0 \arrow[r, tail] & A_{01} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & A_{02} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & A_{0n} \arrow[d, two heads] \\
& 0 \arrow[r, tail] & A_{12} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & \ddots \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & A_{1n} \arrow[d, two heads] \\
& & 0 \arrow[r, tail] & \ddots \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & \vdots \arrow[d, two heads] \\
& & & 0 \arrow[r, tail] & A_{(n-1)n} \arrow[d, two heads] \\
& & & & 0
\end{tikzcd} \tag{4.1}$$

This is a refinement of Quillen's foundational construction $Q(\mathcal{E})$ in [16], which is the category of correspondences in $\mathcal{E}$ of the form $C \twoheadleftarrow B \rightarrowtail A$. Namely, the forgetful functor from the edgewise subdivision $ES(\mathcal{E}) \to N^{\mathcal{E}}(Q(\mathcal{E}))$ is an equivalence, that is, the whole diagram $A \in S_{2n+1}(\mathcal{E})$ of shape (4.1) is uniquely recovered from

$$\begin{tikzcd}[column sep=small]
 & A_{(n-1)(n+1)} \arrow[dl, two heads] \arrow[dr, tail] & & \cdots \arrow[dl, two heads] \arrow[dr, tail] & & A_{0(2n)} \arrow[dl, two heads] \arrow[dr, tail] & \\
A_{n(n+1)} & & A_{(n-1)(n+2)} & \cdots & A_{1(2n)} & & A_{0(2n+1)}
\end{tikzcd} \tag{4.2}$$

by taking successive pullbacks and pushouts in $\mathcal{E}$.

(3) For $k = 2$, the simplicial category $S^{\langle 2\rangle}(\mathcal{E})$ was introduced by Hesselholt-Madsen [14]. An element $A \in S_4^{\langle 2\rangle}(\mathcal{E})$ of its 4-cells is a diagram of the following form.

$$\begin{tikzcd}[column sep=small]
A_{012} \arrow[rr, tail] & & A_{013} \arrow[rr, tail] \arrow[d] \arrow[drr, phantom, "\lrcorner" very near start] & & A_{014} \arrow[d] & & \\
 & & A_{023} \arrow[rr, tail] \arrow[dr, two heads] & & A_{024} \arrow[dr, two heads] \arrow[dd] & & \\
 & & & A_{123} \arrow[rr, tail, crossing over] & & A_{124} \arrow[dd] & \\
 & & & & A_{034} \arrow[dr, two heads] \arrow[ur, phantom, "\ulcorner" very near end] & & \\
 & & & & & A_{134} \arrow[dr, two heads] & \\
 & & & & & & A_{234}
\end{tikzcd} \tag{4.3}$$

Note that the middle square is neither cartesian nor cocartesian. Rather, the diagram consists of bicartesian cubes (cf. Remark 4.5), as indicated in the following picture.

$$\begin{tikzcd}[column sep=small, row sep=small]
A_{012} \arrow[rr, tail] \arrow[dr, two heads] \arrow[dd] & & A_{013} \arrow[rr, tail] \arrow[dr, two heads] \arrow[dd] & & A_{014} \arrow[dr, two heads] \arrow[dd] & & \\
 & 0 \arrow[rr, tail, crossing over] & & 0 \arrow[rr, tail, crossing over] & & 0 \arrow[dd] & \\
0 \arrow[rr, tail] \arrow[dr, two heads] & & A_{023} \arrow[rr, tail] \arrow[dr, two heads] \arrow[dd] & & A_{024} \arrow[dr, two heads] \arrow[dd] & & \\
 & 0 \arrow[rr, tail] \arrow[from=uu, crossing over] & & A_{123} \arrow[rr, tail] \arrow[from=uu, crossing over] & & A_{124} \arrow[dr, two heads] \arrow[dd] & \\
 & & 0 \arrow[rr, tail] \arrow[dr, two heads] & & A_{034} \arrow[dr, two heads] & & 0 \arrow[dd] \\
 & & & 0 \arrow[rr, tail] \arrow[dr, two heads] \arrow[from=uu, crossing over] & & A_{134} \arrow[dr, two heads] & \\
 & & & & 0 \arrow[rr, tail] & & A_{234}
\end{tikzcd} \tag{4.4}$$

Remark 4.5. In general, $S^{\langle k\rangle}(\mathcal{E})$ is composed of $(k+1)$-dimensional bicartesian hypercubes. More precisely, let $A \in S_n^{[k]}(\mathcal{E})$. Then A lies in $S_n^{\langle k]}(\mathcal{E})$ if and only if

$$A_\beta \;=\; \varprojlim_{\substack{\beta < \beta' \\ \beta' - \beta \leq 1}} A_{\beta'} \tag{4.5}$$

for every k-simplex β in Δ^n with $\beta_{k-i} < n - i$ for $0 \leq i \leq k$. Then Lemma 4.6 below implies that the dual condition defines $S_n^{[k\rangle}(\mathcal{E})$ inside $S_n^{[k]}(\mathcal{E})$, so that $A \in S_n^{[k\rangle}(\mathcal{E})$ if and only if

$$\varinjlim_{\substack{\beta' < \beta \\ \beta - \beta' \leq 1}} A_{\beta'} \;=\; A_\beta$$

for every k-simplex β in Δ^n with $i < \beta_i$ for all $0 \le i \le k$. Together, these yield the claim for

$$S_n^{\langle k \rangle}(\mathcal{E}) \simeq S_n^{\langle k]}(\mathcal{E}) \times_{S_n^{[k]}(\mathcal{E})} S_n^{[k\rangle}(\mathcal{E}).$$

In order to see the first statement, note that the sequence of admissible morphisms $A_{d_\bullet^*\gamma}$ corresponding to some $(k+1)$-simplex γ in Δ^n defines a hypercube $\operatorname{conv}(A_{d_\bullet^*\gamma})$, formed by all A_β with $d_{k+1}^*\gamma \le \beta \le d_0^*\gamma$. If the maps (4.5) are isomorphisms, the minimal subhypercubes of $\operatorname{conv}(A_{d_\bullet^*\gamma})$ are cartesian, and hence so is $\operatorname{conv}(A_{d_\bullet^*\gamma})$ as their composition. Therefore,

$$A_{d_{k+1}^*\gamma} \;=\; \varprojlim_{d_{k+1}^*\gamma < \beta \le d_0^*\gamma} A_\beta \;=\; \ker(A_{d_k^*\gamma} \to A_{d_{k-1}^*\gamma}).$$

Conversely, let β be a k-simplex in Δ^n with $\beta_{k-i} < n-i$ for all $0 \le i \le k$. Then we can infer inductively that the hypercube Q_β on all $\beta' \ge \beta$ with $\beta' - \beta \le 1$ is cartesian. First, assume that $|\beta| = \sum \beta_i$ is maximal. Thus, $\beta = (d_0^*)^{n-k}\Delta_n^n - 1 = d_{k+1}^*(d_0^*)^{n-k-1}\Delta_n^n$. But then Q_β is exactly given by the hypercube $\operatorname{conv}(A_{d_\bullet^*(d_0^*)^{n-k-1}\Delta_n^n})$.

In general, consider $\gamma = \beta \amalg \{n\}$. Then $d_{k+1}^*\gamma = \beta \le \beta + 1 \le d_0^*\gamma$, and hence $\operatorname{conv}(A_{d_\bullet^*\gamma})$ contains Q_β entirely. But its complement is covered by hypercubes $Q_{\widetilde{\beta}}$ with $|\widetilde{\beta}| > |\beta|$, which are cartesian by induction. Therefore, so are all of their compositions, and thus so is Q_β.

The following observation makes the inherent symmetry by duality precise.

Lemma 4.6. *The duality on Δ induces equivalences of simplicial categories*

$$S^{\langle k \rangle}(\mathcal{E}) \xrightarrow{\simeq} S^{\langle k \rangle}(\mathcal{E}^{\mathrm{op}}),$$
$$S^{\langle k]}(\mathcal{E}) \xrightarrow{\simeq} S^{[k\rangle}(\mathcal{E}^{\mathrm{op}}),$$
$$S^{[k]}(\mathcal{E}) \xrightarrow{\simeq} S^{[k]}(\mathcal{E}^{\mathrm{op}}).$$

Proof. This is immediate from the definitions. □

Lemma 4.7. *Let $k \ge 0$, $n \ge 1$, $0 \le i \le n$, and let $\mathcal{E}$ be a proto-exact category. The face map*

$$\partial_i \colon S_n^{[k]}(\mathcal{E}) \to S_{n-1}^{[k]}(\mathcal{E})$$

is an isofibration. In particular, the analogous statements hold for $S^{\langle k]}(\mathcal{E})$ and $S^{[k\rangle}(\mathcal{E})$, as well as the higher Waldhausen construction of $\mathcal{E}$.

Proof. Let $\Phi\colon \partial_i(A) \xrightarrow{\sim} B'$ be an isomorphism in $S_{n-1}^{[k]}(\mathcal{E})$. We construct a lift $B \in S_n^{[k]}(\mathcal{E})$ of B' as follows,

$$B\colon ([k] \xrightarrow{\beta} [n]) \longmapsto \begin{cases} A_\beta & \text{if } \beta \notin \operatorname{im}(d_i)_*, \\ B'_\alpha & \text{if } \beta = (d_i)_*\alpha. \end{cases}$$

The map in B for $\beta \le \widetilde{\beta}$ is given by the corresponding arrow in A, resp. B', if $\beta, \widetilde{\beta} \notin \operatorname{im}(d_i)_*$, resp. both $\beta = (d_i)_*\alpha$ and $\widetilde{\beta} = (d_i)_*\widetilde{\alpha}$. Otherwise, we define

$$(B_\beta \to B_{\widetilde{\beta}}) = \begin{cases} (A_\beta \to A_{\widetilde{\beta}} \xrightarrow{\Phi} B'_{\widetilde{\alpha}}) & \text{if } \beta \notin \operatorname{im}(d_i)_* \text{ and } \widetilde{\beta} = (d_i)_*\widetilde{\alpha}, \\ (B'_\alpha \xleftarrow{\Phi} A_\beta \to A_{\widetilde{\beta}}) & \text{if } \beta = (d_i)_*\alpha \text{ and } \widetilde{\beta} \notin \operatorname{im}(d_i)_*. \end{cases}$$

The lifting of Φ itself is then straightforward. □

Remark 4.8. In particular, all the limits with transition maps given by compositions of ∂_i we consider throughout are computed by the respective 1-categorical limits; we will make no further mention of this henceforth.

This also applies to Theorem 4.12, whose proof in turn can be used to provide an alternative argument for Lemma 4.7 by reducing to the classical case $k = 1$ (which further reduces to the case $k = 0$ via the equivalences $S_{n-1}^{\langle 0]}(\mathcal{E}) \simeq S_n^{\langle 1 \rangle}(\mathcal{E}) \simeq S_{n-1}^{[0\rangle}(\mathcal{E})$ from Lemma 4.9 below).

Lemma 4.9. *The path spaces of the one-dimensional Waldhausen construction are given by*

$$P^{\triangleleft}S^{\langle 1 \rangle}(\mathcal{E}) \simeq S^{\langle 0]}(\mathcal{E}) \simeq N^{\mathcal{E}}(\mathcal{E}^{\triangleleft}), \text{ and dually, } P^{\triangleright}S^{\langle 1 \rangle}(\mathcal{E}) \simeq S^{[0\rangle}(\mathcal{E}) \simeq N^{\mathcal{E}}(\mathcal{E}^{\triangleright}).$$

Proof (cf. [8]*, Lemma 2.4.9).* Let $n \geq 1$. We construct an inverse to the forgetful functor

$$P^{\triangleright} S^{\langle 1 \rangle}_{n-1}(\mathcal{E}) \longrightarrow S^{[0\rangle}_{n-1}(\mathcal{E}).$$

Given $A \in S^{[0\rangle}_{n-1}(\mathcal{E})$, we define $\widehat{A} \in S^{\langle 1 \rangle}_n(\mathcal{E}) \simeq P^{\triangleright} S^{\langle 1 \rangle}_{n-1}(\mathcal{E})$ as its right Kan extension along

$$[n-1] \xrightarrow{d_n} [n] \xrightarrow{-\cup\{n\}} C \hookrightarrow \operatorname{Fun}([1],[n]),$$

where $C = \{\beta \in \operatorname{Fun}([1],[n]) \mid \beta_0 = \beta_1 \text{ or } \beta_1 = n\}$ is considered as a full subcategory. Explicitly, this results in extending $A \cong (A_{0n} \twoheadrightarrow \ldots \twoheadrightarrow A_{(n-1)n})$ by zeroes on the diagonal as in (4.1), then taking successive pullbacks to recover the whole Waldhausen cell. □

We are now prepared to state our main result.

Theorem 4.10. *Let $\mathcal{E}$ be a proto-exact category, and $k \geq 0$. The k-dimensional Waldhausen construction $S^{\langle k \rangle}(\mathcal{E})$ is a $2k$-Segal category.*

Proof. The case $k = 0$ is settled by Example 4.4 (1). For $k = 1$, this is one of the main results of [8], namely Proposition 2.4.8. As observed in *op.cit.*, Example 6.3.3, we can apply the path space criterion to Lemma 4.9 to conclude.

In general, this strategy only works under additional assumptions, as explained in §5. Instead, the result follows from Theorem 4.12 below, together with Proposition 2.18. □

In light of work in progress by Bergner, Osorno, Ozornova, Rovelli, and Scheimbauer, another proof for $k = 1$ is provided by Example 4.4 (2), where we have seen that $ES^{\langle 1 \rangle}(\mathcal{E})$ is lower 1-Segal.

An analogue of Theorem 4.10 in the context of stable ∞-categories is a result of work in progress by Dyckerhoff and Jasso.

Definition 4.11. We write $S^{(k)}(\mathcal{E}) \in \operatorname{Cat}_{\Delta^{\times k}}$ for the k-fold iterate of the 1-dimensional Waldhausen construction, where each $S^{\langle 1 \rangle}_n(\mathcal{E})$ carries the point-wise proto-exact structure.

Theorem 4.12. *Let $k \geq 0$, and let $\mathcal{E}$ be a proto-exact category. There is a natural equivalence of simplicial categories between the k-dimensional Waldhausen construction of $\mathcal{E}$ and the total simplicial object of its k-fold Waldhausen construction,*

$$S^{\langle k \rangle}(\mathcal{E}) \xrightarrow{\simeq} TS^{(k)}(\mathcal{E}).$$

Proof. For $k \leq 1$, this is tautological. By the same reasoning as in the proof of Theorem 3.6, it is sufficient to construct, for all $n, k \geq 1$, a natural equivalence of categories

$$S^{\langle k \rangle}_n(\mathcal{E}) \xrightarrow{\simeq} \varprojlim_{\substack{I \cup J = [n] \\ I \leq J}} S^{\langle k-1 \rangle}_I(S^{\langle 1 \rangle}_J(\mathcal{E})).$$

To define the functor, we use that the right-hand side is a full subcategory of

$$\begin{aligned} &\varprojlim_{\substack{I \cup J = [n] \\ I \leq J}} \operatorname{Fun}(\operatorname{Fun}([k-1], I), \operatorname{Fun}(\operatorname{Fun}([1], J), \mathcal{E})) \\ \simeq\ &\varprojlim_{\substack{I \cup J = [n] \\ I \leq J}} \operatorname{Fun}(\operatorname{Fun}([k-1], I) \times \operatorname{Fun}([1], J), \mathcal{E}) \\ \simeq\ &\operatorname{Fun}(\varinjlim_{\substack{I \cup J = [n] \\ I \leq J}} \operatorname{Fun}([k-1], I) \times \operatorname{Fun}([1], J), \mathcal{E}). \end{aligned}$$

Consider the following full subcategory of the indexing category for its elements.

$$\{(\alpha, \varepsilon) \in \varinjlim_{\substack{I \cup J = [n] \\ I \leq J}} \operatorname{Fun}([k-1], I) \times \operatorname{Fun}([1], J) \mid \alpha_{k-1} = \varepsilon_0\}$$

This is equivalent to $\operatorname{Fun}([k],[n])$ via $(\alpha,\varepsilon)\mapsto \alpha\cup\varepsilon$, with inverse induced by the functor

$$\operatorname{Fun}([k],[n]) \longrightarrow \varinjlim_{\substack{I\cup J=[n]\\ I\leq J}} \operatorname{Fun}([k-1],I)\times\operatorname{Fun}([1],J),\ \beta\longmapsto (\beta|_{[k-1]},\beta|_{\{k-1,k\}}).$$

Now let $A\in S_n^{\langle k\rangle}(\mathcal{E})$. Then its left Kan extension $A^!$ along the inclusion

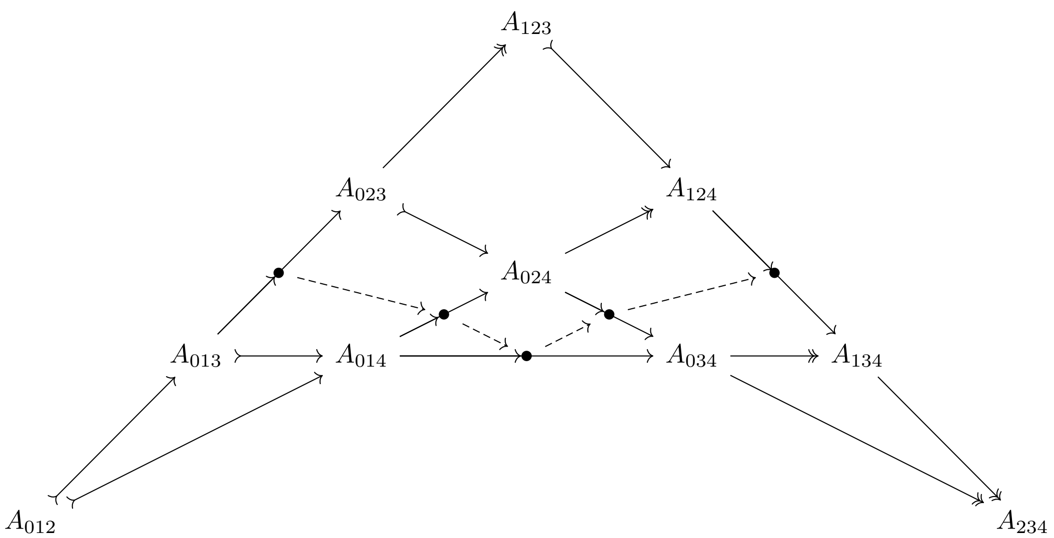
$$\operatorname{Fun}([k],[n]) \hookrightarrow \{(\alpha,\varepsilon)\in \varinjlim_{\substack{I\cup J=[n]\\ I\leq J}} \operatorname{Fun}([k-1],I)\times\operatorname{Fun}([1],J)\mid \alpha_{k-1}=\varepsilon_0 \text{ or } \varepsilon_0=\varepsilon_1\}$$

amounts to an (iterated) extension by zero, as in the proof of Lemma 4.9. Then we define the image $\widehat{A}\in \varprojlim_{\substack{I\cup J=[n]\\ I\leq J}} S_I^{\langle k-1\rangle}(S_J^{\langle 1\rangle}(\mathcal{E}))$ of A as a further left Kan extension to the whole indexing category $\varinjlim_{\substack{I\cup J=[n]\\ I\leq J}} \operatorname{Fun}([k-1],I)\times\operatorname{Fun}([1],J)$, which again is an iterated version of the corresponding construction in the proof of Lemma 4.9. Recall that for every $0\leq i<n$,

$$S^{\langle k-1\rangle}_{\{0,\dots,i\}}(S^{\langle 0]}_{\{i+1,\dots,n\}}(\mathcal{E})) \xrightarrow{\simeq} S^{\langle k-1\rangle}_{\{0,\dots,i\}}(S^{\langle 1\rangle}_{\{i,\dots,n\}}(\mathcal{E}))$$

is an equivalence of categories, which is compatible with the transition maps in the limit. This construction is well-defined, fully faithful, and essentially surjective, which completes the proof. $\square$

Example 4.13. As a special case of Theorem 4.12, we illustrate the equivalence of categories $S_4^{\langle 2\rangle}(\mathcal{E})\simeq S^{(2)}_{\{0,1\},\{1,2,3,4\}}(\mathcal{E})\times_{S^{(2)}_{\{0,1\},\{2,3,4\}}(\mathcal{E})} S^{(2)}_{\{0,1,2\},\{2,3,4\}}(\mathcal{E})\times_{S^{(2)}_{\{0,1,2\},\{3,4\}}(\mathcal{E})} S^{(2)}_{\{0,1,2,3\},\{3,4\}}(\mathcal{E})$ in the following diagram, where the glueing is represented by the two dashed sequences.



Let $k\geq 1$. As can be seen either directly or with the help of Theorem 4.12, the functor

$$\Phi\colon \mathrm{ExCat}\longrightarrow \mathrm{Top}_*,\ \mathcal{E}\longmapsto |S^{\langle k-1\rangle}(\mathcal{E})^\times|, \tag{4.6}$$

satisfies the hypotheses of [13], §1.3. This permits us to draw the following consequences on geometric realizations.

Corollary 4.14 (Additivity)**.** *Let $\mathcal{E}$ be an exact category, $k\geq 1$. There is a weak equivalence*

$$|S^{\langle k\rangle}(S_2(\mathcal{E}))^\times| \xrightarrow{\simeq} |S^{\langle k\rangle}(\mathcal{E})^\times|\times|S^{\langle k\rangle}(\mathcal{E})^\times|$$

induced by the functor $(\partial_2,\partial_0)\colon S_2(\mathcal{E})\to\mathcal{E}\times\mathcal{E}$. In fact, the simplicial space $|S^{\langle k\rangle}(S_\bullet(\mathcal{E}))^\times|$ is a lower 1-Segal space.

Proof. By Theorem 4.12, this is precisely [13], Theorem 1.3.5 (2), with Φ as in (4.6). $\square$

Needless to say, the other versions of additivity in [13], Theorem 1.3.5, hold as well; this also allows us to deduce the following as in *loc.cit.* via the proof of [15], Proposition 3.6.2.

Corollary 4.15 (Delooping)**.** *Let $k \geq 1$, and let $K(\mathcal{E})$ denote the algebraic K-theory space of the exact category $\mathcal{E}$. There is a natural homotopy equivalence*

$$\Omega^k |S^{\langle k \rangle}(\mathcal{E})^\times| \xrightarrow{\simeq} K(\mathcal{E}).$$

Remark 4.16. Similarly to Corollary 3.7, we can also use Theorem 4.12 to immediately reduce to the case $k = 1$ ([23], Theorem 1.4.2 and Proposition 1.3.2, resp. Proposition 1.5.3).

In conclusion, the algebraic K-theory spectrum of $\mathcal{E}$ is given by the sequence of maps

$$|S^{\langle 0 \rangle}(\mathcal{E})^\times| \longrightarrow \Omega|S^{\langle 1 \rangle}(\mathcal{E})^\times| \longrightarrow \Omega^2|S^{\langle 2 \rangle}(\mathcal{E})^\times| \longrightarrow \dots,$$

where, for each $k \geq 0$, the morphism

$$|S^{\langle k \rangle}(\mathcal{E})^\times| \longrightarrow \Omega|S^{\langle k+1 \rangle}(\mathcal{E})^\times|$$

is induced by the inclusions of the cells $\iota_n \colon S^{\langle k \rangle}_n(\mathcal{E}) \hookrightarrow S^{\langle k+1 \rangle}_{n+1}(\mathcal{E})$ as $S^{\langle 1 \rangle}_1(S^{\langle k \rangle}_n(\mathcal{E}))$, extended by zeroes appropriately, for all $n \geq 0$. Namely, [23], Lemma 1.5.2, constructs the map via

$$S^{(k)}(\mathcal{E}) \longrightarrow P^\triangleleft S^{(k+1)}(\mathcal{E}) \longrightarrow S^{(k+1)}(\mathcal{E}), \tag{4.7}$$

inducing a map $S^{(k)}(\mathcal{E}) \to L^\triangleleft S^{(k+1)}(\mathcal{E})$ to the simplicial loop space. Totalizing (4.7) yields

$$S^{\langle k \rangle}(\mathcal{E}) \xrightarrow{\iota} P^\triangleleft S^{\langle k+1 \rangle}(\mathcal{E}) \longrightarrow S^{\langle k+1 \rangle}(\mathcal{E})$$

by Lemma 2.17. As before, this produces the desired map $S^{\langle k \rangle}(\mathcal{E}) \longrightarrow L^\triangleleft S^{\langle k+1 \rangle}(\mathcal{E})$.

5. Stringent base categories

Let $\mathcal{E}$ be a proto-exact category. In this section, we investigate which further higher Segal conditions the higher dimensional Waldhausen construction of $\mathcal{E}$ and its variants satisfy under additional homological assumptions on $\mathcal{E}$, based on the following characterization.

Lemma 5.1. *Let $\mathcal{E}$ be a pointed category. The following conditions are equivalent.*

(*i*) *There exists a proto-exact structure on $\mathcal{E}$, in which a morphism is admissible only if it admits a kernel or a cokernel.*

(*ii*) *The class of all kernel-cokernel pairs defines a proto-exact structure on $\mathcal{E}$, and a morphism in $\mathcal{E}$ is admissible if and only if it admits a kernel and a cokernel.*

(*iii*) *The pushout of a kernel by a cokernel exists in $\mathcal{E}$ and is a kernel again, and the pullback of a cokernel by a kernel exists in $\mathcal{E}$ and is a cokernel again. Furthermore, if a map f in $\mathcal{E}$ admits a kernel and a cokernel, then it is strict, that is, it factors as the composition of a cokernel and a kernel. Equivalently, the natural map*

$$\mathrm{coim}(f) \to \mathrm{im}(f)$$

is an isomorphism.

Proof. Assume (*i*). The proto-exact structure on $\mathcal{E}$ necessarily contains all kernel-cokernel pairs, since kernels and cokernels are admissible morphisms. Moreover, admissible morphisms admit kernels and cokernels, implying (*ii*). The converse is tautological.

Given (*iii*), we need to see that the class of all kernel-cokernel pairs defines a proto-exact structure on $\mathcal{E}$. Indeed, a composition of cokernels $B \twoheadrightarrow B' \twoheadrightarrow C$ admits a kernel

$$\begin{tikzcd} A \arrow[r, two heads] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner"] & A' \arrow[r, two heads] \arrow[d, tail] \arrow[dr, phantom, "\square"] & 0 \arrow[d, tail] \\ B \arrow[r, two heads] & B' \arrow[r, two heads] & C \end{tikzcd}$$

and therefore is a cokernel in $\mathcal{E}$. Then we can conclude that (*iii*) $\Leftrightarrow$ (*ii*) by definition.

Finally, the equivalence in (*iii*) is immediate from the fact that a factorization as in (*iii*) is unique (up to unique isomorphism). $\square$

Definition 5.2. A pointed category $\mathcal{E}$ satisfying the equivalent conditions in Lemma 5.1 will be called a stringent category.

Remark 5.3. In [7], Definition 1.2, a proto-abelian category is defined as a pointed category on which the classes of all monomorphisms and epimorphisms define a proto-exact structure. For us, it will prove convenient to change this terminology slightly by additionally requiring the existence of all kernels and cokernels (rather than introducing another different term). This does not exclude any of the main examples of interest (like Example 5.4 (1) below).

Example 5.4. A pointed category which admits all kernels and cokernels is stringent if and only if it is proto-abelian (in the sense of Remark 5.3). Similarly, a pre-abelian category is stringent if and only if it is abelian. Moreover, $\mathcal{E}$ is stringent if and only if $\mathcal{E}^{\mathrm{op}}$ is.

(1) In particular, the category $\mathrm{Fun}(Q, \mathrm{vect}_{\mathbb{F}_1})$ of representations of any small category Q in (finite) $\mathbb{F}_1$-vector spaces is stringent.
(2) Consider the pointed category $\mathcal{E}$ on a non-zero object V with $\mathrm{End}(V) = \{0, 1, \varepsilon\}$, such that $\varepsilon^2 = 0$. It can easily be verified directly that ε admits neither a kernel nor a cokernel, and thus $\mathcal{E}$ is stringent.
    If F is a field, then the F-linear Cauchy completion of $\mathcal{E}$ is an additive stringent category, namely the category of finite free $F[x]/(x^2)$-modules.

Remark 5.5. An additive category $\mathcal{E}$ is stringent if and only if the class of all kernel-cokernel pairs defines an exact structure on $\mathcal{E}$ and a morphism in $\mathcal{E}$ is admissible if and only if it admits a kernel and a cokernel. Similarly, the other conditions in Lemma 5.1 have evident additive analogues.

Remark 5.6. An additive stringent category $\mathcal{E}$ is in particular weakly idempotent complete, in the sense of [3], Proposition 7.6. In fact, $\mathcal{E}$ satisfies the stronger statement of [19], Proposition 1.1.8, whose proof applies verbatim. Indeed, note that if a composition of the form $A' \twoheadrightarrow A \xrightarrow{f} B$ is admissible, then its cokernel

$$\begin{tikzcd} A' \arrow[r, two heads] \arrow[d, two heads] & A \arrow[r, "f"] \arrow[d, two heads] & B \arrow[d, two heads] \\ 0 \arrow[r, two heads] \arrow[ur, phantom, "\ulcorner\cdot"] & 0 \arrow[r, two heads] \arrow[ur, phantom, "\ulcorner\cdot"] & C \end{tikzcd} \tag{5.1}$$

also defines a cokernel for f, which is therefore strict, and thus admits a kernel. In particular, the snake lemma for admissible morphisms holds in $\mathcal{E}$, as shown in [3], Corollary 8.13.

In fact, the snake lemma holds in any stringent category $\mathcal{E}$. The neat argument presented in *loc.cit.* does not quite apply here (as it ultimately relies on the additive structure of an exact category); however, the proof of [12], Proposition 4.3, does apply.

For this, we need to verify Heller's axioms for $\mathcal{E}$ (cf. [3], Proposition B.1; with the obvious exception of additivity). Since cancellation follows directly from (5.1) by the coimage-image isomorphism, it only remains to prove the following result.

Proposition 5.7. *Let $\mathcal{E}$ be a stringent category, and consider a diagram of the form*

$$\begin{tikzcd} A' \arrow[r, tail] \arrow[d] & B' \arrow[r, two heads] \arrow[d, tail] & C' \arrow[d, tail] \\ A \arrow[r, tail] \arrow[d] & B \arrow[r, two heads] \arrow[d, two heads] & C \arrow[d, two heads] \\ A'' \arrow[r, tail] & B'' \arrow[r, two heads] & C'' \end{tikzcd}$$

in $\mathcal{E}$, where all rows as well as all columns but the first are exact. Then $A' \rightarrowtail A \twoheadrightarrow A''$ is a short exact sequence as well.

Proof. First of all, by cancellation, $A' \rightarrowtail A$. Then we have the following cartesian squares.

$$\begin{tikzcd}
A' \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & B' \arrow[d, two heads] \\
0 \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C' \arrow[d, tail] \\
0 \arrow[r, tail] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[r, tail] \arrow[d, tail] & B' \arrow[d, tail] \\
A \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & B \arrow[d, two heads] \\
0 \arrow[r, tail] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[r, tail] \arrow[d, tail] & A \arrow[d, tail] \\
B' \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & B \arrow[d, two heads] \\
0 \arrow[r, tail] & B''
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[r, tail] \arrow[d, two heads] & A \arrow[d] \\
0 \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A'' \arrow[d, tail] \\
0 \arrow[r, tail] & B''
\end{tikzcd}$$

The first diagram implies that the outer rectangle of the second is a pullback, hence so is its upper square, which agrees with the upper square of the third, implying that the outer rectangle of the fourth is a pullback, and thus its upper square. Therefore,

$$\operatorname{coker}(A' \rightarrowtail A) \xrightarrow{\sim} \operatorname{coker}(\ker(A \to A'')) \xrightarrow{\sim} \ker(\operatorname{coker}(A \to A'')) \rightarrowtail A''.$$

On the other hand, dually, we have the following cocartesian squares,

$$\begin{tikzcd}
B' \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner", very near end] & C' \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & C \arrow[d, two heads] \\
0 \arrow[r] & 0 \arrow[r, tail] & C''
\end{tikzcd}
\qquad
\begin{tikzcd}
B' \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & B \arrow[r, two heads] \arrow[d, two heads] & C \arrow[d, two heads] \\
0 \arrow[r, tail] & B'' \arrow[r, two heads] & C''
\end{tikzcd}$$

which together imply that the right-hand square of the second diagram is again a pushout. But then the dual of Lemma 5.8 below tells us that $A \twoheadrightarrow A''$. $\square$

Lemma 5.8. *Let $\mathcal{E}$ be a stringent category, and consider a pullback square of admissible morphisms in $\mathcal{E}$ of the following form.*

$$\begin{tikzcd}
B_1 \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A_1 \arrow[d] \\
B_0 \arrow[r, tail] & A_0
\end{tikzcd} \tag{5.2}$$

The induced map $C_1 \rightarrowtail C_0$ is an admissible monomorphism, where $C_i = \operatorname{coker}(B_i \rightarrowtail A_i)$.

Proof. We have the following composition of cartesian squares.

$$\begin{tikzcd}
B_1 \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A_1 \arrow[d] \\
B_0 \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & A_0 \arrow[d, two heads] \\
0 \arrow[r, tail] & C_0
\end{tikzcd}$$

Thus, the composition $A_1 \to A_0 \twoheadrightarrow C_0$ is admissible with kernel B_1, and therefore admits a factorization $A_1 \twoheadrightarrow C_1 \rightarrowtail C_0$, which fits uniquely into the diagram

$$\begin{tikzcd}
A_1 \arrow[r, two heads] \arrow[d, tail] & C_1 \arrow[d, tail] \\
A_0 \arrow[r, two heads] & C_0
\end{tikzcd}$$

yielding the claim. $\square$

We are now prepared to state the first main result of this section.

Proposition 5.9. *Suppose $\mathcal{E}$ is a stringent category. Then the simplicial categories $S^{\langle k]}(\mathcal{E})$ and $S^{[k\rangle}(\mathcal{E})$ are lower $(2k-1)$-Segal.*

Remark 5.10. When $k \geq 2$, the assumption in Proposition 5.9 that $\mathcal{E}$ be stringent is necessary, which is illustrated by the following observation, at least in the additive case. Suppose $\mathcal{E}$ is an exact category which is not stringent; then $S^{\langle 1]}(\mathcal{E})$ is not lower 3-Segal.

By [10], Proposition 3.1, there is a morphism $f\colon A \to B$ in $\mathcal{E}$ which is not strict. However, by [3], Remark 8.2, it can be written as the composition of strict morphisms

$$A \xrightarrow{(1,f)} A \oplus B \xrightarrow{(0,1)} B.$$

Now suppose f admits a kernel C and consider the following possible element of $S_4^{\langle 1]}(\mathcal{E})$.

$$\begin{tikzcd}
0 \arrow[r, tail] & 0 \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, "{(1,f)}"] \\
 & 0 \arrow[r, tail] & A \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \oplus B \arrow[d, "{(0,1)}"] \\
 & & 0 \arrow[r, tail] & 0 \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d] \\
 & & & 0 \arrow[r, tail] & B \arrow[d] \\
 & & & & 0
\end{tikzcd} \tag{5.3}$$

Then the triple of diagrams

$$\begin{tikzcd}
0 \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C \arrow[d] \\
0 \arrow[r, tail] & A \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d] \\
 & 0 \arrow[r, tail] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, "{(1,f)}"] \\
0 \arrow[r, tail] & A \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \oplus B \arrow[d] \\
 & 0 \arrow[r, tail] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \oplus B \arrow[d, "{(0,1)}"] \\
0 \arrow[r, tail] & 0 \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d] \\
 & 0 \arrow[r, tail] & B
\end{tikzcd}$$

defines an element in the right-hand side of the lower 3-Segal map for $S_4^{\langle 1]}(\mathcal{E})$. However, it does not lie in its essential image, because the sequence

$$C \rightarrowtail A \xrightarrow{f} B$$

indexed by $\{0, 2, 4\}$ is not left exact (the map f not being strict), so $(5.3) \notin P^{\triangleleft} S_4^{\langle 2\rangle}(\mathcal{E})$.

Dually, since there exist non-strict maps admitting a cokernel in $\mathcal{E}$, the 4-cells of $S^{[1\rangle}(\mathcal{E})$ do not satisfy the lower 3-Segal condition (by Lemma 4.6).

Example 5.11. Let us illustrate the lowest 3-Segal conditions for $S^{\langle 2\rangle}(\mathcal{E})$, which is more conveniently done by depicting an element of its 4-cells as the following projection of (4.3).

$$\begin{tikzcd}
 & & & A_{123} \arrow[dr, tail] & & & \\
 & & A_{023} \arrow[ur, two heads] \arrow[dr, tail, dashed] & & A_{124} \arrow[dr] & & \\
 & & & A_{024} \arrow[ur, two heads, dashed] \arrow[dr, dashed] & & & \\
 & A_{013} \arrow[r, tail] \arrow[uur] & A_{014} \arrow[rr] \arrow[ur, dashed] & & A_{034} \arrow[r, two heads] \arrow[drr, two heads, dashed] & A_{134} \arrow[dr, two heads] & \\
A_{012} \arrow[ur, tail] \arrow[urr, tail, dashed] & & & & & & A_{234}
\end{tikzcd} \tag{5.4}$$

The dashed part marks the image of (5.4) in the right-hand side of the upper 3-Segal map

$$S_4^{\langle 2\rangle}(\mathcal{E}) \longrightarrow S_3^{\langle 2\rangle}(\mathcal{E}) \times_{S_2^{\langle 2\rangle}(\mathcal{E})} S_3^{\langle 2\rangle}(\mathcal{E}).$$

The upper 3-Segal condition says that the whole diagram (5.4) is uniquely recovered from the dashed subdiagram. Note that the complementary statement (for the lower 3-Segal map) is false in general. In fact, it is equivalent to uniquely filling the frame of short exact sequences

$$\begin{tikzcd}
C_4 \arrow[r, tail] \arrow[d, tail] & A_{023} \arrow[r, two heads] \arrow[d, tail, dashed] & A_{123} \arrow[d, tail] \\
C_3 \arrow[r, tail, dashed] \arrow[d, two heads] & A'_{024} \arrow[r, two heads, dashed] \arrow[d, two heads, dashed] & A_{124} \arrow[d, two heads] \\
C_2 \arrow[r, tail] & C_1 \arrow[r, two heads] & C_0
\end{tikzcd} \tag{5.5}$$

where $C_i = \operatorname{coker}(A_{d_3^* d_i^* \Delta_4^4} \rightarrowtail A_{d_2^* d_i^* \Delta_4^4}) \cong \ker(A_{d_1^* d_i^* \Delta_4^4} \twoheadrightarrow A_{d_0^* d_i^* \Delta_4^4})$. However, there is an obstruction to this, which is parametrized by the quotient groupoid

$$[\operatorname{Ext}^1(C_0, C_4)/\operatorname{Hom}(C_0, C_4)],$$

as calculated in [7], Lemma 2.30 and Proposition 2.38, assuming that $\mathcal{E}$ is abelian.

Remark 5.12. Suppose $\mathcal{E}$ is additive. Then Theorem 5.18 does not generalize to the higher dimensional Waldhausen constructions, that is, $S^{\langle k\rangle}(\mathcal{E})$ is not upper $(2k-1)$-Segal for $k \neq 2$. Indeed, the diagram

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & A \arrow[r, "="] \arrow[d, "{(0,1)}"] & A \arrow[d, "="] \\
 & 0 \arrow[r] & A \arrow[r, "{(1,0)}"] \arrow[d] & A \oplus A \arrow[r, "{(1,1)}"'] \arrow[d, "{(0,1)}"'] & A \arrow[d] \\
 & & 0 \arrow[r] & A \arrow[r] \arrow[d] & 0 \arrow[d] \\
 & & & 0 \arrow[r] & 0 \arrow[d] \\
 & & & & 0
\end{tikzcd} \tag{5.6}$$

is an element of the right-hand side of the lower 3-Segal map for $P^{\triangleleft}P^{\triangleright}S_4^{\langle 3\rangle}(\mathcal{E})$, but does not lie in its essential image.

Our next observation will prove essential for our inductive arguments.

Proposition 5.13 (Hyperplane lemma)**.** *Let $1 \leq k \leq l < m \leq n$, and let $\mathcal{E}$ be a stringent category. Then there is a natural functor*

$$\eta^{\triangleleft}_{lm} \colon S_n^{\langle k]}(\mathcal{E}) \longrightarrow S_l^{\langle k-1]}(\mathcal{E}),\ A \longmapsto (\beta \mapsto \operatorname{coker}(A_{\beta\cup\{l\}} \rightarrowtail A_{\beta\cup\{m\}})).$$

Dually, there is a corresponding natural functor

$$\eta^{\triangleright}_{lm} \colon S_n^{[k\rangle}(\mathcal{E}) \longrightarrow S_l^{[k-1\rangle}(\mathcal{E}),\ A \longmapsto (\beta \mapsto \ker(A_{\{n-m\}\cup\beta} \twoheadrightarrow A_{\{n-l\}\cup\beta})).$$

Moreover, both of these restrict to functors on the higher Waldhausen construction,

$$S_n^{\langle k\rangle}(\mathcal{E}) \overset{\eta^{\triangleleft}_{lm}}{\underset{\eta^{\triangleright}_{lm}}{\rightrightarrows}} S_l^{\langle k-1\rangle}(\mathcal{E}).$$

Proof. Let γ be a k-simplex in Δ^l, and $\gamma' = \gamma \amalg \{m\}$. If $l \in \gamma$, then the sequence $\eta^{\triangleleft}_{lm}(A)_{d^*_\bullet\gamma}$ is given by

$$\operatorname{coker}(A_{d^*_{k+1}\gamma'} \rightarrowtail A_{d^*_k\gamma'}) \rightarrowtail A_{d^*_{k-1}\gamma'} \longrightarrow A_{d^*_{k-2}\gamma'} \longrightarrow \ldots \longrightarrow A_{d^*_0\gamma'}$$

which of course is indeed left exact. Furthermore, it is exact if and only if A lies in $S_n^{\langle k\rangle}(\mathcal{E})$, by definition.

Now assume that $l \notin \gamma$. Then, for each vertex $0 < i < k$, let us write

$$\begin{tikzcd}
(A_{d_{i+1}^*\gamma\cup\{l\}} \twoheadrightarrow)\ B_{i+1} \arrow[r, tail] \arrow[d, dashed] & A_{d_i^*\gamma\cup\{l\}} \arrow[r, two heads] \arrow[d, dashed] & B_i\ (\rightarrowtail A_{d_{i-1}^*\gamma\cup\{l\}}) \arrow[d, dashed] \\
(A_{d_{i+1}^*\gamma\cup\{m\}} \twoheadrightarrow)\ C_{i+1} \arrow[r, tail] & A_{d_i^*\gamma\cup\{m\}} \arrow[r, two heads] & C_i\ (\rightarrowtail A_{d_{i-1}^*\gamma\cup\{m\}})
\end{tikzcd}$$

for the corresponding short exact sequence. Taking cokernels yields a diagram as follows.

$$\begin{tikzcd}
B_{i+1} \arrow[r, tail] \arrow[d, tail] & C_{i+1} \arrow[r, two heads] \arrow[d, tail] & D_{i+1} \arrow[d] \\
A_{d_i^*\gamma\cup\{l\}} \arrow[r, tail] \arrow[d, two heads] & A_{d_i^*\gamma\cup\{m\}} \arrow[r, two heads] \arrow[d, two heads] & \eta^{\triangleleft}_{lm}(A)_{d_i^*\gamma} \arrow[d] \\
B_i \arrow[r, tail] & C_i \arrow[r, two heads] & D_i
\end{tikzcd}$$

By the snake lemma, the right vertical sequence is short exact. Note that if $A \in S_n^{\langle k\rangle}(\mathcal{E})$,

$$B_1 \xrightarrow{\simeq} A_{d_0^*\gamma\cup\{l\}} \text{ and } C_1 \xrightarrow{\simeq} A_{d_0^*\gamma\cup\{m\}}$$

which immediately implies also $D_1 \xrightarrow{\simeq} \eta^{\triangleleft}_{lm}(A)_{d_0^*\gamma}$ by definition. It remains to prove that

$$\eta^{\triangleleft}_{lm}(A)_{d_k^*\gamma} \longrightarrow \eta^{\triangleleft}_{lm}(A)_{d_{k-1}^*\gamma}$$

is an admissible monomorphism. In order to see this, we may show that the diagram

$$\begin{tikzcd}
A_{d_k^*\gamma\cup\{l\}} \arrow[r, tail] \arrow[d] & A_{d_k^*\gamma\cup\{m\}} \arrow[d] \\
A_{d_{k-1}^*\gamma\cup\{l\}} \arrow[r, tail] & A_{d_{k-1}^*\gamma\cup\{m\}}
\end{tikzcd} \tag{5.7}$$

is pullback, by Lemma 5.8. In fact, we claim that it is the composition of pullback diagrams

$$\begin{tikzcd}
A_{d_k^*\gamma\cup\{l\}} \arrow[r, tail] \arrow[d] & A_{d_k^*\gamma\cup\{l+1\}} \arrow[r, tail] \arrow[d] & \dots \arrow[r, tail] & A_{d_k^*\gamma\cup\{m-1\}} \arrow[r, tail] \arrow[d] & A_{d_k^*\gamma\cup\{m\}} \arrow[d] \\
A_{d_{k-1}^*\gamma\cup\{l\}} \arrow[r, tail] & A_{d_{k-1}^*\gamma\cup\{l+1\}} \arrow[r, tail] & \dots \arrow[r, tail] & A_{d_{k-1}^*\gamma\cup\{m-1\}} \arrow[r, tail] & A_{d_{k-1}^*\gamma\cup\{m\}}.
\end{tikzcd}$$

To prove this claim, for each $l \le j < m$, we have the diagram

$$\begin{tikzcd}
A_{d_k^*\gamma\cup\{j\}} \arrow[r, tail] \arrow[d] & A_{d_k^*\gamma\cup\{j+1\}} \arrow[d] \\
A_{d_{k-1}^*\gamma\cup\{j\}} \arrow[r, tail] \arrow[d] & A_{d_{k-1}^*\gamma\cup\{j+1\}} \arrow[d] \\
0 \arrow[r, tail] & A_{d_{k-1}^*d_k^*\gamma\cup\{j,j+1\}}
\end{tikzcd}$$

in which the lower and outer rectangles are pullback, and therefore, so is the upper.

Finally, the functor $\eta^{\triangleright}_{lm}$ is given by the map $\eta^{\triangleleft}_{lm}$ for $\mathcal{E}^{\mathrm{op}}$, via Lemma 4.6. □

The following result constitutes a generalization of Lemma 4.9.

Proposition 5.14. *Let $k \ge 1$, and assume that $\mathcal{E}$ is a stringent category. Then there are equivalences of simplicial categories*

$$P^{\triangleleft}S^{\langle k\rangle}(\mathcal{E}) \xrightarrow{\simeq} S^{\langle k-1]}(\mathcal{E}),\ A \longmapsto A_{[0]\oplus -},$$
$$P^{\triangleright}S^{\langle k\rangle}(\mathcal{E}) \xrightarrow{\simeq} S^{[k-1\rangle}(\mathcal{E}),\ A \longmapsto A_{-\oplus[0]},$$

induced by the forgetful functors. For $k \geq 2$, there is an equivalence

$$P^{\triangleleft}P^{\triangleright}S^{\langle k\rangle}(\mathcal{E}) \xrightarrow{\simeq} S^{[k-2]}(\mathcal{E}),\ A \longmapsto A_{[0]\oplus-\oplus[0]}.$$

Proof. We prove the second statement first. For a diagram $A \in S_n^{[k-1\rangle}(\mathcal{E})$, we construct its image in $P^{\triangleright}S_n^{\langle k\rangle}(\mathcal{E}) \simeq S_{n+1}^{\langle k\rangle}(\mathcal{E})$ under the inverse functor as a right Kan extension. Namely, we extend by zero appropriately, and then into the kth dimension, as follows.

$$\begin{tikzcd}
\mathrm{Fun}([k-1],[n]) \arrow[rr, "A"] \arrow[d, "\iota"'] & & \mathcal{E} \\
\mathrm{Fun}([k-1],[n+1]) \arrow[d, "\mathrm{incl}"'] & & \\
\mathrm{Cyl}(\iota|_{\Delta^n_{k-1}}) \arrow[uurr, dashed, "A^!"] \arrow[d, "\lambda"'] & & \\
\mathrm{Fun}([k],[n+1]) \arrow[uuurr, dashed, "\widehat{A}"'] & &
\end{tikzcd} \tag{5.8}$$

Here, we have set $\iota = (d_{n+1})_*$, and the category $\mathrm{Cyl}(\iota|_{\Delta^n_{k-1}})$ is the cograph of its restriction to the skeleton. The functor λ is defined by $(s_{k-1})_*$ on Δ^n_{k-1}, and on $\mathrm{Fun}([k-1],[n+1])$, it maps

$$\alpha \mapsto \alpha \cup \{n+1\}.$$

Explicitly, $\widehat{A}$ is given by the diagram

$$\beta \longmapsto \varprojlim_{\beta \leq \lambda(\alpha)} A^!_{\alpha} \ \cong \ \begin{cases} A_{\beta \smallsetminus \{n+1\}} & \text{if } n+1 \in \beta, \\ \ker(A_{d_k^*\beta} \to A_{d_{k-1}^*\beta}) & \text{otherwise.} \end{cases}$$

Indeed, $d_k^*\beta$ is initial amongst those objects of the indexing category of the limit which come from $\mathrm{Fun}([k-1],[n+1])$. If $n+1 \in \beta$, then this is the only contribution. Otherwise, there are additionally the objects of the form

$$[\alpha] \in \Delta^n_{k-1} \text{ with } d_{k-1}^*\beta \leq \alpha.$$

Therefore, in that case, the limit reduces to just the pullback

$$\widehat{A}_\beta \cong \varprojlim \left[\begin{array}{ccc} & & A^!_{d_k^*\beta} \\ & & \big\downarrow \\ A^!_{[d_{k-1}^*\beta]} & \longrightarrow & A^!_{d_{k-1}^*\beta} \end{array} \right] \cong \varprojlim \left[\begin{array}{ccc} & & A_{d_k^*\beta} \\ & & \big\downarrow \\ 0 & \longrightarrow & A_{d_{k-1}^*\beta} \end{array} \right] = \ker(A_{d_k^*\beta} \to A_{d_{k-1}^*\beta}).$$

Now let γ be a $(k+1)$-simplex in Δ^{n+1}. We claim that the corresponding sequence $\widehat{A}_{d_\bullet^*\gamma}$ is exact. If $n+1 \in \gamma$, then this is simply given by

$$\ker(A_{d_k^* d_{k+1}^*\gamma} \to A_{d_{k-1}^* d_{k+1}^*\gamma}) \rightarrowtail A_{d_k^*\gamma \smallsetminus \{n+1\}} \longrightarrow \dots \longrightarrow A_{d_1^*\gamma \smallsetminus \{n+1\}} \twoheadrightarrow A_{d_0^*\gamma \smallsetminus \{n+1\}}$$

which is an exact sequence in $\mathcal{E}$ by definition.

Otherwise, the relevant sequence is given by

$$
\begin{tikzcd}
\ker(A_{d_k^* d_{k+1}^*\gamma} \to A_{d_{k-1}^* d_{k+1}^*\gamma}) \arrow[d] \\
\ker(A_{d_k^* d_k^*\gamma} \to A_{d_{k-1}^* d_k^*\gamma}) \arrow[d] \\
\ker(A_{d_k^* d_{k-1}^*\gamma} \to A_{d_{k-1}^* d_{k-1}^*\gamma}) \arrow[d] \\
\ker(A_{d_k^* d_{k-2}^*\gamma} \to A_{d_{k-1}^* d_{k-2}^*\gamma}) \arrow[d] \\
\vdots \arrow[d] \\
\ker(A_{d_k^* d_0^*\gamma} \to A_{d_{k-1}^* d_0^*\gamma})
\end{tikzcd}
\quad \text{or equivalently,} \quad
\begin{tikzcd}
\ker(A_{d_k^* d_{k+1}^*\gamma} \to A_{d_{k-1}^* d_{k+1}^*\gamma}) \arrow[d] \\
\ker(A_{d_k^* d_{k+1}^*\gamma} \to A_{d_{k-1}^* d_k^*\gamma}) \arrow[d] \\
\ker(A_{d_{k-1}^* d_{k+1}^*\gamma} \to A_{d_{k-1}^* d_k^*\gamma}) \arrow[d] \\
\ker(A_{d_{k-2}^* d_{k+1}^*\gamma} \to A_{d_{k-2}^* d_k^*\gamma}) \arrow[d] \\
\vdots \arrow[d] \\
\ker(A_{d_0^* d_{k+1}^*\gamma} \to A_{d_0^* d_k^*\gamma}).
\end{tikzcd}
$$

We prove exactness inductively. The first part of the sequence fits into a diagram of the form

$$
\begin{tikzcd}
\widehat{A}_{d_{k+1}^*\gamma} \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \widehat{A}_{d_k^*\gamma} \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A_{d_k^* d_{k+1}^*\gamma} \arrow[d] \\
0 \arrow[r, tail] & \widehat{A}_{d_{k-1}^*\gamma} \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A_{d_{k-1}^* d_{k+1}^*\gamma} \arrow[d] \\
& 0 \arrow[r, tail] & A_{d_{k-1}^* d_k^*\gamma}.
\end{tikzcd}
$$

By definition, the bottom left square is pullback, so we can pull it back to the top and then to the left, since each outer rectangle is a pullback square by construction. Thus,

$$\widehat{A}_{d_{k+1}^*\gamma} \rightarrowtail \widehat{A}_{d_k^*\gamma} \longrightarrow \widehat{A}_{d_{k-1}^*\gamma}$$

is a left exact sequence. Now, for each $0 < i < k$, let us write

$$
\begin{tikzcd}
(A_{d_{i+1}^* d_{k+1}^*\gamma} \twoheadrightarrow)\ B_{i+1} \arrow[r, tail] \arrow[d, dashed] & A_{d_i^* d_{k+1}^*\gamma} \arrow[r, two heads] \arrow[d, dashed] & B_i\ (\rightarrowtail A_{d_{i-1}^* d_{k+1}^*\gamma}) \arrow[d, dashed] \\
(A_{d_{i+1}^* d_k^*\gamma} \twoheadrightarrow)\ C_{i+1} \arrow[r, tail] & A_{d_i^* d_k^*\gamma} \arrow[r, two heads] & C_i\ (\rightarrowtail A_{d_{i-1}^* d_k^*\gamma})
\end{tikzcd}
\tag{5.9}
$$

for the corresponding short exact sequences at the ith vertex of $d_{k+1}^*\gamma$ and $d_k^*\gamma$, respectively. First, we show that $B_k \to C_k$ is an admissible epimorphism. But we have

$$B_k = \operatorname{coker}(\widehat{A}_{d_{k+1}^*\gamma} \rightarrowtail A_{d_k^* d_{k+1}^*\gamma}), \text{ and } C_k = \operatorname{coker}(\widehat{A}_{d_k^*\gamma} \rightarrowtail A_{d_k^* d_k^*\gamma}).$$

Therefore, they fit into a diagram of the following form, which yields the claim.

$$
\begin{tikzcd}
\widehat{A}_{d_{k+1}^*\gamma} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner"] & \widehat{A}_{d_k^*\gamma} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner"] & A_{d_k^* d_{k+1}^*\gamma} \arrow[d, two heads] \\
0 \arrow[r, tail] & B_k' \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner"] & B_k \arrow[d, two heads] \\
& 0 \arrow[r, tail] & C_k
\end{tikzcd}
$$

In particular, $B'_k = \ker(B_k \twoheadrightarrow C_k)$. Next, we show that $B_{k-1} \to C_{k-1}$ admits a kernel B'_{k-1} in $\mathcal{E}$. In fact, consider the following diagram.

$$\begin{tikzcd}
B_k \arrow[r, tail] \arrow[d, two heads] & A_{d^*_{k-1}d^*_{k+1}\gamma} \arrow[r, two heads] \arrow[d, two heads] & B_{k-1} \arrow[d, dashed, two heads, "(3)"] \\
C_k \arrow[r, dashed, tail, "(1)"] \arrow[d, "="'] & E \arrow[r, dashed, two heads, "(1')"] \arrow[d, tail] & D \arrow[d, dashed, tail, "(2)"] \\
C_k \arrow[r, tail] \arrow[d, two heads] & A_{d^*_{k-1}d^*_k\gamma} \arrow[r, two heads] \arrow[d, two heads] & C_{k-1} \arrow[d, dashed, two heads, "(2')"] \\
0 \arrow[r, tail] & C'_{k-1} \arrow[r, "="] & C'_{k-1}
\end{tikzcd}$$

We have (1) by Remark 5.6, and (1′) is its cokernel. The snake lemma yields (2) and (2′), and (3) is obtained dually to (1).

Now the snake lemma implies that the top row of the following diagram is short exact.

$$\begin{tikzcd}
B'_k \arrow[r, tail] \arrow[d, tail] & \widehat{A}_{d^*_{k-1}\gamma} \arrow[r] \arrow[d, tail] & B'_{k-1} \arrow[d, tail] \\
B_k \arrow[r, tail] \arrow[d, two heads] & A_{d^*_{k-1}d^*_{k+1}\gamma} \arrow[r, two heads] \arrow[d] & B_{k-1} \arrow[d] \\
C_k \arrow[r, tail] & A_{d^*_{k-1}d^*_k\gamma} \arrow[r, two heads] & C_{k-1}
\end{tikzcd}$$

In particular, this settles the case $k = 2$. For $k \geq 3$, we can rewrite the diagram

$$\begin{tikzcd}
B_2 \arrow[r, tail] \arrow[d] & A_{d^*_1 d^*_{k+1}\gamma} \arrow[r, two heads] \arrow[d] & B_1 = A_{d^*_0 d^*_{k+1}\gamma} \arrow[d] \\
C_2 \arrow[r, tail] & A_{d^*_1 d^*_k\gamma} \arrow[r, two heads] & C_1 = A_{d^*_0 d^*_k\gamma}
\end{tikzcd}$$

in terms of the hyperplane lemma (Proposition 5.13), namely as the upper part of the short exact sequence of acyclic sequences

$$\begin{tikzcd}
\eta^{\triangleright}_{(n-\gamma_1)(n-\gamma_0)}(A)_{d^*_{k-1}\alpha} \arrow[r, tail] \arrow[d] & A_{\{\gamma_0\}\cup d^*_{k-1}\alpha} \arrow[r, two heads] \arrow[d] & A_{\{\gamma_1\}\cup d^*_{k-1}\alpha} \arrow[d] \\
\eta^{\triangleright}_{(n-\gamma_1)(n-\gamma_0)}(A)_{d^*_{k-2}\alpha} \arrow[r, tail] \arrow[d] & A_{\{\gamma_0\}\cup d^*_{k-2}\alpha} \arrow[r, two heads] \arrow[d] & A_{\{\gamma_1\}\cup d^*_{k-2}\alpha} \arrow[d] \\
\eta^{\triangleright}_{(n-\gamma_1)(n-\gamma_0)}(A)_{d^*_{k-3}\alpha} \arrow[r, tail] \arrow[d] & A_{\{\gamma_0\}\cup d^*_{k-3}\alpha} \arrow[r, two heads] \arrow[d] & A_{\{\gamma_1\}\cup d^*_{k-3}\alpha} \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd} \tag{5.10}$$

where $\alpha = d^*_0 d^*_1\gamma$. In particular, $B_2 \to C_2$ is an admissible morphism. Applying the snake lemma to the third morphism of short exact sequences in (5.10) tells us that the map

$$C'_2 = \operatorname{coker}(B_2 \to C_2) \longrightarrow \operatorname{coker}(A_{d^*_1 d^*_{k+1}\gamma} \to A_{d^*_1 d^*_k\gamma})$$

is an admissible monomorphism, and therefore, by applying it to the first, that $\widehat{A}_{d^*_1\gamma} \twoheadrightarrow \widehat{A}_{d^*_0\gamma}$.

Finally, we can iterate the argument, realizing (5.9) as the upper part of the diagram

$$\begin{tikzcd}
\eta^{(i)}(A)_{d^*_{k-i}\alpha^{(i)}} \arrow[r, tail] \arrow[d] & A_{\{\gamma_0,\dots,\gamma_{i-1}\}\cup d^*_{k-i}\alpha^{(i)}} \arrow[r, two heads] \arrow[d] & \eta^{(i-1)}(A)_{d^*_{k-i+1}\alpha^{(i-1)}} \arrow[d] \\
\eta^{(i)}(A)_{d^*_{k-i-1}\alpha^{(i)}} \arrow[r, tail] \arrow[d] & A_{\{\gamma_0,\dots,\gamma_{i-1}\}\cup d^*_{k-i-1}\alpha^{(i)}} \arrow[r, two heads] \arrow[d] & \eta^{(i-1)}(A)_{d^*_{k-i}\alpha^{(i-1)}} \arrow[d] \\
\eta^{(i)}(A)_{d^*_{k-i-2}\alpha^{(i)}} \arrow[r, tail] \arrow[d] & A_{\{\gamma_0,\dots,\gamma_{i-1}\}\cup d^*_{k-i-2}\alpha^{(i)}} \arrow[r, two heads] \arrow[d] & \eta^{(i-1)}(A)_{d^*_{k-i-1}\alpha^{(i-1)}} \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

where $\eta^{(i)} = \eta^{\triangleright}_{(n-\gamma_i)(n-\gamma_{i-1})} \circ \dots \circ \eta^{\triangleright}_{(n-\gamma_1)(n-\gamma_0)}$ and $\alpha^{(i)} = d^*_0 \dots d^*_i \gamma$. Then the sequence

$$B'_{i+1} \rightarrowtail \widehat{A}_{d^*_i\gamma} \longrightarrow B'_i$$

is the beginning of the corresponding long exact snake, where $B'_i = \ker(B_i \to C_i)$, and is therefore a short exact sequence, as above.

Finally, the equivalence $P^{\triangleleft}S^{\langle k\rangle}(\mathcal{E}) \xrightarrow{\simeq} S^{\langle k-1]}(\mathcal{E})$ follows via Lemma 4.6 from the one we have proven above. Furthermore, if $k \geq 2$, we obtain $P^{\triangleleft}P^{\triangleright}S^{\langle k\rangle}(\mathcal{E}) \xrightarrow{\simeq} S^{[k-2]}(\mathcal{E})$ as an immediate consequence of the two. Namely, let $A \in S^{[k-2]}_n(\mathcal{E})$. Then the left Kan extension analogous to (5.8) produces a diagram

$$\widehat{A} \in S^{[k-1\rangle}_{n+1}(\mathcal{E}) = P^{\triangleleft}S^{[k-1\rangle}_n(\mathcal{E}) \simeq P^{\triangleleft}P^{\triangleright}S^{\langle k\rangle}_n(\mathcal{E}),$$

as all arguments above apply verbatim to show that $\widehat{A}$ consists of right exact sequences. $\square$

Remark 5.15. Proposition 5.14 can be seen as a higher analogue of the third isomorphism theorem, in that the equivalence of categories $S^{\langle k-1]}_{k+1}(\mathcal{E}) \xrightarrow{\simeq} P^{\triangleleft}S^{\langle k\rangle}_{k+1}(\mathcal{E}) = S^{\langle k\rangle}_{k+2}(\mathcal{E})$ boils down to the following statement. Given a configuration of left exact sequences of the form

$$\begin{tikzcd}
A^{k+1}_k \arrow[r, tail] \arrow[dr, tail] & A^{k+1}_{k-1} \arrow[r] \arrow[d, tail] & \cdots \arrow[r] & A^{k+1}_1 \arrow[r] \arrow[d, tail] & A^{k+1}_0 \arrow[d, tail] \\
& A^k_{k-1} \arrow[r] \arrow[dr] & \cdots \arrow[r] & A^k_1 \arrow[r] \arrow[d] & A^k_0 \arrow[d] \\
& & \cdots \arrow[r] & A^{k-1}_1 \arrow[r] \arrow[d] & A^{k-1}_0 \arrow[d] \\
& & & \vdots \arrow[dr] & \vdots \arrow[d] \\
& & & & A^0_0
\end{tikzcd}$$

where $A_i^j = A_{d_i^* d_j^* \Delta_{k+1}^{k+1}}$ in the previous notation, the induced maps between the cokernels

$$\operatorname{coker}(A_1^{k+1} \to A_0^{k+1}) \longrightarrow \operatorname{coker}(A_1^k \to A_0^k) \longrightarrow \ldots \longrightarrow \operatorname{coker}(A_1^0 \to A_0^0)$$

constitute an exact sequence in $\mathcal{E}$.

Corollary 5.16. *If $\mathcal{E}$ is stringent, there are natural equivalences of simplicial categories*

$$\begin{aligned} S^{\langle k]}(\mathcal{E}) &\xrightarrow{\simeq} T(S^{\langle 0]}S^{(k)}(\mathcal{E})), \\ S^{[k\rangle}(\mathcal{E}) &\xrightarrow{\simeq} T(S^{(k)}S^{[0\rangle}(\mathcal{E})), \\ S^{[k]}(\mathcal{E}) &\xrightarrow{\simeq} T(S^{\langle 0]}S^{(k)}S^{[0\rangle}(\mathcal{E})). \end{aligned}$$

Proof. This is an immediate consequence of Proposition 5.14 and Theorem 4.12, as

$$S^{\langle k]}(\mathcal{E}) \simeq P^{\triangleleft}S^{\langle k+1\rangle}(\mathcal{E}) \simeq P^{\triangleleft}TS^{(k+1)}(\mathcal{E}) \simeq TP^{\triangleleft}S^{(k+1)}(\mathcal{E}) \simeq T(S^{\langle 0]}S^{(k)}(\mathcal{E})),$$

by Lemma 2.17. □

Example 5.17. Let us consider the situation of (2.6). We see that its essential image cannot contain $\pi_0(S^{\langle 1]}(\mathcal{E})^\times)$ for every stringent category $\mathcal{E}$, as the corresponding double category would have precisely one object, which precludes the existence of a bijection on the higher cells in general. One non-trivial exception is $\mathcal{E} = \mathrm{vect}_{\mathbb{F}_1}$ where $S^{\langle 1\rangle}(\mathcal{E})^\times$ is lower 1-Segal.

Theorem 5.18. *Let $\mathcal{E}$ be a proto-exact category. The two-dimensional Waldhausen construction $S^{\langle 2\rangle}(\mathcal{E})$ is an upper 3-Segal category if and only if $\mathcal{E}$ is proto-abelian.*

Proof. By the path space criterion, it suffices to show that $P^{\triangleleft}P^{\triangleright}S^{\langle 2\rangle}(\mathcal{E})$ is lower 1-Segal. But Proposition 5.14 below shows that for all $n \geq 2$, the forgetful functor

$$P^{\triangleleft}P^{\triangleright}S^{\langle 2\rangle}_{n-2}(\mathcal{E}) \longrightarrow S^{[0]}_{n-2}(\mathcal{E}),\ A \longmapsto (A_{01n} \to A_{02n} \to \cdots \to A_{0(n-1)n}),$$

is an equivalence of categories, identifying the double path space $P^{\triangleleft}P^{\triangleright}S^{\langle 2\rangle}(\mathcal{E}) \xrightarrow{\simeq} N^{\mathcal{E}}(\mathcal{E})$ with the categorified nerve of $\mathcal{E}$, cf. Example 4.4 (1).

Conversely, the lower 1-Segal condition for $P^{\triangleleft}P^{\triangleright}S^{\langle 2\rangle}(\mathcal{E}) \simeq S^{[0]}(\mathcal{E})$ requires admissible morphisms in $\mathcal{E}$ be closed under composition, hence $\mathcal{E}$ must be proto-abelian already. □

We are now prepared to prove our main result. In particular, by Proposition 5.14 as well as Lemma 4.6, the path space criterion provides a new proof of Theorem 4.10...

Proof of Proposition 5.9. Let $n \geq 2k$. We show inductively that the lower $(2k-1)$-Segal map

$$S_n^{\langle k-1]}(\mathcal{E}) \longrightarrow \varprojlim_{I \in \mathcal{L}([n], 2k-1)} S_I^{\langle k-1]}(\mathcal{E}) \tag{5.11}$$

is an equivalence. Throughout, for $0 \leq i \leq n$, let δ_i refer to the ith face map of Δ_n^n, even when applied to any subsimplex of it. That is, δ_i^* removes the vertex i, and $\widehat{\delta}_i^*$ adjoins it.

First, consider the case $n = 2k$. By Lemma 5.19, the only k-simplex in Δ^{2k} not contained in an even subset of $[2k]$ of cardinality $2k$ already is

$$\varepsilon = \{0, 2, \ldots, 2k\} = \delta_{2k-1}^* \delta_{2k-3}^* \cdots \delta_1^* \Delta_{2k}^{2k}.$$

But if A lies in the right-hand side of (5.11), then we can form the unique compositions

$$\begin{tikzcd}[column sep=small] A_{\delta_{2k}^*\varepsilon} \arrow[rr, dashed] \arrow[dr, tail] & & A_{\delta_{2k-2}^*\varepsilon} \arrow[rr, dashed] \arrow[dr] & & \ldots \arrow[rr, dashed] \arrow[dr] & & A_{\delta_0^*\varepsilon} \\ & A_{\delta_{2k}^*\widehat{\delta}_{2k-1}^*\delta_{2k-2}^*\varepsilon} \arrow[ur, tail] & & A_{\delta_{2k-2}^*\widehat{\delta}_{2k-3}^*\delta_{2k-4}^*\varepsilon} \arrow[ur] & \ldots & A_{\delta_2^*\widehat{\delta}_1^*\delta_0^*\varepsilon} \arrow[ur] & \end{tikzcd}$$

completing A to an element of $S_{2k}^{\langle k-1]}(\mathcal{E})$. It remains to be shown that the resulting sequence

$$A_{\delta_{2k}^*\varepsilon} \longrightarrow A_{\delta_{2k-2}^*\varepsilon} \longrightarrow \ldots \longrightarrow A_{\delta_0^*\varepsilon}$$

is left exact. We proceed by induction. The case $k=2$ is settled by Theorem 5.18. In general, since $A_{\delta^*_{2k}\varepsilon} \rightarrowtail A_{\delta^*_{2k-2}\varepsilon}$ is an admissible monomorphism (as a composition of such), it suffices to prove that

$$\operatorname{coker}(A_{\delta^*_{2k}\varepsilon} \rightarrowtail A_{\delta^*_{2k-2}\varepsilon}) \longrightarrow A_{\delta^*_{2k-4}\varepsilon} \longrightarrow \ldots \longrightarrow A_{\delta^*_0\varepsilon} \tag{5.12}$$

is a left exact sequence. For this, we use the hyperplane lemma. Namely, the functor

$$\eta^{\triangleleft}_{(2k-2)2k} \colon S^{\langle k-1]}_{2k}(\mathcal{E}) \longrightarrow S^{\langle k-2]}_{2k-2}(\mathcal{E})$$

constructed in Proposition 5.13 is compatible with the corresponding lower Segal maps on both sides, in that it induces a commutative diagram of the following form.

$$\begin{tikzcd}
S^{\langle k-1]}_{2k}(\mathcal{E}) \arrow[r] \arrow[d, "\eta^{\triangleleft}_{(2k-2)2k}"] & \varprojlim_{I\in\mathcal{L}([2k],2k-1)} S^{\langle k-1]}_I(\mathcal{E}) \arrow[d, dashed, "\eta^{\triangleleft}_{(2k-2)2k}"] \\
S^{\langle k-2]}_{2k-2}(\mathcal{E}) \arrow[r] & \varprojlim_{J\in\mathcal{L}([2k-2],2k-3)} S^{\langle k-2]}_J(\mathcal{E})
\end{tikzcd}$$

Indeed, this is because we have $d_0^*d_0^*\varepsilon = \delta^*_{2k-3}\delta^*_{2k-5}\cdots\delta^*_1\Delta^{2k-2}_{2k-2}$. But then, by induction, the lower horizontal map is an equivalence, which by the above means precisely that (5.12) is a left exact sequence.

In order to prove the Segal conditions for the higher cells $S^{\langle k-1]}_n(\mathcal{E})$, we once again employ induction, now on the dimension n. If A lies in the right-hand side of the lower $(2k-1)$-Segal map for $S^{\langle k-1]}_n(\mathcal{E})$, we first need to see that taking compositions completes A to a well-defined diagram of shape $\operatorname{Fun}([k-1],[n])$ in $\mathcal{E}$.

By Lemma 5.19, we need only consider sequences indexed by simplices γ with all vertices separated by gaps. For gaps $i \in [n]$ of size 1, there is (as before) a unique composition,

$$\begin{tikzcd}
A_{\delta^*_{i+1}\gamma} \arrow[rr, dashed] \arrow[dr] & & A_{\delta^*_{i-1}\gamma}. \\
& A_{\delta^*_{i+1}\widehat{\delta}^*_i\delta^*_{i-1}\gamma} \arrow[ur] &
\end{tikzcd}$$

For a gap of γ of size $l+1$, say $\{i,\ldots,i+l\} \subseteq [n]$, each $0 \le j \le l$ defines the composition

$$\begin{tikzcd}
A_{\delta^*_{i+l+1}\gamma} \arrow[rr, dashed] \arrow[dr] & & A_{\delta^*_{i-1}\gamma}. \\
& A_{\delta^*_{i+l+1}\widehat{\delta}^*_{i+j}\delta^*_{i-1}\gamma} \arrow[ur] &
\end{tikzcd}$$

By induction, all possible compositions can be reduced to one of these. On the other hand, they all agree, since for all $0 \le j < j' \le l$, the following diagram commutes.

$$\begin{tikzcd}
A_{\delta^*_{i+l+1}\gamma} \arrow[r] \arrow[d] & A_{\delta^*_{i+l+1}\widehat{\delta}^*_{i+j'}\delta^*_{i-1}\gamma} \arrow[d] \\
A_{\delta^*_{i+l+1}\widehat{\delta}^*_{i+j}\delta^*_{i-1}\gamma} \arrow[r] \arrow[ur] & A_{\delta^*_{i-1}\gamma}
\end{tikzcd}$$

Finally, we apply induction to obtain the remaining exactness conditions for the completed diagram of A. Namely, the sequence indexed by γ is left exact, since $\partial_i(A)$ lies in the right-hand side of the lower $(2k-1)$-Segal map of $S^{\langle k-1]}_{n-1}(\mathcal{E})$, for any gap i of γ. □

Lemma 5.19. *Let $n \ge 2k$. Let γ be a k-subsimplex of Δ^n with a pair of adjacent simplices. Then γ is contained in an even subset $I \subseteq [n]$ of cardinality $\#I = 2k$.*

Proof. For $n = 2k$, this is clear. For the induction step, we can assume that $n \in \gamma$, otherwise the statement follows tautologically from the induction hypothesis. Let $0 < m < n$ be the maximal gap of γ. By induction, $(\gamma \cup \{m\}) \smallsetminus \{n\}$ is contained in an even subset $I' \subseteq [n-1]$ with $\#I' = 2k$. But then γ is contained in $I = (I' \smallsetminus \{m\}) \cup \{n\}$, which is even in $[n]$. □

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