Simply definable failures of weak compactness

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Incontro Italiano Insiemi e Modelli 2018
Turin, 22.02.2018

Introduction

The work presented in this talk is motivated by the following results:

- A theorem of Hung and Negrepontis showing that weakly compact cardinals can be characterized through *generalized descriptive set theory*.
- Recent work of Andretta and Motto Ros that considers variations of this characterization in determinacy models.

Given a set X and $n < \omega$, we let $[X]^n$ denote the set of all n-element subsets of X.

Given a function c with domain $[X]^n$ and $H \subseteq X$, we say that H is *c-homogeneous* if $c \restriction [H]^n$ is constant.

An uncountable cardinal κ is *weakly compact* if and only if for every function $c : [\kappa]^2 \longrightarrow 2$, there is a c-homogeneous subset of κ of cardinality κ.

Given an uncountable regular cardinal κ, the *generalized Baire space of* κ is the set ${}^\kappa\kappa$ of all functions from κ to κ equipped with the topology whose basic open sets are of the form

$$N_s \ = \ \{x \in {}^\kappa\kappa \mid s \subseteq x\}$$

for some $s : \alpha \longrightarrow \kappa$ with $\alpha < \kappa$.

The *generalized Cantor space of* κ is the subspace ${}^\kappa 2$ of ${}^\kappa\kappa$ consisting of all binary functions.

Theorem (Hung–Negrepontis)

The following statements are equivalent for every uncountable regular cardinal κ*:*

- κ *is weakly compact.*
- *The spaces* ${}^\kappa\kappa$ *and* ${}^\kappa 2$ *are not homeomorphic.*

Theorem (Andretta–Motto Ros)

The theory $\mathbf{ZF} + \mathbf{DC} + \mathbf{AD}$ *proves that the spaces* ${}^{\omega_1}\omega_1$ *and* ${}^{\omega_1}2$ *are not homeomorphic.*

With the help of results of Woodin on the Π_2-maximality of the $\mathbb{P}_{max}$-extension of $\mathrm{L}(\mathbb{R})$, it is possible to use the above theorem to prove results on the complexity of homeomorphisms witnessing failures of weak compactness at ω_1.

Remember that a formula φ in the language $\mathcal{L}_\in = \{\in\}$ of set theory is a Σ_0-*formula* if it is contained in the smallest collection of $\mathcal{L}_\in$-formulas that contains all atomic $\mathcal{L}_\in$-formulas and is closed under $\neg$, $\wedge$ and $\exists x \in v$.

Moreover, an $\mathcal{L}_\in$-formula is a Σ_{n+1}-*formula* if it is of the form $\exists x \; \neg\varphi(x)$ for some Σ_n-formula φ.

Corollary

Assume that there are infinitely many Woodin cardinals with a measurable cardinal above them all. Then no homeomorphism between the spaces ${}^{\omega_1}\omega_1$ *and* ${}^{\omega_1}2$ *is definable by a* Σ_1*-formula with parameters in* $\mathrm{H}(\omega_1) \cup \{\omega_1\}$.

Proof.

Assume that there is a Σ_1-formula φ that defines a homeomorphism $h : {}^{\omega_1}\omega_1 \longrightarrow {}^{\omega_1}2$ using the parameters ω_1 and $z \in \mathrm{H}(\omega_1)$.

If G be $\mathbb{P}_{max}$-generic over $\mathrm{L}(\mathbb{R})$, then the Π_2-maximality of $\mathrm{L}(\mathbb{R})[G]$ implies that, in $\mathrm{L}(\mathbb{R})[G]$, the formula φ and the parameters ω_1 and z define a homeomorphism $g : {}^{\omega_1}\omega_1 \longrightarrow {}^{\omega_1}2$.

Since $\mathbb{P}_{max}$ is weakly homogeneous in $\mathrm{L}(\mathbb{R})$, the map $g \restriction ({}^{\omega_1}\omega_1)^{\mathrm{L}(\mathbb{R})}$ is definable in $\mathrm{L}(\mathbb{R})$ and this restriction is an homeomorphism of ${}^{\omega_1}\omega_1$ and ${}^{\omega_1}2$ in $\mathrm{L}(\mathbb{R})$.

This contradicts the above result of Andretta and Motto Ros. $\square$

Motivated by above result, we want to study the influence of canonical extensions of $\mathbf{ZFC}$ (e.g. large cardinal axioms or forcing axioms) on the possible complexities of homeomorphisms witnessing failures of weak compactness.

Since most of these extension are compatible with the assumption $\mathrm{V} = \mathrm{HOD}$, the following proposition shows that this question is most interesting for Σ_1-definitions.

Proposition

If $\mathrm{V} = \mathrm{HOD}$ holds, then the following statements are equivalent for every uncountable regular cardinal κ:

- *κ is not weakly compact.*
- *There is a homeomorphism between ${}^{\kappa}\kappa$ and ${}^{\kappa}2$ that is definable by a Σ_2-formula with parameter κ.*

The Σ_n-partition property

Motivated by the above observation, we study restrictions of weak compactness to the definable context.

Definition

Given $0 < n < \omega$, an uncountable regular cardinal κ has the $\mathbf{\Sigma}_n$-*partition property* if for every function $c : [\kappa]^2 \longrightarrow 2$ that is definable by a Σ_n-formula with parameters in $\mathrm{H}(\kappa) \cup \{\kappa\}$, there exists a c-homogeneous subset of κ of cardinality κ.

The following observation connects the above notion with the questions discussed in this talk.

Lemma

Let κ be an uncountable regular cardinal with the $\mathbf{\Sigma}_n$-partition property. Then no homeomorphism between ${}^{\kappa}\kappa$ and ${}^{\kappa}2$ is definable by a Σ_n-formula with parameters in $\mathrm{H}(\kappa) \cup \{\kappa\}$.

The above lemma is a consequence of the following result.

Lemma

Given an uncountable regular cardinal κ, the following statements are equivalent for every $0 < n < \omega$:

- *κ has the $\mathbf{\Sigma}_n$-partition property.*
- *If $\iota : \kappa \longrightarrow {}^{<\kappa}2$ is an injection that is definable by a Σ_n-formula with parameters in $\mathrm{H}(\kappa) \cup \{\kappa\}$, then there is an $x \in {}^{\kappa}2$ such that the set*

  $$\{\alpha < \kappa \mid \exists \beta < \kappa \;\; x \restriction \alpha \subseteq \iota(\beta)\}$$

  is unbounded in κ.

Using the above characterization, we can now provide more examples of non-weakly compact cardinals with the property that no homeomorphism witnessing the failure of weak compactness is Σ_1-definable.

Proposition

If κ is a weakly compact cardinal, then every $\mathbf{\Pi}^1_1$-statement that holds in $\langle \mathrm{V}_\kappa, \in \rangle$ reflects to an inaccessible cardinal less than κ with the $\mathbf{\Sigma}_1$-partition property.

Proposition

If $\mathrm{V} = \mathrm{L}$, κ is inaccessible with Σ_1-partition property, $\lambda < \kappa$ regular and G $\mathrm{Col}(\lambda, {<}\kappa)$-generic over V, then κ has the $\mathbf{\Sigma}_1$-partition property in $\mathrm{V}[G]$.

Assuming the existence of certain definable well-orderings, it is also possible to prove a converse of the above implication.

Definition

Given sets A and z, a well-ordering $\lhd$ of A is a *good* $\Sigma_n(z)$*-well-ordering* if the set of all proper initial segments of $\lhd$ is definable by a Σ_n-formula with parameter z.

Note that, in L and K^{DJ}, good Σ_1-well-orderings of $\mathrm{H}(\kappa)$ of length κ exist for every infinite cardinal κ.

Proposition

Let κ be an uncountable regular cardinal such that there is a good $\Sigma_n(\kappa, z)$-well-ordering of $\mathrm{H}(\kappa)$ of length κ for some $z \in \mathrm{H}(\kappa)$.

- *If κ not inaccessible, then κ does not have the $\mathbf{\Sigma}_n$-partition property.*
- *If κ does not have $\mathbf{\Sigma}_n$-partition property, then there is a homeomorphism between ${}^\kappa\kappa$ and ${}^\kappa 2$ that is definable by a Σ_n-formula with parameters in $\mathrm{H}(\kappa) \cup \{\kappa\}$.*

Corollary

If $\mathrm{V} = \mathrm{L}$ holds, then the following statements are equivalent for every uncountable regular cardinal:

- *No homeomorphism between ${}^\kappa\kappa$ and ${}^\kappa 2$ is definable by a Σ_1-formula with parameters in $\mathrm{H}(\kappa \cup \{\kappa\})$.*
- *κ has the Σ_1-partition property (and hence κ is inaccessible).*

The Σ_n-club property

Definition

An uncountable regular cardinal κ has the $\mathbf{\Sigma}_n$-*club property* if for every function $c : [\kappa]^m \longrightarrow \alpha$ with $\alpha < \kappa$ that is definable by a Σ_n-formula with parameters in $\mathrm{H}(\kappa) \cup \{\kappa\}$, there is a c-homogeneous club subset of κ.

Theorem

Assume that one of the following assumptions holds:

- *There is a measurable cardinal above a Woodin cardinal.*
- *There is a measurable cardinal and a precipitous ideal on ω_1.*
- $\mathbf{BMM}$ *holds and the nonstationary ideal on ω_1 is precipitous.*
- *Woodin's Axiom $(*)$ holds.*

Then ω_1 has the $\mathbf{\Sigma}_1$-club property.

Lemma

The following statements are equivalent for every uncountable regular cardinal κ:

- κ *has the* $\mathbf{\Sigma}_n$*-club property.*
- *If* $A \subseteq \kappa$ *has the property that the set* $\{A\}$ *is definable by a* Σ_n*-formula with parameters in* $\mathrm{H}(\kappa) \cup \{\kappa\}$*, then* A *either contains a club or is disjoint from such a subset.*

Using this characterization, it is easy to derive the following statements:

- If κ has the $\mathbf{\Sigma}_1$-club property, then either $\kappa = \omega_1$ or κ is a limit cardinal.
- If κ has the $\mathbf{\Sigma}_2$-club property, then $\kappa = \omega_1$.
- In the $\mathbb{P}_{max}$-extension of $\mathrm{L}(\mathbb{R})$, the cardinal ω_1 has the $\mathbf{\Sigma}_n$-club property for every $n < \omega$.

Moreover, this characterization shows that the above theorem is a direct consequence of the following result.

Theorem (L.–Schindler–Schlicht)

Assume that either $M_1^\#(A)$ exists for every $A \subseteq \omega_1$ or there is a measurable cardinal and a precipitous ideal on ω_1.

Then the following statements hold for every Σ_1-formula $\varphi(v_0, v_1, v_2)$ and all $z \in \mathrm{H}(\omega_1)$:

- *If there is $A \subseteq \omega_1$ stationary with $\varphi(A, \omega_1, z)$, then there is an element B of the club filter on ω_1 with $\varphi(B, \omega_1, z)$.*
- *If there is $A \subseteq \omega_1$ costationary with $\varphi(A, \omega_1, z)$, then there is an element B of the non-stationary ideal on ω_1 with $\varphi(B, \omega_1, z)$.*

The proof of this theorem uses iterated generic ultrapowers and Woodin's countable stationary tower forcing.

Ideas from the proof of the above lemma can also be used to show that certain (non-weakly compact) large cardinals have the $\mathbf{\Sigma}_1$-club property.

Definition (Sharpe & Welch)

An uncountable cardinal κ is *iterable* if for every $A \subseteq \kappa$, there is a transitive model M of ZFC$^-$ of cardinality κ and a weakly amenable M-ultrafilter U on κ such that $A \in M$ and $\langle M, \in, U \rangle$ is iterable.

- Iterable cardinals and stationary limits of iterable cardinals have the $\mathbf{\Sigma}_1$-club property.
- Regular limits of measurable cardinals have the $\mathbf{\Sigma}_1$-club property.
- If κ is iterable and G is either $\mathrm{Add}(\omega, \kappa)$- or $\mathrm{Col}(\omega, \kappa)$-generic over V, then κ has the $\mathbf{\Sigma}_1$-club property in $\mathrm{V}[G]$.

Definable colourings of $[\omega_2]^2$

Somewhat surprisingly, it turns out that that Σ_1-definability behaves completely different at the second uncountable cardinal ω_2.

First of all, it is easy to see that neither large cardinal axioms nor forcing axioms imply that ω_2 has the $\mathbf{\Sigma}_1$-partition property.

Proposition

$\mathbf{BPFA}$ *implies that there is a homeomorphism between* ${}^{\omega_2}\omega_2$ *and* ${}^{\omega_2}2$ *that is definable by a* Σ_1*-formula with parameter in* $\mathrm{H}(\omega_2) \cup \{\omega_2\}$*.*

By the above results, this statement is a direct consequence of the following theorem.

Theorem (Caicedo–Veličković)

$\mathbf{BPFA}$ *implies that for some* $z \in \mathrm{H}(\omega_2)$*, there is a good* $\Sigma_1(\omega_2, z)$*-well-ordering of* $\mathrm{H}(\omega_2)$ *of length* ω_2*.*

The above Σ_1-formula witnessing the non-weak compactness of ω_2 uses a subset of ω_1 as a parameter.

Since a lot of information can be coded into such subsets, it is natural to also consider Σ_1-formulas that only used the cardinal ω_2 as a parameter.

Again, it turns out that the influence of large cardinals and forcings axioms on Σ_1-definability at ω_2 completely differs from the situation at ω_1 and large cardinals.

Theorem

Assume that $\mathbf{BPFA}$ holds and that there is a well-ordering of the reals that is definable over the structure $\langle \mathrm{H}(\omega_2), \in \rangle$ by a formula without parameters.

- *There is a function $c : [\omega_2]^2 \longrightarrow 2$ that is definable by a Σ_1-formula with parameter ω_2 such that every c-homogeneous subset of ω_2 is countable.*
- *There is a homeomorphism between ${}^{\omega_2}\omega_2$ and ${}^{\omega_2}2$ that is definable by a Σ_1-formula with parameter ω_2.*

Note that results of Asperó and Larson show that the above assumptions are compatible with $\mathbf{PFA}^{++}$, $\mathbf{MM}^{+\omega}$ and all large cardinal assumptions that are preserved by small forcings.

The following lemma is the main ingredient used in the proof of the above result.

Lemma

$\mathbf{BPFA}$ *implies that the set* $\{\mathrm{H}(\omega_2)\}$ *is definable by a* Σ_1*-formula with parameter* ω_2.

The above statement is a direct consequence of the following result.

Theorem (Caicedo–Veličković)

There is a finite fragment $\mathbf{F}$ *of* $\mathbf{ZFC}$ *with the property that* $\mathbf{BPFA}$ *implies that every transitive model* M *of* $\mathbf{F} + \mathbf{BPFA}$ *with* $\omega_2 = \omega_2^M$ *contains* $\mathcal{P}(\omega_1)$.

Successors of singular cardinals

Again, rather surprisingly, $\mathbf{ZFC}$ proves the existence of definable failures of weak compactness for successor of singular strong limit cardinals of uncountable cofinality.

The following result can be extracted from the proof of a theorem of Shelah showing that $\mathrm{L}(\mathcal{P}(\kappa))$ is a model of $\mathbf{ZFC}$ for such cardinals κ.

Theorem

Successors of singular strong limit cardinals of uncountable cofinality do not have the $\mathbf{\Sigma}_2$-partition property.

In contrast, a theorem of Cummings, Friedman, Magidor, Rinot and Sinapova shows that the existence of a singular strong limit cardinal λ of countable cofinality with the property that λ^+ has the $\mathbf{\Sigma}_n$-partition property for all $n < \omega$ is consistent.

Open Questions

Question

Is it provable in $\mathbf{ZFC}$ that successors of singular strong limit cardinals of uncountable cofinality do not have the $\mathbf{\Sigma}_1$-partition property?

Question

Does $\mathbf{MM}^{++}$ imply that for every function $c : [\omega_2]^2 \longrightarrow 2$ that is definable by a Σ_1-formula with parameter ω_2, there is an uncountable c-homogeneous subset of ω_2?

Question

Does $\mathbf{MM}^{++}$ imply that no homeomorphism between ${}^{\omega_2}\omega_2$ and ${}^{\omega_2}2$ is definable by a Σ_1-formula with parameter ω_1?

Thank you for listening!