The influence of closed maximality principles on generalized Baire spaces

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Σ^1_1-subsets of generalized Baire spaces

Let κ be an infinite cardinal, ${}^{\kappa}\kappa$ be the set of all functions $f : \kappa \longrightarrow \kappa$ and ${}^{<\kappa}\kappa$ be the set of all functions f with $\mathrm{dom}(f) \in \kappa$ and $\mathrm{ran}(f) \subseteq \kappa$.

The *generalized Baire space* of κ is the set ${}^{\kappa}\kappa$ equipped with the topology whose basic open sets are of the form

$$U_s = \{f \in {}^{\kappa}\kappa \mid s \subseteq f\}$$

for some $s \in {}^{<\kappa}\kappa$. Note that closed sets in this topology are of the form

$$[T] = \{f \in {}^{\kappa}\kappa \mid (\forall \alpha < \kappa)\ f \upharpoonright \alpha \in T\}$$

for some subtree T of ${}^{<\kappa}\kappa$.

We call a subset of $({}^{\kappa}\kappa)^n$ a $\mathbf{\Sigma}^1_1$-subset if it is the projection of a closed subset of $({}^{\kappa}\kappa)^{n+1}$. A subset is a $\mathbf{\Pi}^1_1$-subset if it is the compliment of a $\mathbf{\Sigma}^1_1$-subset and it is a $\mathbf{\Delta}^1_1$-subset if it is both a $\mathbf{\Sigma}^1_1$- and a $\mathbf{\Pi}^1_1$-subset.

We want to study the generalized Baire spaces of uncountable regular cardinals κ with $\kappa = \kappa^{<\kappa}$ and the $\mathbf{\Sigma}^1_1$-subsets of these spaces. The following proposition shows that this class is both interesting and rich.

Proposition

Let κ be an uncountable regular cardinal with $\kappa = \kappa^{<\kappa}$. A subset A of ${}^{\kappa}\kappa$ is a $\mathbf{\Sigma}^1_1$-subset if and only if it is definable over the structure $\langle \mathrm{H}(\kappa^+), \in \rangle$ by a Σ_1-formula with parameters.

It is a well-known phenomenon that many basic and interesting questions about $\mathbf{\Sigma}^1_1$- and $\mathbf{\Pi}^1_1$-subsets of generalized Baire spaces of regular uncountable cardinals κ with $\kappa = \kappa^{<\kappa}$ are independent from the axioms of set theory plus large cardinal axioms. We will discuss three examples of such questions.

Complexity of the λ-club-filter

Given an uncountable regular cardinal κ and an infinite regular cardinal $\lambda < \kappa$, we define $\mathrm{Club}(S^\kappa_\lambda)$ to be the set of all characteristic functions of subsets $X \subseteq \kappa$ containing a λ-club subset of κ, i.e. a subset C that is unbounded in κ and contains all of its limit points of cofinality λ.

It is easy to see that $\mathrm{Club}(S^\kappa_\lambda)$ is a $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa$. But the question whether this set can be a $\mathbf{\Pi}^1_1$-subset turns out to be independent from the axioms of $\mathbf{ZFC}$ plus large cardinal axioms.

Theorem (Mekler/Shelah, Hyttinen/Rautila, Friedman/Hyttinen/Kulikov)

Assume (GCH). *Let* ν *be an infinite regular cardinal and* $\kappa = \nu^+$. *Then there is a partial order* $\mathbb{P}$ *such that forcing with* $\mathbb{P}$ *preserves cardinalities and cofinalities and* $\mathrm{Club}(S^\kappa_\nu)$ *is a* $\mathbf{\Delta}^1_1$*-subset of* ${}^\kappa\kappa$ *in every* $\mathbb{P}$*-generic extension of the ground model.*

Theorem (Friedman/Hyttinen/Kulikov)

If κ *is a regular uncountable cardinal with* $\kappa = \kappa^{<\kappa}$, $\lambda < \kappa$ *is a regular cardinal and* G *is* $\mathrm{Add}(\kappa, \kappa^+)$*-generic over* V, *then* $\mathrm{Club}(S^\kappa_\lambda)$ *is not a* $\mathbf{\Delta}^1_1$*-subset of* ${}^\kappa\kappa$ *in* $\mathrm{V}[G]$.

Lengths of Σ^1_1-definable well-orders

Definition

Given a cardinal κ, we call a well-order $\langle A, \prec \rangle$ a $\mathbf{\Sigma}^1_1$-*well-ordering of a subset of* ${}^\kappa\kappa$ if "$\prec$" is a $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa \times {}^\kappa\kappa$.

We are interested in the relation between 2^κ and the least upper bound of the order-types of such well-orders.

The following results show that the axioms of $\mathbf{ZFC}$ plus large cardinal axioms prove only trivial statements about this relation in the case of uncountable regular cardinals κ with $\kappa = \kappa^{<\kappa}$.

Theorem (L.)

Let κ be a regular uncountable cardinal with $\kappa = \kappa^{<\kappa}$. For every subset A of ${}^\kappa\kappa$, there is a partial order $\mathbb{P}$ that satisfies the following statements.

- *$\mathbb{P}$ is $<\kappa$-closed, satisfies the κ^+-chain condition and has cardinality at most 2^κ.*
- *If G is $\mathbb{P}$-generic over V, then A is a $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa$ in $\mathrm{V}[G]$.*

Theorem (L.)

Assume that κ is a regular uncountable cardinal with $\kappa = \kappa^{<\kappa}$, $\nu > \kappa$ is a cardinal and G is $\mathrm{Add}(\kappa, \nu)$-generic over V. If $\langle A, \lessdot \rangle$ is a $\mathbf{\Sigma}^1_1$-well-ordering of a subset of ${}^\kappa\kappa$ in $\mathrm{V}[G]$, then $A \neq ({}^\kappa\kappa)^{\mathrm{V}[G]}$ and the order-type of $\langle A, \lessdot \rangle$ has cardinality at most $(2^\kappa)^{\mathrm{V}}$ in $\mathrm{V}[G]$.

The bounding and the dominating number of $\langle \mathcal{TO}_\kappa, \leq \rangle$

Given an infinite cardinal κ, we let $\mathcal{T}_\kappa$ denote the class of all trees of cardinality and height κ and $\mathcal{TO}_\kappa$ denote the class of all trees in $\mathcal{T}_\kappa$ without a branch of length κ. We can easily identify $\mathcal{TO}_\kappa$ with a $\mathbf{\Pi}^1_1$-subset of ${}^\kappa\kappa$.

Let $\mathbb{T}_0$ and $\mathbb{T}_1$ be elements of $\mathcal{T}_\kappa$. We say that $\mathbb{T}_0$ *is order-preserving embeddable into* $\mathbb{T}_1$ (abbreviated by $\mathbb{T}_0 \leq \mathbb{T}_1$) if there is a function $f : \mathbb{T}_0 \longrightarrow \mathbb{T}_1$ such that

$$t_0 <_{\mathbb{T}_0} t_1 \longrightarrow f(t_0) <_{\mathbb{T}_1} f(t_1)$$

holds for all $t_0, t_1 \in \mathbb{T}_0$. Note that f need not be injective. This ordering is $\mathbf{\Sigma}^1_1$-definable with respect to the above identification.

We are interested in the structure of the resulting partial order $\langle \mathcal{TO}_\kappa, \leq \rangle$. In particular, we want to determine the size of the following cardinal invariants.

- The *bounding number* $\mathfrak{b}_{\langle \mathcal{TO}_\kappa, \leq \rangle}$ is the smallest cardinality of a subset $B \subseteq \mathcal{TO}_\kappa$ such that there is no tree $T \in \mathcal{TO}_\kappa$ with $S \leq T$ for all $S \in B$.
- The *dominating number* $\mathfrak{d}_{\langle \mathcal{TO}_\kappa, \leq \rangle}$ is the smallest cardinality of a subset $C \subseteq \mathcal{TO}_\kappa$ such that for every $S \in \mathcal{TO}_\kappa$ there is a $T \in C$ with $S \leq T$.

The following theorem shows that it is not possible to prove non-trivial statements about the relation of these cardinals and 2^κ from the axioms of $\mathbf{ZFC}$ plus large cardinal axioms.

Theorem (Mekler/Väänänen)

Assume (CH). *If* ν *is a regular cardinal with* $\omega_1 < \nu \leq 2^{\omega_1}$, *then there is a partial order* $\mathbb{P}$ *such that forcing with* $\mathbb{P}$ *preserves cardinalities, cofinalities and the value of* 2^μ *for every cardinal* μ *and*

$$\mathfrak{b}_{\langle \mathcal{TO}_{\omega_1}, \leq \rangle} = \mathfrak{d}_{\langle \mathcal{TO}_{\omega_1}, \leq \rangle} = \nu$$

holds in every $\mathbb{P}$*-generic extension of the ground model.*

Observation (Schlicht)

If κ is an uncountable regular cardinal with $\kappa = \kappa^{<\kappa}$, ν is a cardinal and G is $\mathrm{Add}(\kappa, \nu)$-generic over V, then

$$\mathfrak{b}^{\mathrm{V}[G]}_{\langle \mathcal{TO}_\kappa, \leq \rangle} \leq \mathfrak{b}^{\mathrm{V}}_{\langle \mathcal{TO}_\kappa, \leq \rangle} \quad \textit{and} \quad \nu \leq \mathfrak{d}^{\mathrm{V}[G]}_{\langle \mathcal{TO}_\kappa, \leq \rangle}.$$

Question

Are there axioms that decide *many interesting* questions about Σ^1_1-subsets of ${}^{\kappa}\kappa$ for uncountable regular κ with $\kappa = \kappa^{<\kappa}$?

In the following, we will show that forcing axioms called *closed maximality principle* are examples of such axioms.

Maximality principles

Definition

A sentence φ in the language of set theory is *forceably necessary* if there is a partial order $\mathbb{P}$ such that $\mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}} \Vdash \varphi$ holds whenever $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a partial order.

Example

The sentence "$\omega_1 > \omega_1^{\mathrm{L}}$" is forceably necessary.

Definition

The *Maximality Principle* $\mathbf{MP}$ is the scheme of axioms stating that every forceably neccessary sentence is true.

This principle was introduced by Jonathan Stavi and Jouko Väänänen and independently by Joel Hamkins. The theory $\mathbf{ZFC} + \mathbf{MP}$ was shown to be equiconsistent with $\mathbf{ZFC}$.

We will discuss variations of this principle that allow formulas with parameters and restrict the class of partial orders considered.

Definition

Fix a pair $\Gamma = \langle \varphi_\Gamma, p_\Gamma \rangle$ where $\varphi_\Gamma \equiv \varphi_\Gamma(v_0, v_1)$ is a formula and p_Γ is a parameter.

- Given a formula $\psi \equiv \psi(v_0, \dots, v_{n-1})$ and parameters $x_0, \dots, x_{n-1}$, we say that the statement $\psi(x_0, \dots, x_{n-1})$ is Γ-*forceably necessary* if there is a partial order $\mathbb{P}$ with $\varphi_\Gamma(\mathbb{P}, p_\Gamma)$ such that

$$\mathbb{1}_{\mathbb{P} * \dot{\mathbb{Q}}} \Vdash \psi(\check{x}_0, \dots, \check{x}_{n-1})$$

  holds whenever $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a partial order with $\mathbb{1}_\mathbb{P} \Vdash \varphi_\Gamma(\dot{\mathbb{Q}}, \check{p}_\Gamma)$.
- Given a class P and $n < \omega$, the axiom $\mathbf{MP}^n_\Gamma(P)$ is the statement that every Γ-forceably necessary Σ_n-statement with parameters in P is true.

We present an example to show how such axioms can be used. In the following, let κ denote an uncountable regular cardinal with $\kappa = \kappa^{<\kappa}$.

Assume that the pair $\Gamma = \langle \varphi, \kappa \rangle$ satisfies $\varphi(\mathrm{Add}(\kappa, (2^\kappa)^+), \kappa)$ and

$$\mathbb{1}_{\mathrm{Add}(\kappa,(2^\kappa)^+)} \Vdash \forall \mathbb{Q} \Big[\mathbb{Q} \textit{ is a partial order with } \varphi(\mathbb{Q}, \check{\kappa}) \\ \longrightarrow \mathbb{Q} \textit{ is } {<}\check{\kappa}\textit{-closed and preserves all cardinals} \Big].$$

If $\mathbf{MP}^3_\Gamma(\mathrm{H}(\kappa^+))$ holds and $A = p[T]$ is a $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa$ of cardinality 2^κ, then there is a ${<}\kappa$-closed partial order $\mathbb{P}$ with $\mathbb{1}_\mathbb{P} \Vdash$ "$p[\check{T}] \neq \check{A}$", because otherwise the Σ_3-statement "$|p[T]| < 2^\kappa$" would be Γ-forceably necessary and therefore true.

Lemma

Let $A = p[T]$ be a $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa$ such that $\mathbb{1}_\mathbb{P} \Vdash$ "$p[\check{T}] \neq \check{A}$" holds for some ${<}\kappa$-closed partial order $\mathbb{P}$, then A contains a perfect subset, i.e. ${}^\kappa 2$ embeds continuously into A.

Therefore $\mathbf{MP}^3_\Gamma(\mathrm{H}(\kappa^+))$ implies that every $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa$ of cardinality 2^κ contains a perfect subset.

Closed maximality principles

Let $\Gamma_{<\kappa-cl.}$ be the pair defining the class of all $<\kappa$-closed partial orders.

The princples $\mathbf{MP}^n_{\Gamma_{<\kappa-cl.}}(\mathrm{H}(\kappa^+))$ have been extensively studied by Gunter Fuchs.

Theorem (Fuchs)

Let M be a set-sized transitive model of $\mathbf{ZFC}$ and κ be a regular cardinal with $\kappa = \kappa^{<\kappa}$ in M.

- *If $\kappa < \delta \in M$ is regular in M with $\langle \mathrm{V}^M_\delta, \in\rangle \prec \langle M, \in\rangle$ and G is $\mathrm{Col}(\kappa, {<}\delta)^M$-generic over M, then $\mathbf{MP}^n_{\Gamma_{<\kappa-cl.}}(\mathrm{H}(\kappa^+))$ holds in $\langle M[G], \in\rangle$ for all $n < \omega$.*
- *If $\mathbf{MP}^n_{\Gamma_{<\kappa-cl.}}(\mathrm{H}(\kappa^+))$ holds in $\langle M, \in\rangle$ for all $n < \omega$ and $\delta = (\kappa^+)^M$, then $\langle \mathrm{L}_\delta, \in\rangle \prec \langle \mathrm{L}_{\mathrm{M}\cap\mathrm{On}}, \in\rangle$.*

The axiom $\mathbf{MP}^2_{\Gamma_{<\kappa-cl.}}(\mathrm{H}(\kappa^+))$ settles the first two questions proposed above.

Theorem

Let κ be an uncountable regular cardinal with $\kappa = \kappa^{<\kappa}$ and assume that $\mathbf{MP}^2_{\Gamma_{<\kappa-cl.}}(\mathrm{H}(\kappa^+))$ holds.

- *The least upper bound for the order-types of $\mathbf{\Sigma}^1_1$-well-orderings of subsets of ${}^\kappa\kappa$ is equal to κ^+.*
- *If $\lambda < \kappa$ is a regular cardinal, then $\mathrm{Club}(S^\kappa_\lambda)$ is not a $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa$.*

Observation

The statements $\mathfrak{b}_{\langle \mathcal{TO}_\kappa, \leq \rangle} = \mathfrak{d}_{\langle \mathcal{TO}_\kappa, \leq \rangle}$ and $\mathfrak{b}_{\langle \mathcal{TO}_\kappa, \leq \rangle} < 2^\kappa$ are not decided by $\mathbf{MP}^n_{\Gamma_{<\kappa-cl.}}(\mathrm{H}(\kappa^+))$ for all $n < \omega$.

Maximality principles with more parameters

Question

Is it possible to have such maximality principles for statements with parameters in $\mathrm{H}(2^\kappa)$ with $2^\kappa > \kappa^+$ if we only allow classes of forcings that preserve cardinals, like classes of $<\kappa$-closed forcings that satisfy the κ^+-chain condition?

Note that $\mathbf{MP}^1_\Gamma(\mathrm{H}(\nu))$ implies $\mathbf{FA}_{<\nu}(\mathbb{P})$ for every partial order $\mathbb{P}$ of cardinality less than ν with $\varphi_\Gamma(\mathbb{P}, p_\Gamma)$, i.e. for every collection $\mathcal{D}$ of less than ν-many dense subsets of $\mathbb{P}$, there is a $\mathcal{D}$-generic filter in $\mathbb{P}$.

Saharon Shelah showed that there is a $< \kappa$-closed partial order $\mathbb{P}$ of cardinality κ^+ satisfying the κ^+-chain condition such that $\mathbf{FA}_{\kappa^+}(\mathbb{P})$ fails.

Therefore we have to look for strengthenings of the above properties.

Definition

Given a cardinal κ, we call a partial order $\mathbb{P}$ $<\kappa$-coupling if there is a function $c : \mathbb{P} \longrightarrow \kappa$ such that for all $\leq_{\mathbb{P}}$-descending chains $\langle p_\alpha \mid \alpha < \lambda \rangle$ and $\langle q_\alpha \mid \alpha < \lambda \rangle$ in $\mathbb{P}$ with $\lambda < \kappa$, $c(p_\alpha) = c(q_\alpha)$ and p_α and q_α are compatible for all $\alpha < \lambda$ there is a condition r in $\mathbb{P}$ with $r \leq_{\mathbb{P}} p_\alpha, q_\alpha$ for all $\alpha < \lambda$.

Observation

If $\mathbb{P}$ is $<\kappa$-coupling, then $\mathbb{P}$ is $<\kappa$-closed.

Observation

If $\mathbb{P}$ is $<\kappa$-closed and well-met, then $\mathbb{P}$ is $<\kappa$-coupling.
In particular, $\mathrm{Add}(\kappa, \nu)$ is $<\kappa$-coupling.

Definition

Given partial orders $\mathbb{P}$ and $\mathbb{Q}$, we say that $\mathbb{P}$ *is antichain reducible to* $\mathbb{Q}$ if there is a function $r : \mathbb{P} \longrightarrow \mathbb{Q}$ that maps pairs of incompatible conditions in $\mathbb{P}$ to incompatible conditions in $\mathbb{Q}$.

Observation

If $\mathbb{P}$ is antichain reducible to $\mathbb{Q}$ and $\mathbb{Q}$ satisfies the ν-chain condition, then $\mathbb{P}$ satisfies the ν-chain condition.

Theorem

Forcing iterations with $<\kappa$-support of $<\kappa$-coupling forcings that are antichain reducible to partial orders of the form $\mathrm{Add}(\kappa, \nu)$ satisfy the κ^+-chain condition.

Let Γ_κ be the pair defining the class of all $<\kappa$-coupling partial orders that are antichain reducible to a partial order of the form $\mathrm{Add}(\kappa,\nu)$.

Theorem

Let M be a set-sized transitive model of $\mathbf{ZFC}$ and κ be a regular cardinal with $\kappa = \kappa^{<\kappa}$ in M.

- *If $\kappa < \delta \in M$ is regular in M with $\langle \mathrm{V}^M_\delta, \in\rangle \prec \langle M, \in\rangle$, then there is a partial order $\mathbb{P} \in M$ such that the following statements hold.*
  - *$\mathbb{P}$ is $<\kappa$-closed and satisfies the κ^+-chain condition in M.*
  - *If G is $\mathbb{P}$-generic over M, then $\delta = (2^\kappa)^{M[G]}$ and $\mathbf{MP}^n_{\Gamma_\kappa}(\mathrm{H}(\delta))$ holds in $\langle M[G], \in\rangle$ for all $n < \omega$.*
- *If $\delta = (2^\kappa)^M$ and $\mathbf{MP}^n_{\Gamma_\kappa}(\mathrm{H}(\delta))$ holds in $\langle M, \in\rangle$ for all $n < \omega$, then δ is regular in M and $\langle \mathrm{L}_\delta, \in\rangle \prec \langle \mathrm{L}_{\mathrm{M}\cap\mathrm{On}}, \in\rangle$.*

It turns out that the axiom $\mathbf{MP}^3_{\Gamma_\kappa}(\mathrm{H}(2^\kappa))$ settles all of the above questions.

Theorem

Let κ be an uncountable regular cardinal with $\kappa = \kappa^{<\kappa}$ and assume that $\mathbf{MP}^3_{\Gamma_\kappa}(\mathrm{H}(2^\kappa))$ holds.

- *The least upper bound for the order-types of $\mathbf{\Sigma}^1_1$-well-orderings of subsets of ${}^\kappa\kappa$ is equal to 2^κ.*
- *If $\lambda < \kappa$ is a regular cardinal, then $\mathrm{Club}(S^\kappa_\lambda)$ is not a $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa$.*
- $\mathfrak{b}_{\langle \mathcal{TO}_\kappa, \leq \rangle} = \mathfrak{d}_{\langle \mathcal{TO}_\kappa, \leq \rangle} = 2^\kappa$.

The above statements can be derived from certain structural properties of $\mathbf{\Sigma}^1_1$-subsets that follow from $\mathbf{MP}^3_{\Gamma_\kappa}(\mathrm{H}(2^\kappa))$.

In the following, I want to show how to derive the first statement.

Fix an uncountable regular cardinal κ with $\kappa = \kappa^{<\kappa}$.

We consider degrees of forcing absoluteness under $<\kappa$-closed forcings.

Proposition

If $\mathbb{P}$ is a $<\kappa$-closed partial order, then $\mathbf{\Sigma}^1_1({}^\kappa\kappa)$-absoluteness holds for $\mathbb{P}$, i.e. $\mathrm{H}(\kappa^+)^{\mathrm{V}} \prec_{\Sigma_1} \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ holds whenever G is $\mathbb{P}$-generic over V.

Proposition

Let Γ be a pair defining a class of $<\kappa$-closed partial orders and assume that $\mathbf{MP}^2_\Gamma(\mathrm{H}(\kappa^+))$ holds. If $\mathbb{P}$ is a partial order with $\varphi_\Gamma(\mathbb{P}, p_\Gamma)$, then $\mathbf{\Sigma}^1_2({}^\kappa\kappa)$-absoluteness holds for $\mathbb{P}$, i.e. we have $\mathrm{H}(\kappa^+)^{\mathrm{V}} \prec_{\Sigma_2} \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ whenever G is $\mathbb{P}$-generic over V.

Remark

There is a $<\kappa$-coupling partial order $\mathbb{P}$ that is antichain reducible to $\mathrm{Add}(\kappa, (2^\kappa))$ such that $\mathbf{\Sigma}^1_3({}^\kappa\kappa)$-absoluteness fails for $\mathbb{P}$.

Our upper bound for the length of $\mathbf{\Sigma}^1_1$-well-orderings will be a consequence of the following lemma.

Lemma

Assume that $\mathbf{\Sigma}^1_2({}^\kappa\kappa)$-absoluteness holds for $\mathrm{Add}(\kappa, 1)$. If $\langle A, \lessdot\rangle$ is a $\mathbf{\Sigma}^1_1$-well-ordering of a subset of ${}^\kappa\kappa$, then A contains no perfect subset.

Assume, toward a contradiction, that A has a perfect subset.

Pick trees S and T with $A = p[S]$ and $\prec = p[T]$. Set $\mathbb{P} = \mathrm{Add}(\kappa, 1)$ and let G be $\mathbb{P}$-generic over V.

By $\mathbf{\Sigma}^1_2({}^\kappa\kappa)$-absoluteness, $\prec^* = p[T]^{\mathrm{V}[G]}$ is a well-ordering of $A^* = p[S]^{\mathrm{V}[G]}$ and $A \subsetneq A^*$. By the homogeneity of $\mathbb{P}$, there is a $\mathbb{P}$-name $\dot{x}_0$ with $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{x}_0 \in p[\check{S}] \setminus \check{A}$".

Pick $G_{0,0} \times G_{0,1} \in \mathrm{V}[G]$ that is $(\mathbb{P} \times \mathbb{P})$-generic over V with $\mathrm{V}[G] = \mathrm{V}[G_{0,0}][G_{0,1}]$. Then $\dot{x}_0^{G_{0,0}} \neq \dot{x}_0^{G_{0,1}}$ and we may assume $\dot{x}_0^{G_{0,1}} \prec^* \dot{x}_0^{G_{0,0}} =: y_0$.

As above, the homogeneity of $\mathbb{P}$ implies that there is a $\mathbb{P}$-name $\dot{x}_0 \in \mathrm{V}[G_{0,0}]$ such that $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{x}_1 \notin \dot{\mathrm{V}} \wedge \langle \dot{x}_1, \check{y}_0 \rangle \in p[\check{T}]$". Pick $G_{1,0} \times G_{1,1} \in \mathrm{V}[G]$ that is $(\mathbb{P} \times \mathbb{P})$-generic over $\mathrm{V}[G_{0,0}]$ with $\mathrm{V}[G] = \mathrm{V}[G_{0,0}][G_{1,0}][G_{1,1}]$. Then $\dot{x}_1^{G_{1,0}} \neq \dot{x}_1^{G_{1,1}}$ and we may assume $\dot{x}_1^{G_{1,1}} \prec^* \dot{x}_1^{G_{1,0}} =: y_1 \prec^* y_0$.

By repeating this process, we can construct a $\prec^*$-descending sequence $\langle y_n \in A^* \mid n < \omega \rangle$ in $\mathrm{V}[G]$, a contradiction. $\square$

Lemma

Assume that $\mathbf{\Sigma}^1_2({}^\kappa\kappa)$-absoluteness holds for $\mathrm{Add}(\kappa, 1)$. If $\langle A, \prec\rangle$ is a $\mathbf{\Sigma}^1_1$-well-ordering of a subset of ${}^\kappa\kappa$, then A contains no perfect subset.

Corollary

If $\mathbf{MP}^3_{\Gamma_\kappa}(\mathrm{H}(\kappa^+))$ holds, then every $\mathbf{\Sigma}^1_1$-well-ordering of a subset of ${}^\kappa\kappa$ has length less than 2^κ.

The following theorem allows us to show that 2^κ is the correct upper bound in the above situation.

Theorem (L.)

Let κ be an uncountable regular cardinal with $\kappa = \kappa^{<\kappa}$ and A be a subset of ${}^\kappa\kappa$. Then there is a partial order $\mathbb{P}(A)$ with the following properties.

- $\mathbb{P}(A)$ *is* $<\kappa$*-coupling and is antichain reducible to* $\mathrm{Add}(\kappa, 2^\kappa)$.
- *If G is $\mathbb{P}(A)$-generic over* V*, $\mathbb{Q}$ is a σ-closed partial order in* $\mathrm{V}[G]$ *that preserves the regularity of κ and H is $\mathbb{Q}$-generic over* $\mathrm{V}[G]$*, then A is a $\mathbf{\Sigma}^1_1$-subset in* $\mathrm{V}[G][H]$.

Corollary

If $\mathbf{MP}^2_{\Gamma_\kappa}(\mathrm{H}(2^\kappa))$ holds, then every subset of ${}^\kappa\kappa$ of cardinality less than 2^κ is a $\mathbf{\Sigma}^1_1$-subset.

We can also use the above result to show that our maximality principle determines the value of the bounding and the dominating number of $\langle \mathcal{TO}_\kappa, \leq \rangle$. We need the following lemma.

Lemma (Boundedness Lemma, Mekler/Väänänen)

If A is a $\mathbf{\Sigma}^1_1$-subset of $\mathcal{TO}_\kappa$, then there is an element T of $\mathcal{TO}_\kappa$ with $S \leq T$ for all $S \in A$.

Corollary

If $\mathbf{MP}^2_{\Gamma_\kappa}(\mathrm{H}(2^\kappa))$ holds, then $\mathfrak{b}_{\langle \mathcal{TO}_\kappa, \leq \rangle} = \mathfrak{d}_{\langle \mathcal{TO}_\kappa, \leq \rangle} = 2^\kappa$.

Thank you for listening!