Σ^1_1-definability at uncountable regular cardinals

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Let κ be an infinite cardinal, ${}^{\kappa}\kappa$ be the set of all functions $f : \kappa \longrightarrow \kappa$ and ${}^{<\kappa}\kappa$ be the set of all functions f with $\mathrm{dom}(f) \in \kappa$ and $\mathrm{ran}(f) \subseteq \kappa$.

The *generalized Baire space* of κ is the set ${}^{\kappa}\kappa$ equipped with the topology whose basic open sets are of the form

$$U_s = \{f \in {}^{\kappa}\kappa \mid s \subseteq f\}$$

for some $s \in {}^{<\kappa}\kappa$. Note that closed sets in this topology are of the form

$$[T] = \{f \in {}^{\kappa}\kappa \mid (\forall \alpha < \kappa)\ f \upharpoonright \alpha \in T\}$$

for some subtree T of ${}^{<\kappa}\kappa$.

We call a subset of $({}^{\kappa}\kappa)^n$ a $\mathbf{\Sigma}^1_1$-subset if it is the projection of a closed subset of $({}^{\kappa}\kappa)^{n+1}$. Given $0 < 1 < \omega$, we define $\mathbf{\Sigma}^1_n$-, $\mathbf{\Pi}^1_n$- and $\mathbf{\Delta}^1_n$-subsets in the usual way.

We want to study the generalized Baire spaces of uncountable regular cardinals κ with $\kappa = \kappa^{<\kappa}$ and the $\mathbf{\Sigma}^1_1$-subsets of these spaces.

The following proposition shows that this class is both interesting and rich.

Proposition

Let κ be an uncountable regular cardinal with $\kappa = \kappa^{<\kappa}$. The following statements are equivalent for a subset A of ${}^{\kappa}\kappa$.

- *A is a $\mathbf{\Sigma}^1_1$-subset of ${}^{\kappa}\kappa$.*
- *A is definable in the structure $\langle \mathrm{H}(\kappa^+), \in \rangle$ by a Σ_1-formula with parameters.*

The initial motivation of this work was to find generalizations of the following coding result due to Leo Harrington to uncountable regular cardinals κ and $<\kappa$-closed forcings that satisfy the κ^+-chain condition.

Theorem (L. Harrington, 1977)

Assume $\omega_1 = \omega_1^{\mathrm{L}}$. For every subset A of ${}^{\omega}\omega$, there is a partial order $\mathbb{P}$ with the following properties.

- *$\mathbb{P}$ satisfies the countable chain condition.*
- *If G is $\mathbb{P}$-generic over V, then A is a $\mathbf{\Pi}^1_2$-subset of ${}^{\omega}\omega$ in $\mathrm{V}[G]$.*

This is achieved by the following result.

Theorem (P.L., 2012)

Let κ be a regular uncountable cardinal with $\kappa = \kappa^{<\kappa}$. For every subset A of ${}^{\kappa}\kappa$, there is a partial order $\mathbb{P}$ that satisfies the following statements.

- *$\mathbb{P}$ is $<\kappa$-closed, satisfies the κ^+-chain condition and has cardinality 2^{κ}.*
- *If G is $\mathbb{P}$-generic over V, then A is a $\mathbf{\Delta}^1_1$-subset of ${}^{\kappa}\kappa$ in $\mathrm{V}[G]$.*

In contrast to previous coding results for subsets of ${}^{\kappa}\kappa$, we do not need to assume that "$2^{\kappa} = \kappa^{+}$" holds.

The proof of this result relies on a technique called *generic tree coding*. In the remainder of this talk, I want to give a brief introduction to this technique. The following theorem sums up its properties.

Theorem

Let κ be a regular uncountable cardinal with $\kappa = \kappa^{<\kappa}$. For every subset A of ${}^{\kappa}\kappa$, there is a partial order $\mathbb{P}$ that satisfies the following statements.

- *$\mathbb{P}$ is $<\kappa$-closed, satisfies the κ^{+}-chain condition and has cardinality 2^{κ}.*
- *If $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name with*

$$\mathbb{1}_{\mathbb{P}} \Vdash \text{“}\dot{\mathbb{Q}} \text{ is a } \sigma\text{-strategically closed partial order and forcing with } \dot{\mathbb{Q}} \text{ preserves the regularity of } \check{\kappa}\text{”}.$$

  and $G * H$ is $(\mathbb{P} * \dot{\mathbb{Q}})$-generic over V, then A is a $\mathbf{\Sigma}^1_1$-subset of ${}^{\kappa}\kappa$ in $\mathrm{V}[G][H]$.

Given a nonempty subset A of ${}^{\kappa}\kappa$ and an enumeration $\langle s_\beta \mid \beta < \kappa\rangle$ of ${}^{<\kappa}\kappa$, we define $\mathbb{P}$ to be the partial order consisting of conditions

$$p = \langle T_p, f_p, h_p\rangle$$

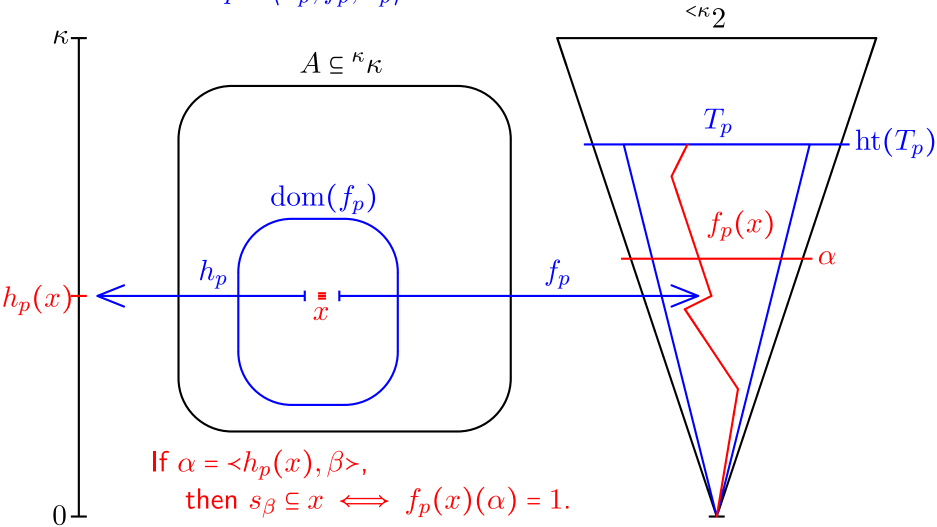
with the following properties.

- T_p is a subtree of ${}^{<\kappa}2$ that satisfies the following statements.
  - T_p has cardinality less than κ.
  - If $t \in T_p$ with $\mathrm{lh}(t) + 1 < \mathrm{ht}(T_p)$, then t has two immediate successors in T_p.
- $f_p : A \xrightarrow{part} [T_p]$ is a partial function such that $\mathrm{dom}(f_p)$ is a nonempty set of cardinality less than κ.
- $h_p : A \xrightarrow{part} \kappa$ is a partial function with the following properties.
  - $\mathrm{dom}(h_p) = \mathrm{dom}(f_p)$.
  - For all $x \in \mathrm{dom}(h_p)$ and $\alpha, \beta < \mathrm{ht}(T_p)$ with $\alpha = \prec h_p(x), \beta \succ$, we have

$$s_\beta \subseteq x \iff f_p(x)(\alpha) = 1.$$

We order $\mathbb{P}$ by end-extensions of trees and extensions of branches and functions.

$p = \langle T_p, f_p, h_p \rangle \in \mathbb{P}$



If $\alpha = \prec h_p(x), \beta \succ$,
then $s_\beta \subseteq x \iff f_p(x)(\alpha) = 1$.

If $G * H$ is $(\mathbb{P} * \dot{\mathbb{Q}})$-generic over V, then the following statements hold.

- $T = \bigcup\{T_p \mid p \in G\}$ is a subtree of ${}^{<\kappa}2$ of height κ with $[T] \cap \mathrm{V} = \varnothing$.
- If we define $F(x) = \bigcup\{f_p(x) \mid p \in G,\ x \in \mathrm{dom}(x)\}$ for all $x \in A$, then $F : A \longrightarrow [T]^{\mathrm{V}[G][H]}$ is a bijection.
- If we define $H = \bigcup\{h_p \mid p \in G\}$, then $H : A \longrightarrow \kappa$ is a function with

$$s_\beta \subseteq x \iff F(x)(\prec H(x), \beta \succ) = 1$$

for all $x \in A$ and $\beta < \kappa$.

This yields the following $\mathbf{\Sigma}^1_1$-definition of A in $\mathrm{V}[G][H]$:

$$x \in A \iff (\exists y \in [T])(\exists \gamma < \kappa)(\forall \beta < \kappa)\left[s_\beta \subseteq x \longleftrightarrow y(\prec \gamma, \beta \succ) = 1\right].$$

This result has several applications. As above, we let κ denote a regular uncountable cardinal with $\kappa = \kappa^{<\kappa}$.

- If A is an arbitrary set, then there is a $<\kappa$-closed partial order $\mathbb{P}$ such that $\mathbb{P}$ satisfies the κ^+-chain condition and $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\check{A} \in \mathrm{L}(\mathcal{P}(\check{\kappa}))$".
- Generic absoluteness for $\mathbf{\Sigma}^1_3$-formulas over ${}^{\kappa}\kappa$ under $<\kappa$-closed forcings that satisfy the κ^+-chain condition is inconsistent. (It is consistent to have such absoluteness for $\mathbf{\Sigma}^1_2$-formulas over ${}^{\kappa}\kappa$).
- There is a $<\kappa$-closed partial order $\mathbb{P}$ satisfying the κ^+-chain condition such that forcing with $\mathbb{P}$ preserves the value of 2^{κ} and adds a $\mathbf{\Delta}^1_2$-definable well-ordering of ${}^{\kappa}\kappa$.

I close by presenting a version of the above result for large cardinals.

Theorem (S. Friedman & P.L.)

There is a $\mathbf{ZFC}$*-preserving class forcing* $\mathbb{P}$ *definable without parameters that satisfies the following statements.*

- *Let* κ *be a cardinal with the property that there is no singular limit of inaccessible cardinals* ν *with* $\nu^+ < \kappa \leq 2^\nu$*. Then forcing with* $\mathbb{P}$ *does not collapse* κ *and, if* κ *is regular, then* $\mathbb{P}$ *preserves the regularity of* κ*.*
- $\mathbb{P}$ *preserves the inaccessibility of inaccessible cardinals and the supercompactness of supercompact cardinals.*
- *If* α *is an inaccessible cardinal and* G *is* $\mathbb{P}$ *generic over* V*, then* $(2^\alpha)^{\mathrm{V}} = (2^\alpha)^{\mathrm{V}[\mathrm{G}]}$*.*
- *If* κ *is an inaccessible cardinal and* A *is a subset of* ${}^{\kappa}\kappa$*, then there is a condition* p *in* $\mathbb{P}$ *with the property that* A *is a* $\mathbf{\Sigma}^1_1$*-subset of* ${}^{\kappa}\kappa$ *in* $\mathrm{V}[\mathrm{G}]$ *whenever* G *is* $\mathbb{P}$*-generic over* V *with* $p \in \mathrm{G}$*.*

Thank you for listening!