Fragments of the forcing theorem for class forcings

Philipp Moritz Lücke
(joint work with P. Holy, R. Krapf, A. Njegomir & P. Schlicht)

Mathematisches Institut
Rheinische Friedrich-Wilhelms-Universität Bonn
http://www.math.uni-bonn.de/people/pluecke/

Turin, 08 May 2015

Overview

- Introduction
- Examples of class forcings
- A failure of the definability lemma
- A failure of the truth lemma
- Boolean completions
- Open questions

Introduction

Class forcing and the forcing theorem

Paul Cohen's method of forcing provides us with a powerful tool to construct new models of set theory. One way to generalize this technique is to allow partial orders that are proper classes and require generic filters to intersect all dense subclasses of these partial orders. This approach allows us to construct an even greater variety of models of (fragments of) $\mathbf{ZFC}$.

Since the *Forcing theorem* is the fundamental result in the theory of set forcing and its proof does not generalize to class forcings, it is natural to ask whether certain fragments of this statement also hold for all class forcings.

Our setting

We outline the setting of this talk.

- We work in a model of $\mathbf{ZFC}$ and fix a countable transitive models M of $\mathbf{ZF}^-$ and a partial order $\mathbb{P} \subseteq M$ that is definable over M.
- We say that a filter G on $\mathbb{P}$ is M-generic if G meet every dense subset of $\mathbb{P}$ that is definable over M.
- We let $M^{\mathbb{P}}$ denote the collection of all $\mathbb{P}$-names contained in $M$and, given an M-generic filter G on $\mathbb{P}$, we define $M[G] = \{\sigma^G \mid \sigma \in M^{\mathbb{P}}\}$ to be the corresponding class generic extension of M.
- Given a formula $\varphi(v_0, \ldots, v_{n-1})$ in the language $\mathcal{L}_\in$ of set theory, a condition p in $\mathbb{P}$ and $\sigma_0, \ldots, \sigma_{n-1} \in M^{\mathbb{P}}$, we let

  $$p \Vdash_{\mathbb{P}}^{M} \varphi(\sigma_0, \ldots, \sigma_{n-1})$$

  denote the statement that $\varphi(\sigma_0^G, \ldots, \sigma_{n-1}^G)$ holds in $M[G]$, whenever G is an M-generic filter on $\mathbb{P}$ with $p \in G$.

Fragments of the forcing theorem

Given a countable transitive model M of some $\mathbf{ZF}^-$, a partial order $\mathbb{P}$ definable over M and an $\mathcal{L}_\in$-formula $\varphi(v_0,\ldots,v_{n-1})$, we will consider the following fragments of the forcing theorem for class forcings.

- We say that $\mathbb{P}$ *satisfies the definability lemma for* φ *over* M if the set

$$\{\langle p,\sigma_0,\ldots,\sigma_{n-1}\rangle\in\mathbb{P}\times M^{\mathbb{P}}\times\ldots\times M^{\mathbb{P}}\mid p\Vdash_{\mathbb{P}}^M\varphi(\sigma_0,\ldots,\sigma_{n-1})\}$$

  is definable over M.

- We say that $\mathbb{P}$ *satisfies the truth lemma for* φ *over* M if for all $\sigma_0,\ldots,\sigma_{n-1}\in M^{\mathbb{P}}$ and every M-generic filter G on $\mathbb{P}$ with the property that $\varphi(\sigma_0^G,\ldots,\sigma_{n-1}^G)$ holds in $M[G]$, there is a $p\in G$ with $p\Vdash_{\mathbb{P}}^M\varphi(\sigma_0,\ldots,\sigma_{n-1})$.

- We say that $\mathbb{P}$ *satisfies the forcing theorem for* φ if $\mathbb{P}$ satisfies the definability and the truth lemma for φ over M.

We start by presenting two positive results.

First, we observe that a careful mimicking of the forcing theorem for set forcings yields the following result that shows that a failure of the forcing theorem yields a failure of the forcing theorem for atomic formulas.

Theorem

Let M be a countable transitive model of $\mathbf{ZF}^-$ and $\mathbb{P}$ be a partial order that is definable over M. If $\mathbb{P}$ satisfies the definability lemma for the formula "$v_0 \subseteq v_1$" over M, then $\mathbb{P}$ satisfies the forcing theorem for all $\mathcal{L}_\in$-formulas over M.

Next, we consider definable boolean completions of class forcings.

- Let $\mathbb{B}$ be a boolean algebra that is definable over M. We say that $\mathbb{B}$ is *M-complete* if $\sup_{\mathbb{B}} A$ exists for every $A \subseteq \mathbb{B}$ with $A \in M$.
- We say that $\mathbb{P}$ *has a boolean completion in M* if there is a boolean algebra $\mathbb{B}$ such that $\mathbb{P}$ is a dense suborder of $\mathbb{B}$, $\mathbb{B}$ is definable over M and $\mathbb{B}$ is M-complete.

The next result shows that the existence of a boolean completion is equivalent to the validity of the forcing theorem.

Theorem

Let M be a countable transitive model of $\mathbf{ZF}^-$ and $\mathbb{P}$ be a partial order that is a class in M. If either the power set axiom holds in M of there is a well-ordering of M that is definable in M, then the following statements are equivalent.

- *$\mathbb{P}$ satisfies the forcing theorem for all $\mathcal{L}_\in$-formulas over M.*
- *$\mathbb{P}$ has a boolean completion in M.*

We will later sketch a proof of this result that shows that both statements are equivalent to the definability of the forcing relation for the quantifier-free infinitary language $\mathcal{L}_{\mathrm{On},0}$, allowing set-sized conjunctions and disjunctions.

In the following, we present results showing that all of the properties considered above can fail for class forcings.

The first result shows that there always is a class forcing that does not satisfy the definability lemma. The proof of this result relies on a class forcing defined by Sy Friedman that we will discuss in detail later.

Theorem

Let M be a countable transitive model of $\mathbf{ZF}^-$. Then there is a partial order $\mathbb{P}$ such that $\mathbb{P}$ is definable over M and $\mathbb{P}$ does not satisfy the forcing theorem for atomic formulae over M.

The next result shows that even stronger failures of the definability lemma are possible.

The proof of the following result relies on the notion of *pointwise definable models*, i.e. first-order structures $\mathcal{M}$ with the property that every element of the domain of $\mathcal{M}$ is definable in $\mathcal{M}$ by a formula without parameters. This concept was studied in depth by Hamkins, Linetsky and Reitz. Note that the existence of a transitive model of $\mathbf{ZFC}$ yields the existence of a countable transitive model of $\mathbf{ZFC}$ that is pointwise definable.

We will use this concept to show that there can be class forcing whose forcing relation is not only non-definable over the ground model but also not amenable to the ground model.

Theorem

Let M be a countable transitive model of $\mathbf{ZF}^-$ that is pointwise definable. Then there is a partial order $\mathbb{P}$ such that $\mathbb{P}$ is definable over M and the set

$$\{\langle\sigma,\tau\rangle \in M^{\mathbb{P}} \times M^{\mathbb{P}} \mid \sigma,\tau \in \mathrm{L}_{\omega\cdot 2},\ \mathbb{1} \Vdash_{\mathbb{P}}^{M} \text{“}\sigma = \tau\text{”}\}$$

is not an element of M.

Finally, we consider failures of the truth lemma. The proof of the following result combines results about class forcing over models of Kelley-Morse set theory with Friedman's forcing used in the proof of the above theorem and a class forcing constructed by Hamkins, Linetsky and Reitz that can be used to obtain pointwise definable generic extension.

Theorem

Assume that there is an inaccessible cardinal. Then there is a countable transitive model M of $\mathbf{ZFC}$, a partial order $\mathbb{P}$ and an $\mathcal{L}_\in$-formula φ such that $\mathbb{P}$ is definable over M and $\mathbb{P}$ does not satisfy the truth lemma for φ over M.

We will later sketch proofs for all three negative results.

Examples of class forcings

Collapses

We present some examples of class forcings to emphasize the differences between set and class forcing.

We start by considering class-sized collapses.

Definition

Let M be a countable transitive model of $\mathbf{ZF}^-$ with $\alpha = M \cap \mathrm{On}$.

- Let $\mathrm{Col}(\omega, \mathrm{On})^M$ denote the partial order whose conditions are finite partial functions $p : \omega \xrightarrow{par} \alpha$ ordered by reverse inclusion.
- Define $\mathrm{Col}_*(\omega, \mathrm{On})^M$ to be the suborder of $\mathrm{Col}(\omega, \mathrm{On})^M$ consisting of all conditions p with $\mathrm{dom}(p) \in \omega$.

Note that all of these partial orders are definable over the corresponding model M.

Lemma

Let M be a countable transitive model of $\mathbf{ZF}^-$.

- *If G is an M-generic filter on $\mathrm{Col}(\omega, \mathrm{On})^M$, then for every ordinal in M there is a surjection from a subset of ω onto that ordinal in $M[G]$.*
- *If G is an M-generic filter on $\mathrm{Col}_*(\omega, \mathrm{On})^M$, then $M = M[G]$.*
- *The model M contains no non-trivial maximal antichain in $\mathrm{Col}(\omega, \mathrm{On})^M$ or $\mathrm{Col}_*(\omega, \mathrm{On})^M$.*
- *If M is a model of $\mathbf{ZFC}$, then M contains no complete suborder of $\mathrm{Col}(\omega, \mathrm{On})^M$ or $\mathrm{Col}_*(\omega, \mathrm{On})^M$.*

Proof.

(*1*) Pick $\lambda \in M \cap \mathrm{On}$. Given $\alpha \in M \cap \mathrm{On}$, define

$$D_\alpha = \{p \in \mathrm{Col}(\omega, \mathrm{On})^M \mid \exists n \in \mathrm{dom}(p)\ p(n) = \alpha\}.$$

Then each D_α is dense and definable over M. This implies that, if G is $\mathrm{Col}(\omega, \mathrm{On})$-generic over M, then for every $\alpha \in M \cap \mathrm{On}$ there is an $n < \omega$ with $\{\langle n, \alpha\rangle\} \in G$. This shows that

$$\sigma = \{\langle \mathsf{op}(\check{n}, \check{\alpha}), \{\langle n, \alpha\rangle\}\rangle \mid \alpha < \lambda,\ n < \omega\}$$

is a name for a surjection from a subset of ω onto λ.
(2) Let σ be a $\mathrm{Col}_*(\omega, \mathrm{On})^M$-name in M. Then $\mathrm{ran}(p) \subseteq \mathrm{rank}(\sigma)$ holds for every condition p in $\mathrm{tc}(\sigma) \cap \mathrm{Col}_*(\omega, \mathrm{On})^M$. If we define

$$D = \{p \in \mathrm{Col}_*(\omega, \mathrm{On})^M \mid \mathrm{rank}(\sigma) \in \mathrm{ran}(p)\},$$

then D is dense and definable over M. If $p \in D \cap G$, then p completely determines σ^G, because p either extends or is incompatible to any condition contained in $\mathrm{tc}(\sigma)$. Hence $\sigma^G \in M$. □

The above computations show that, in contrast to forcing with set-sized partial orders, forcing with dense suborders of class forcing can produce different generic extensions. Note that in our setting, it is still true that generic filters correspond to generic filters on dense suborders.

Corollary

If M is a countable transitive model of $\mathbf{ZF}$, then there are partial orders $\mathbb{P}$ and $\mathbb{Q}$ definable over M such that $\mathbb{Q}$ is a dense suborder of $\mathbb{P}$ and $M = M[G \cap \mathbb{Q}] \subsetneq M[G]$ whenever G is an M-generic filter on $\mathbb{P}$. $\square$

It can be shown that the above partial orders satisfy the forcing theorem.

Friedman Coding

The following class forcing was defined by Sy Friedman.

Definition

Let M be a countable transitive model of $\mathbf{ZF}^-$. Define $\mathbb{F}^M$ to be the partial order whose conditions are triples $p = \langle d_p, e_p, f_p \rangle$ satisfying

- d_p is a finite subset of ω,
- e_p is a binary acyclic relation on d_p,
- f_p is an injective function with $\mathrm{dom}(f_p) \in \{\varnothing, d_p\}$ and $\mathrm{ran}(f_p) \subseteq M$,
- if $\mathrm{dom}(f_p) = d_p$ and $i, j \in d_p$, then we have $i\ e_p\ j$ if and only if $f_p(i) \in f_p(j)$,

and whose ordering is given by

$$p \leq_{\mathbb{F}^M} q \iff d_q \subseteq d_p \ \wedge \ e_p \cap (d_q \times d_q) = e_q \ \wedge \ f_q \subseteq f_p.$$

Again, it is easy to see that $\mathbb{F}^M$ is definable over M.

The following lemma is the key to all density arguments concerned with $\mathbb{F}$.

Lemma

The set of all conditions p in $\mathbb{F}^M$ with $\mathrm{dom}(f_p) = d_p$ is dense.

Given $i, j \in \omega$ with $i \neq j$, define $p_{i,j}$ to be the condition in $\mathbb{F}^M$ with $d_{p_{i,j}} = \{i, j\}$, $e_{p_{i,j}} = \{\langle i, j\rangle\}$ and $f_{p_{i,j}} = \varnothing$.

Lemma

Let be a M countable transitive model of $\mathbf{ZF}^-$, G be an M-generic filter on $\mathbb{F}^M$, $E = \bigcup_{p \in G} e_p$ and $F = \bigcup_{p \in G} f_p$. Set

$$\dot{E} \;=\; \{\langle \mathsf{op}(\check{i}, \check{j}), p_{i,j}\rangle \mid i, j \in \omega,\ i \neq j\} \;\in\; M^{\mathbb{F}^M}.$$

Then $E = \dot{E}^G \in M[G]$ is a binary relation E on ω and F is an isomorphism of the models $\langle \omega, E\rangle$ and $\langle M, \in\rangle$.

A failure of the definability lemma

We will use the properties of the forcing $\mathbb{F}^M$ mentioned above to show that the forcing relation of $\mathbb{F}^M$ is not first-order definable over M.

Let $Fml_1 \subseteq \omega$ denote the set of all Gödel numbers for $\mathcal{L}_\in$-formulas with one free variable.

Definition

A relation $T \subseteq Fml_1 \times M$ is a *first-order truth predicate for* M if

$$\langle \ulcorner \varphi \urcorner, x\rangle \in T \iff \langle M, \in\rangle \models \varphi(x)$$

holds for every $\ulcorner \varphi \urcorner \in Fml_1$ and every $x \in M$.

Let G be an M-generic filter on $\mathbb{F}^M$ and define E and F as above. Then

$$T \ = \ \{\langle \ulcorner \varphi \urcorner, x\rangle \in Fml_1 \times M \mid \langle \omega, E\rangle \models \varphi(F^{-1}(x))\} \subseteq M$$

is a first-order truth predicate for M and, by Tarski's Undefinability Theorem, T cannot be defined over M by first-order formulae. In the following, we will show that a first-order definition of the forcing relation for $\mathbb{F}^M$ would lead to a first-order definition of T.

The principal ingredient for this is the next lemma.

Lemma

For every formula $\varphi(v_0, \ldots, v_{k-1})$ *and for all sequences* $\vec{n} = n_0, \ldots, n_{k-1}$ *of natural numbers, there are* $\mathbb{F}^M$*-names*

$$\mu_{\ulcorner\varphi\urcorner}(\vec{n}), \nu_{\ulcorner\varphi\urcorner}(\vec{n}), \pi_{\ulcorner\varphi\urcorner}(\vec{n}), \sigma_{\ulcorner\varphi\urcorner}(\vec{n}) \in \mathrm{L}_{\omega\cdot 2} \subseteq M$$

such that the following statements hold, whenever G *is an* M*-generic filter on* $\mathbb{F}^M$ *and* $E = \dot{E}^G$ *is the relation on* ω *induced by* G*.*

- $\langle\omega, E\rangle \models \varphi(v_0, \ldots, v_{k-1})$ *if and only if* $\sigma_{\ulcorner\varphi\urcorner}(\vec{n})^G = \mu_{\ulcorner\varphi\urcorner}(\vec{n})^G$*.*
- $\langle\omega, E\rangle \models \neg\varphi(v_0, \ldots, v_{k-1})$ *if and only if* $\sigma_{\ulcorner\varphi\urcorner}(\vec{n})^G = \nu_{\ulcorner\varphi\urcorner}(\vec{n})^G$*.*
- $\langle\omega, E\rangle \models \varphi(v_0, \ldots, v_{k-1})$ *if and only if* $\pi_{\ulcorner\varphi\urcorner}(\vec{n})^G \in \sigma_{\ulcorner\varphi\urcorner}(\vec{n})^G$*.*

Moreover, the map $[\ulcorner\varphi\urcorner \mapsto \langle\mu_{\ulcorner\varphi\urcorner}(\cdot), \nu_{\ulcorner\varphi\urcorner}(\cdot), \pi_{\ulcorner\varphi\urcorner}(\cdot), \sigma_{\ulcorner\varphi\urcorner}(\cdot)\rangle]$ *is an element of* M*.*

Theorem

Let M be a countable transitive model of $\mathbf{ZF}^-$. Then the partial order $\mathbb{F}^M$ does not satisfy the forcing theorem for atomic formulae over M.

Proof.

Assume, towards a contradiction, that the set $\{\langle p, \sigma, \tau\rangle \mid p \Vdash^M_{\mathbb{F}^M} \text{“}\sigma = \tau\text{”}\}$ is definable over M. For $x \in M$, let

$$p_x = \langle\{0\}, \varnothing, \{\langle 0, x\rangle\}\rangle$$

be the condition forcing that the induced isomorphism between ω and M maps 0 to x. Then the set

$$T = \{\langle \ulcorner\varphi\urcorner, x\rangle \in Fml_1 \times M \mid p_x \Vdash^M_{\mathbb{F}^M} \text{“}\sigma_{\ulcorner\varphi\urcorner}(0) = \mu_{\ulcorner\varphi\urcorner}(0)\text{”}\}$$

is also definable over M and this set is a first-order truth predcate for M. This contradictions Tarksi's theorem on the undefinability of truth. □

Theorem

Let M be a countable transitive model of $\mathbf{ZF}^-$ that is pointwise definable. Then the set

$$A \ = \ \{\langle\sigma,\tau\rangle \in M^{\mathbb{F}^M} \times M^{\mathbb{F}^M} \mid \sigma,\tau \in \mathrm{L}_{\omega\cdot 2},\ \mathbb{1} \Vdash^M_{\mathbb{F}^M} \text{“}\sigma = \tau\text{”}\}$$

is not an element of M.

Sketch of the proof.

Assume that $A \in M$. Then the set

$$T \ = \ \{\ulcorner\varphi\urcorner \in Fml_0 \mid \langle\sigma_{\ulcorner\varphi\urcorner}, \mu_{\ulcorner\sigma\urcorner}\rangle \in A\} \ = \ \{\ulcorner\varphi\urcorner \in Fml_0 \mid M \vDash \varphi\}$$

is also an element of M. Define O to be the set of all $\ulcorner\varphi(v)\urcorner \in Fml_1$ with $\ulcorner\exists!\alpha \in \mathrm{On}\ \varphi(\alpha)\urcorner \in T$. Then $O \in M$ and, by our assumption, the relation

$$\ulcorner\varphi(v)\urcorner \prec \ulcorner\psi(v)\urcorner \iff \ulcorner\exists\alpha,\beta \in \mathrm{On}\ [\alpha < \beta \wedge \varphi(\alpha) \wedge \psi(\beta)]\urcorner \in T$$

is M and well-orders O in order-type $\mathrm{On} \cap M$, a contradiction. $\square$

A failure of the truth lemma

We will sketch the proof of the following result.

Theorem

Assume that there is an inaccessible cardinal. Then there is a countable transitive model M of $\mathbf{ZFC}$, a partial order $\mathbb{P}$ and an $\mathcal{L}_\in$-formula φ such that $\mathbb{P}$ is definable over M and $\mathbb{P}$ does not satisfy the truth lemma for φ over M.

Our arguments use the fact that the above assumption yields a countable transitive model M of $\mathbf{ZFC} + \mathrm{V} = \mathrm{L}$ that is the first-order part of a countable model of *Kelley-Morse set theory* KM (second order set theory with full class comprehension).

We will use an argument of Hamkins, Linetsky and Reitz to show that there is a class forcing $\mathbb{C}$ with the property in a $\mathbb{C}$-generic extensions of the model M all ordinals are definable by formulas without parameters while in other extensions this statement fails. We will then use Friedman's forcing $\mathbb{F}^M$ to make this statement first-order expressible.

Define $\mathbb{C}$ to be the class forcing whose conditions are pairs $p = \langle s_p, \vec{q}_p \rangle$ such that $s_p : \alpha \longrightarrow 2$ for some $\alpha \in \mathrm{On}$ and $\vec{q}_p$ is in the Easton support product $\prod_{s_p(\alpha)=1} \mathrm{Add}(\omega_{\alpha\cdot 2+1}, \omega_{\alpha\cdot 2+3})$ and whose ordering is the canonical one.

Lemma

If M is a countable transitive model of $\mathbf{ZFC} + \mathrm{V} = \mathrm{L}$, then there is an M-generic filter G on $\mathbb{C}^M$ such that every ordinal in $M[G]$ is definable in $M[G]$ by a formula without parameters.

Sketch of the proof.

Let $\langle D_n \mid n < \omega \rangle$ enumerate all dense subsets of $\mathbb{C}^M$ that are definable in M and let $\langle \alpha_n \mid n < \omega \rangle$ enumerate $M \cap \mathrm{On}$. Note that for every $n < \omega$, there is $\beta_n \in M \cap \mathrm{On}$ such that D_n is defined in M by a formula with parameter β_n. Construct a descending sequence $\langle p_n \mid n < \omega \rangle$ in $\mathbb{C}^M$ with $p_0 = \mathbb{1}_{\mathbb{C}^M}$ in the following way: Assume p_n is already constructed. Set $t = s_{p_n} {}^\frown \langle 0 \rangle^{\alpha_n} {}^\frown \langle 1 \rangle {}^\frown \langle 0 \rangle^{\beta_n} {}^\frown \langle 1 \rangle$ and let p_{n+1} be $<_{\mathrm{L}}$-minimal in D_n with $p_{n+1} \leq_{\mathbb{C}^M} \langle t, \vec{q} \rangle$. If G is an M-generic filter on $\mathbb{C}^M$, then $S = \bigcup_{p \in G} s_p$ is lightface definable in $M[G]$ and this allows us to inductively show that the elements α_n, β_n and p_n are lightface definable in $M[G]$ for all $n < \omega$. $\square$

The class forcing $\mathbb{C}$ is *tame*, i.e. forcing with $\mathbb{C}$ preserves $\mathbf{ZFC}$ and $\mathbb{C}$ satisfies the forcing theorem.

Moreover, if M is the first-order part of a countable model of *Kelley-Morse set theory* KM (second order set theory with full class comprehension) and G is a filter on $\mathbb{C}^M$ that hits all second order objects that are dense in $\mathbb{C}^M$, then $M[G]$ is the first-order part of a KM model by a result of Antos-Kuby. In particular, there are ordinals in the extension $M[G]$ that are not definable by a first-order formula without parameters, because $M[G]$ contains a first-order truth predicate.

Finally, we can construct a two-step forcing iteration $\mathbb{C} * \dot{\mathbb{F}}$ such that an M-generic filter $G * H$ on this forcing corresponds to an M-generic filter G on $\mathbb{C}^M$ and an $M[G]$-generic filter H on $\mathbb{F}^{M[G]}$.

Sketch of the proof.

Let M be a countable transitive model of $\mathbf{ZFC} + \mathrm{V} = \mathrm{L}$ that is the first-order part of a countable KM model. Let $\dot{E}$ denote the canonical name for the relation on ω added by $\mathbb{C} * \dot{\mathbb{F}}$ (i.e. the models $\langle M[G], \in\rangle$ and $\langle \omega, \dot{E}^{G*H}\rangle$ are isomorphic whenever $G * H$ is M-generic on $\mathbb{C} * \dot{\mathbb{F}}$).

By the above remark, there is there is an M-generic filter $G_0 * H_0$ on $\mathbb{C} * \dot{\mathbb{F}}$ such that in the model $\langle \omega, \dot{E}^{G_0 * H_0}\rangle$, every ordinal is lightface definable and, for every condition $\langle p, \dot{q}\rangle$ in $\mathbb{C} * \dot{\mathbb{F}}$, there is an M-generic filter $G * H$ such that $\langle \omega, \dot{E}^{G*H}\rangle$ has ordinals that are not lightface definable.

By a variation of the above construction of $\mathbb{F}^M$-names, we can show that the statement "*in* $\langle \omega, \dot{E}^{G*H}\rangle$*, every ordinal is lightface definable*" can be expressed by a uniform first-order statement in $(\mathbb{C} * \dot{\mathbb{F}}$-generic extension and the above remarks show that the truth lemma fails for this statement. □

Definable boolean completions

In the following, we sketch the proof of the following positive result.

Theorem

Let M be a countable transitive model of $\mathbf{ZF}^-$ and $\mathbb{P}$ be a partial order that is a class in M. If either the power set axiom holds in M of there is a well-ordering of M that is definable in M, then the following statements are equivalent.

- $\mathbb{P}$ *satisfies the forcing theorem for all $\mathcal{L}_\in$-formulas over M.*
- $\mathbb{P}$ *has a boolean completion in M.*

The proof of this result makes use of the infinitary quantifier-free language $\mathcal{L}_{\mathrm{On},0}$ whose atomic formulas are of the form "$v_0 \in v_1$", "$v_0 = v_1$" and "$v \in G$".

Lemma

Assume that M satifies the above assumptions and $\mathbb{P}$ is a class forcing. If $\mathbb{P}$ satisfies the forcing theorem over M, then the forcing relation for $\mathcal{L}_{\mathrm{On},0}$-formulas is uniformly definable over M.

Sketch of the implication "$\Rightarrow$" in the theorem.

Assume $\mathbb{P}$ satisfies the forcing theorem over M. By Lemma, the forcing relation for $\mathcal{L}_{\mathrm{On},0}$-formulas is uniformly definable over M.

We define a Boolean algebra $\mathbb{B}$ in the following way: Let $\bar{\mathbb{B}}$ consist of the infinitary formulae in the forcing language of $\mathbb{P}$ with the atomic formulae $\sigma \in \tau, \sigma = \tau$ and $\check{p} \in \dot{G}$ for $\sigma, \tau \in M^{\mathbb{P}}$ and $p \in \mathbb{P}$. Suprema and infima are just set-sized disjunctions and conjunctions of formulae and complements are just negations. In order to obtain a complete boolean algebra from $\bar{\mathbb{B}}$, consider the equivalence relation

$$\varphi \approx \psi \iff \mathbb{1} \Vdash_{\mathbb{P}}^{M} \varphi \leftrightarrow \psi.$$

By our assumptions, there is a definable boolean algebra $\mathbb{B}$ and a surjective map $\pi : \bar{\mathbb{B}} \longrightarrow \mathbb{B}$ such that $\pi(\varphi) = \pi(\psi) \Leftrightarrow \varphi \approx \psi$. Now we can extend the boolean operations onto $\mathbb{B}$ in the obvious way and define $0_{\mathbb{B}} = \pi(0 \neq 0)$ and $\mathbb{1}_{\mathbb{B}} = \pi(0 = 0)$. Clearly, $\mathbb{B}$ is a complete boolean algebra. We identify $p \in \mathbb{P}$ with the formula $\pi(\check{p} \in \dot{G})$ thus obtaining that the dense embedding $\mathbb{P} \longrightarrow \mathbb{B}$ is definable. $\square$

Sketch of the implication "$\Leftarrow$" in the theorem.

Conversely, assume that $\mathbb{P}$ has a boolean completion $\mathbb{B}(\mathbb{P})$. Then we can assign truth values $[\![\varphi]\!] \in \mathbb{B}(\mathbb{P})$ to $\mathcal{L}_{\mathrm{On},0}$-statements in a canonical way and we get

$$M[G] \vDash \varphi \iff \exists p \in G\ (p \leq_{\mathbb{B}(\mathbb{P})} [\![\varphi]\!]),$$

whenever G is $\mathbb{B}(\mathbb{P})$-generic over M.

By a previous result, this shows that $\mathbb{B}(\mathbb{P})$ satisfies the forcing theorem over M. Moreover, a variation of the proof of the above result shows that we also get a definable forcing relation for first-order formulas using class names as relations. Hence we can talk about the intermediate $\mathbb{P}$-generic extension in the forcing language of $\mathbb{B}(\mathbb{P})$ and we can conclude that the forcing theorem for $\mathbb{P}$ holds over M. $\square$

Open questions

Question

Is there always a class forcing whose forcing relation is not amenable to the ground model?

Question

Is there always a class forcing that does not satisfy the truth lemma?

Question

Does Friedman's forcing $\mathbb{F}^M$ always satisfy the truth lemma?

Thank you for listening!