Automorphism groups and set theory

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If G is an infinite group, then the group $\mathrm{Aut}(G)$ of all automorphisms of G can be *a complicated object* from the set-theoretical point of view.

In this talk, I want to introduce a class of interesting groups with the property that basic statements about the algebraic structure of the corresponding automorphism groups are not decided by the standard axioms of set theory.

Definition

A collection $\mathcal{U}$ of sets of natural numbers is a *non-principal ultrafilter over* $\mathbb{N}$ if the following statemtents hold.

- $\mathcal{U}$ contains no finite sets.
- If $X \subseteq \mathbb{N}$, then either $X \in \mathcal{U}$ or $\mathbb{N} \setminus X \in \mathcal{U}$.
- If $X, Y \in \mathcal{U}$, then $X \cap Y \in \mathcal{U}$.
- If $X \in \mathcal{U}$ and $X \subseteq Y \subseteq \mathbb{N}$, then $Y \in \mathcal{U}$.

If $\mathcal{U}$ is a non-principal ultrafilter over $\mathbb{N}$, then we define

$$N_{\mathcal{U}} = \left\{ (\pi_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} \mathrm{Sym}(n) \;\middle|\; \{ n \in \mathbb{N} \mid \pi_n = \mathrm{id}_n \} \in \mathcal{U} \right\}.$$

It is easy to see that $N_{\mathcal{U}}$ is a normal subgroup of $\prod_{n \in \mathbb{N}} \mathrm{Sym}(n)$. The *ultraproduct of all finite symmetric groups with respect to* $\mathcal{U}$ is the group

$$G_{\mathcal{U}} = \prod_{n \in \mathbb{N}} \mathrm{Sym}(n) \Big/ N_{\mathcal{U}}.$$

Proposition

Let $\mathcal{U}$ *be a non-principal ultrafilter over* $\mathbb{N}$*. If the Continuum Hypothesis holds, then* $G_{\mathcal{U}}$ *is a saturated structure of cardinality* $\aleph_1$ *and* $\mathrm{Aut}(G_{\mathcal{U}})$ *has cardinality* $2^{\aleph_1} > \aleph_1$*.*

Corollary

Let $\mathcal{U}$ *be a non-principal ultrafilter over* $\mathbb{N}$*. If the Continuum Hypothesis holds, then* $G_{\mathcal{U}}$ *has non-inner automorphisms, i.e.* $G_{\mathcal{U}}$ *has an automorphism that is not induced by conjugation with an element of* $G_{\mathcal{U}}$*.*

If $n \neq 6$, then every automorphism of $\mathrm{Sym}(n)$ is inner; and consequently, it appears to be difficult to exhibit *explicit* non-inner automorphisms of groups of the form $G_{\mathcal{U}}$.

The following theorem shows that there is a good reason for these difficulties.

Theorem (P.L. & Simon Thomas)

It is consistent with the axioms of set theory that there is a non-principal ultrafilter $\mathcal{U}$ over $\mathbb{N}$ such that every automorphism of $G_{\mathcal{U}}$ is inner.

I want to conclude this talk with a result showing that groups of the form $G_{\mathcal{U}}$ can have non-inner automorphisms if the Continuum Hypothesis fails. Remember that *Cantor space* is the product space of countably infinitely many copies of the discrete space with two elements. The statement

> "*the intersection of less than continuum-many dense open subsets of Cantor space is non-empty*"

is independent from the axioms of set theory.

Theorem (P.L. & Saharon Shelah)

Assume that the intersection of less than continuum-many dense open subsets of Cantor space is non-empty. Then there is a non-principal ultrafilter $\mathcal{U}$ over $\mathbb{N}$ such that $G_{\mathcal{U}}$ has a non-inner automorphism.