# How tall is the automorphism tower of a centerless group?

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If G is an infinite group, then the group $\mathrm{Aut(G)}$ of all automorphisms of $G$ can be a very complex object; not only from the point of view of algebra, but also in a set-theoretical sense.

I will introduce an algebraic construction that illustrates this phenomenon. This construction is called the *automorphism tower*.

Let $G$ be a group. If $g$ is an element of $G$, then the map

$$\iota_g : G \longrightarrow G;\ h \longmapsto g \circ h \circ g^{-1}.$$

is an automorphism of $G$ and we call $\iota_g$ the *inner automorphism of $G$* corresponding to $g$. We let $\mathrm{Inn(G)}$ denote the group of all inner automorphisms of $G$.

The map

$$\iota_G : G \longrightarrow \mathrm{Aut(G)};\ g \longmapsto \iota_g$$

is a homomorphism of groups with $\ker(\iota_G) = \mathrm{C}(G)$.

Given $g \in G$ and $\pi \in \mathrm{Aut(G)}$, an easy computation shows that

$$\iota_{\pi(g)} = \pi \circ \iota_g \circ \pi^{-1}$$

holds and this implies that $\mathrm{Inn(G)}$ is a normal subgroup of $\mathrm{Aut(G)}$.

If $G$ is a group with trivial center, then $\iota_G$ is an embedding of groups and the above equality implies that

$$\mathrm{C}_{\mathrm{Aut(G)}}(\mathrm{Inn(G)}) = \{\mathrm{id_G}\}$$

holds. In particular, $\mathrm{Aut(G)}$ is a group with trivial center in this case.

By iterating this process, we construct the *automorphism tower of a centerless group* $G$.

## Definition

Let $G$ be a group with trivial center. We call a sequence $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ of groups *an automorphism tower of* $G$ if the following statements hold.

- $G = G_0$.
- If $\alpha \in \mathrm{On}$, then $G_\alpha$ is a normal subgroup of $G_{\alpha+1}$ and the induced homomorphism

$$\varphi_\alpha : G_{\alpha+1} \longrightarrow \mathrm{Aut(G_\alpha)};\ g \mapsto \iota_g \restriction G_\alpha$$

  is an isomorphism.
- If $\alpha \in \mathrm{Lim}$, then $G_\alpha = \bigcup\{G_\beta \mid \beta < \alpha\}$.

In this definition, we replaced $\mathrm{Aut(G_\alpha)}$ by an isomorphic copy $G_{\alpha+1}$ that contains $G_\alpha$ as a normal subgroup. This allows us to take unions at limit stages. Without this isomorphic correction, we would have to take direct limits at limit stages.

By induction, we can construct such a tower for each centerless group and it is easy to show that each group $G_\alpha$ in such a tower is uniquely determined up to an isomorphism which is the identity on $G$. We can therefore speak of *the* $\alpha$-th group $G_\alpha$ in the automorphism tower of a centerless group $G$.

It is natural to ask whether the automorphism tower of every centerless group eventually *terminates* in the sense that there is an ordinal $\alpha$ with $G_\alpha = G_{\alpha+1}$ and therefore $G_\alpha = G_\beta$ for all $\beta \geq \alpha$.

A classical result due to Helmut Wielandt shows that the automorphism tower of every finite centerless group terminates.

**Theorem (Wielandt, 1939)**

*If $G$ is a finite group with trivial center, then there is an $n < \omega$ with $G_n = G_{n+1}$.*

Does this result generalize to infinite centerless groups?

## Example

Consider the *infinite dihedral group*

$$D_\infty \;=\; \langle a, b \mid a^2 = b^2 = \mathbb{1} \rangle.$$

Then

- $\mathrm{Inn}(\mathrm{D}_\infty) \subsetneq \mathrm{Aut}(\mathrm{D}_\infty)$.
- $\mathrm{Aut}(\mathrm{D}_\infty) \cong \mathrm{D}_\infty$.

In particular, the automorphism tower of $D_\infty$ does not terminate after finitely many steps.

## Theorem (Hulse, 1970)

*The automorphism tower of $D_\infty$ terminates after exactly $\omega + 1$ steps.*

Simon Thomas showed that the automorphism tower of every centerless group eventually terminates by proving the following result. An application of Fodor's Lemma (and hence of the *Axiom of Choice*) lies at the heart of this argument.

**Theorem (Thomas, 1985 & 1998)**

*If* $G$ *is an infinite centerless group of cardinality* $\kappa$*, then there is an* $\alpha < (2^\kappa)^+$ *with* $G_\alpha = G_{\alpha+1}$*.*

The above result allows us to make the following definitions.

## Definition

Given a centerless group $G$, we let $\tau(G)$ denote the least ordinal $\alpha$ with $G_\alpha = G_{\alpha+1}$. We call this ordinal the *height of the automorphism tower of* $G$.

If $\kappa$ is an infinite cardinal, then we define

$$\tau_\kappa = \mathrm{lub}\{\tau(G) \mid G \textit{ is a centerless group of cardinality } \kappa\}.$$

A result of Simon Thomas shows that for every infinite cardinal $\kappa$ and every ordinal $\alpha < \kappa^+$ there is a group $G$ of cardinality $\kappa$ such that $\tau(G) = \alpha$. Since there are only $2^\kappa$-many centerless groups of cardinality $\kappa$ and $(2^\kappa)^+$ is a regular cardinal, we can combine the above results to see that

$$\kappa^+ \leq \tau_\kappa < (2^\kappa)^+$$

holds for every infinite cardinal $\kappa$.

The following open problem is the motivation for my work on this topic.

**Problem (The automorphism tower problem)**

*Find a model $\mathcal{M}$ of $\mathrm{ZFC}$ and an infinite cardinal $\kappa$ in $\mathcal{M}$ such that it is possible to compute the exact value of $\tau_\kappa$ in $\mathcal{M}$.*

# Upper bounds for the heights of automorphism towers

Remember that

$$\kappa^+ \;\leq\; \tau_\kappa \;<\; (2^\kappa)^+$$

holds for every infinite cardinal $\kappa$.

The aim of my work was to find better upper bounds for $\tau_\kappa$ that are uniformly definable in parameter $\kappa$.

The following consistency result shows that there is no better estimate for $\tau_\kappa$ by cardinals.

## Theorem (Just, Shelah & Thomas, 1999)

*Assume that the* $(\mathrm{GCH})$ *holds in the ground model* $\mathrm{V}$. *Let* $\kappa$ *and* $\nu$ *be cardinals with*

$$\omega < \kappa = \mathrm{cof}(\kappa) < \mathrm{cof}(\nu).$$

*If* $\alpha < \nu^+$, *then there is a forcing extension* $\mathrm{V}[H]$ *of* $\mathrm{V}$ *with the following properties.*

- *Cardinalities and cofinalities are preserved.*
- $(2^\kappa)^{\mathrm{V}[H]} = \nu$.
- *There is a centerless group* $G \in \mathrm{V}[H]$ *such that* $\tau(G) = \alpha$ *holds in* $\mathrm{V}[H]$.

## Corollary

*It is consistent with the axioms of set theory that there is a cardinal* $\kappa$ *with* $\tau_\kappa > 2^\kappa$.

Does the statement "$\tau_\kappa \geq 2^\kappa$" follow from the axioms of $\mathrm{ZFC}$?

**Theorem (Thomas, 1998)**

*It is consistent with the axioms of* $\mathrm{ZFC}$ *that* $\tau_\kappa < 2^\kappa$ *holds for every infinite regular cardinal* $\kappa$.

These results show that a better upper bound for $\tau_\kappa$ should be a uniformly definable *ordinal* in the interval $[\kappa^+, (2^\kappa)^+]$ that is consistently smaller than $2^\kappa$ for regular $\kappa$.

The work of Itay Kaplan and Saharon Shelah provides an example of such a bound.

## Definition

Given a set $X$, we let $\mathrm{L(X)}$ denote the least inner model containing $X$ and we define

$$\theta_X = \mathrm{lub}\{\alpha \in \mathrm{On} \mid (\exists f \in \mathrm{L}(X))\ f : X \longrightarrow \alpha \textit{ is a surjection}\}.$$

## Theorem (Kaplan & Shelah, 2009)

*If $\kappa$ is an infinite cardinal, then $\tau_\kappa < \theta_{\mathcal{P}(\kappa)}$.*

By refining methods developed in the proof of this result, it is possible to find a better upper bound for $\tau_\kappa$ with the help of *abstract recursion theory*, i.e. the *theory of admissible sets*.

To state this result, we need to introduce some concepts from this theory.

## Definition

A set $\mathbb{A}$ is *admissible* if it has the following properties.

- $\mathbb{A}$ is nonempty, transitive and closed under pairing and union.
- The structure $\langle \mathbb{A}, \in \rangle$ satisfies $\Delta_0$-*Separation*.
- The structure $\langle \mathbb{A}, \in \rangle$ satisfies $\Delta_0$-*Collection*.

Given a set $x$, an ordinal $\alpha$ is $x$-admissible if there is an admissible set $\mathbb{A}$ with $x \in \mathbb{A}$ and $\alpha = \mathbb{A} \cap \mathrm{On}$.

Admissibility has two important consequences:

- The $\boldsymbol{\Sigma}_1$-Recursion Theorem.
- The $\boldsymbol{\Sigma}_1$-Boundedness Theorem.

We are now ready to state the result.

**Theorem (L.)**

*Let $\kappa$ be an infinite cardinal and $\alpha$ be $\mathcal{P}(\kappa)$-admissible. Then either $\tau_\kappa = \alpha + 1$ or $\tau_\kappa < \alpha$.*

Since there are many $\mathcal{P}(\kappa)$-admissible ordinals below $\theta_{\mathcal{P}(\kappa)}$, this result improves the Kaplan-Shelah-bound.

The upper bound for $\tau_\omega$ produced by this theorem was independently derived by Howard Becker using methods from descriptive set theory.

In the following, I outline the ideas behind the proof of this statement.

*Let $\kappa$ be an infinite cardinal, $\mathbb{A}$ be an admissible set with $\mathcal{P}(\kappa) \in \mathbb{A}$, $\alpha = \mathbb{A} \cap \mathrm{On}$, $G$ be a centerless group of cardinality $\kappa$ and $g \in G_{\alpha+1}$.*

- *Show that the function $[\beta \mapsto G_\beta]$ restricted to $\alpha$ is $\mathbf{\Sigma}_1$-definable in $\langle \mathbb{A}, \in \rangle$ with the help of the Recursion Theorem and a representation of the automorphism tower of $G$ as an inductive definition on a structure contained in $\mathbb{A}$.*
- *Show that $\iota_g$ is $\mathbf{\Sigma}_1$-definable in $\langle \mathbb{A}, \in \rangle$ and use the Boundedness Theorem to show that for every $\beta < \alpha$ there is a $\gamma < \alpha$ with $\iota_g$"$G_\beta \subseteq G_\gamma$.*
- *Use the above to find a $\beta < \alpha$ with $\iota_g \restriction G_\beta \in \mathrm{Aut(G_\beta)}$. Since the centralizer of $G_0$ in $G_{\alpha+1}$ is trivial, we can conclude $g \in G_{\beta+1} \subseteq G_\alpha$.*

*This argument shows that $\tau_\kappa \leq \alpha + 1$.*

- *Assume $\tau_\kappa \leq \alpha$. Then we can use the Recursion Theorem to show that the function that sends a centerless group with domain $\kappa$ to the height of its automorphism tower is $\mathbf{\Sigma}_1$-definable in $\langle \mathbb{A}, \in \rangle$. By the Boundedness Theorem, we have $\tau_\kappa < \alpha$.*

# Changing the heights of automorphism towers

One of the reasons why it is so difficult to compute the value of $\tau_\kappa$ is that there are groups whose automorphism tower heights highly depend on the model of set theory in which they are computed.

**Theorem (Thomas, 1998)**

- *There is a centerless group* $G$ *with* $\tau(G) = 0$ *such that* $\tau(G) \geq 1$ *holds in a cardinality and cofinality preserving forcing extension of the ground model.*
- *There is a centerless group* $H$ *with* $\tau(H) = 2$ *such that* $\tau(H) = 1$ *holds in every forcing extension of the ground model that adds a new real number.*

The following result suggests that there is no nontrivial correlation between the heights of automorphism towers of a centerless group computed in different models of set theory.

### Theorem (Hamkins & Thomas, 2000)

*It is consistent with the axioms of set theory that for every infinite cardinal $\kappa$ and every ordinal $\alpha < \kappa$, there exists a centerless group $G$ with the following properties.*

- $\tau(G) = \alpha$.
- *If $0 < \beta < \kappa$, then $\tau(G) = \beta$ holds in a cardinality and cofinality preserving forcing extension of the ground model.*

Gunter Fuchs and I strengthened this result by construction groups whose automorphism tower can be iteratively changed by forcing.

### Theorem (Fuchs & L., 2012)

*It is consistent with the axioms of set theory that for every infinite cardinal $\kappa$ there is a centerless group $G$ with $\tau(G) = 0$ and the property that for every function $s : \kappa \longrightarrow (\kappa \setminus \{0\})$ there is a sequence $\langle \mathrm{V}[H_\alpha] \mid 0 < \alpha < \kappa \rangle$ of cardinality and cofinality preserving forcing extension of the ground model $\mathrm{V}$ such that the following statements hold.*

- *If $0 < \alpha < \beta < \kappa$, then $\mathrm{V}[H_\beta]$ is a forcing extension of $\mathrm{V}[H_\alpha]$.*
- *If $\alpha < \kappa$, then $\tau(G) = s(\alpha)$ holds in $\mathrm{V}[H_{\alpha+1}]$.*
- *If $0 < \alpha < \kappa$ is a limit ordinal, then $\tau(G) = 1$ holds in $\mathrm{V}[H_\alpha]$.*

In another direction, it is also possible to construct models of set theory that contain groups with *unbounded potential automorphism tower height*.

**Theorem (L., 2012)**

*It is consistent with the axioms of set theory that there exists a centerless group* $G$ *with the following property: if* $\alpha \in \mathrm{On}$, *then* $\tau(G) \geq \alpha$ *holds in a cardinality and cofinality preserving forcing extension of the ground model.*

All groups appearing in the above non-absoluteness results are uncountable.

By using a technique from the theory of *Polish groups* called *automatic continuity*, it is possible to show that this assumption is necessary.

**Theorem (L., 2012)**

*Let $\mathcal{M}$ be a transitive model of $\mathrm{ZFC}$ and $G \in \mathcal{M}$ be a centerless group that is countable in $\mathcal{M}$. If $\tau(G) > 1$ holds in $\mathcal{M}$, then $\tau(G) > 1$ holds.*

# Thank you for listening!