The set-theoretic essence of automorphism towers

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Introduction

In this talk, I will introduce an algebraic construction whose properties highly depend on the model of set theory in which the construction takes place.

This construction is called the *automorphism tower*.

The construction

Let G be a group with trivial centre. For each $g \in G$, the map

$$\iota_g : G \longrightarrow G;\ h \mapsto g \circ h \circ g^{-1}$$

is an automorphism of G and is called the *inner automorphism corresponding to* g. Clearly, $\iota_g = \mathrm{id}_G$ if and only if $g = \mathbb{1}_G$. The map

$$\iota_G : G \longrightarrow \mathrm{Aut}(G);\ g \mapsto \iota_g$$

is an embedding of groups that maps G onto the subgroup $\mathrm{Inn}(G)$ of all inner automorphisms of G. An easy computation shows that $\pi \circ \iota_g \circ \pi^{-1} = \iota_{\pi(g)}$ holds for all $g \in G$ and $\pi \in \mathrm{Aut}(G)$. This shows that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$ and $\mathrm{Aut}(G)$ is also a group with trivial centre.

By iterating this process, we inductively construct the automorphism tower of G.

Definition

A sequence $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ of groups is *the automorphism tower of a centreless group* G if the following statements hold.

- $G_0 = G$.
- For all $\alpha \in \mathrm{On}$, G_α is a normal subgroup of $G_{\alpha+1}$ and the induced homomorphism
$$\varphi_\alpha : G_{\alpha+1} \longrightarrow \mathrm{Aut}(G_\alpha);\ g \mapsto \iota_g \restriction G_\alpha$$
is an isomorphism.
- For all $\alpha \in \mathrm{Lim}$, $G_\alpha = \bigcup\{G_\beta \mid \beta < \alpha\}$.

In this definition, we replaced $\mathrm{Aut}(G_\alpha)$ by an isomorphic copy $G_{\alpha+1}$ that contains G_α as a normal subgroup. This allows us to take unions at limit stages.

Given a centreless group G, we can construct an automorphism tower of G and it is unique up to isomorphisms that induce the identity on G.

We say that the automorphism tower of a centerless group terminates if there is an $\alpha \in \mathrm{On}$ with $G_\alpha = G_{\alpha+1}$ and therefore $G_\alpha = G_\beta$ for all $\beta \geq \alpha$. It is natural to ask whether the automorphism tower of every centreless group finally terminates. Simon Thomas used Fodor's Lemma to answer this question.

Theorem (Thomas, 1998)

If G is an infinite centreless group of cardinality κ, then there is an $\alpha < (2^\kappa)^+$ with $G_\alpha = G_{\alpha+1}$.

This result allows us to make the following definitions.

Definition

Given a centreless group G, we let $\tau(G)$ denote the least ordinal α satisfying $G_\alpha = G_{\alpha+1}$ and call this ordinal the *height of the automorphism tower of G*. For every infinite cardinal κ, we define

$$\tau_\kappa = \mathrm{lub}\{\tau(G) \mid G \textit{ is a centreless group of cardinality } \kappa\}.$$

There are only 2^κ-many centreless groups of cardinality κ and this shows that Simon Thomas' result implies "$\tau_\kappa < (2^\kappa)^+$" for all infinite cardinals κ. Simon Thomas also proved "$\tau_\kappa \geq \kappa^+$".

Given a regular uncountable cardinal κ with "$\kappa = \kappa^{<\kappa}$", a result of Winfried Just, Saharon Shelah and Simon Thomas shows that "$\tau_\kappa > 2^\kappa$" holds in a cardinal- and cofinality-preserving forcing extension and Simon Thomas showed that "$\tau_\kappa < 2^\kappa$" holds in a forcing extension that adds many Cohen-subsets of κ.

The following problem is still open.

Problem (The automorphism tower problem)

Find a model M of ZFC and an infinite cardinal κ in M such that it is possible to "compute" the exact value of τ_κ in M.

Non-absoluteness results

One of the reasons why it is so difficult to compute the value of τ_κ is that there are groups whose automorphism tower heights depend on the model of set theory in which they are computed.

Theorem (Thomas, 1998)

- *There is a partial order* $\mathbb{P}$ *satisfying the countable chain condition and a centreless group* G *with "*$\tau(G) = 0$*" and* $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ *"*$\tau(\check{G}) \geq 1$*".*
- *There is a centreless group* H *with "*$\tau(H) = 2$*" and* $\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}}$ *"*$\tau(\check{H}) = 1$*" for every notion of forcing* $\mathbb{Q}$ *that adds a new real.*

Therefore, you always have to take into account the set-theoretic background in which the computation of $\tau(G)$ takes place. This shows that the automorphism tower construction contains a *set-theoretic essence* (this formulation is due to Joel David Hamkins).

The following results show that this set-theoretic influence can be very strong.

Groups with highly malleable automorphism towers

Theorem (Hamkins & Thomas, 2000)

It is consistent with the axioms of ZFC *that for every infinite cardinal* κ *and every ordinal* $\alpha < \kappa$*, there exists a centreless group* G *with the following properties.*

- $\tau(G) = \alpha$.
- *Given* $0 < \beta < \kappa$*, there exists a partial order* $\mathbb{P}$ *preserving cofinalities and cardinalities with* $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ *"*$\tau(\check{G}) = \check{\beta}$*".*

Gunter Fuchs and Joel David Hamkins showed that groups with the above properties exist in Gödel's constructible universe L.

Using techniques developed in the proofs of the above results, it is possible to construct ZFC-models containing groups whose automorphism tower height can changed again and again by passing to another model of set theory.

Theorem (Fuchs & L., 2010)

It is consistent with the axioms of ZFC that for every infinite cardinal κ there is a centreless group G with "$\tau(G) = 0$" and the property that for every function $s : \kappa \longrightarrow (\kappa \setminus \{0\})$ there is a sequence $\langle \mathbb{P}_\alpha \mid 0 < \alpha < \kappa \rangle$ of partial orders such that the following statements hold.

- *For all $0 < \alpha < \kappa$, $\mathbb{P}_\alpha$ preserves cardinalities and cofinalities.*
- *For all $0 < \alpha < \beta < \kappa$, there is a partial order $\mathbb{Q}$ with $\mathbb{P}_\beta = \mathbb{P}_\alpha \times \mathbb{Q}$.*
- *For all $\alpha < \kappa$, we have $\mathbb{1}_{\mathbb{P}_{\alpha+1}} \Vdash_{\mathbb{P}_{\alpha+1}}$ "$\tau(\check{G}) = \check{s}(\check{\alpha})$".*
- *If $0 < \alpha < \kappa$ is a limit ordinal, then $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash_{\mathbb{P}_\alpha}$ "$\tau(\check{G}) = 1$".*

In another direction, it is also possible that there groups whose automorphism tower can be changed to arbitrary heights by forcing with a partial order that preserves cardinalities and cofinalities.

Theorem (L., 2011)

It is consistent with the axioms of ZFC *that there is a centreless group* G *with the property that for every ordinal* α *there is a partial order* $\mathbb{P}$ *satisfying the following statements.*

- $\mathbb{P}$ *preserves cardinalities and cofinalities.*
- $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\tau(\check{G}) \geq \check{\alpha}$".

Thank you for listening!