THE SET-THEORETIC ESSENCE OF AUTOMORPHISM TOWERS

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One focus of my research is the use of set-theoretic results and methods in the study of infinite groups. A prominent example of such an interactions between algebra and set theory is the so-called *automorphism tower problem*. I give a short introduction to this group-theoretic construction and present some results that show that it has a *set-theoretic essence* (this formulation is due to Joel Hamkins, see [Ham02]).

If G is a centerless group, then there is a natural embedding $\iota_G : G \longrightarrow \mathrm{Aut}(G)$ mapping G to the subgroup $\mathrm{Inn}(G)$ of inner automorphisms of G by sending each element g to the corresponding inner automorphism ι_g defined by $\iota_g(h) = ghg^{-1}$. An easy computation shows that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$, $\mathrm{C}_{\mathrm{Aut}(G)}(\mathrm{Inn}(G)) = \{\mathrm{id}_G\}$ and $\mathrm{Aut}(G)$ is also a group with trivial center. Moreover, we may assume that $\mathrm{Aut}(G)$ contains G as a normal subgroup by identifying G with $\mathrm{Inn}(G)$. We iterate this process to construct the automorphism tower $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ of G by setting $G_0 = G$, $G_{\alpha+1} = \mathrm{Aut}(G_\alpha)$ containing G_α as a normal subgroup and $G_\lambda = \bigcup_{\alpha<\lambda} G_\alpha$ for all limit ordinals λ. In [Tho98], Simon Thomas showed that the automorphism tower of every infinite centerless group of cardinality κ *terminates* in less than $(2^\kappa)^+$-many steps, i.e. there is an ordinal $\alpha < (2^\kappa)^+$ with $G_\alpha = G_\beta$ for all $\beta \geq \alpha$. We let $\tau(G)$ denote the least ordinal with this property and call it the *height of the automorphism tower of* G.

Although the definition of automorphism towers is purely algebraic, it has a set-theoretic essence, since there are groups whose automorphism tower heights depend on the model of set theory in which they are computed. In [Tho98], Simon Thomas showed that the height of an automorphism tower can both increase and decrease when we pass to a forcing extension of the ground model V. Moreover, Joel Hamkins and Simon Thomas showed in [HT00] that it is consistent to have a cardinal κ with the property that for every $\alpha < \kappa$ there is a centerless group G with $\tau(G) = \alpha$ and for every $0 < \beta < \kappa$ there is a cardinality and cofinality preserving forcing-extensions $\mathrm{V}[H_\beta]$ of V with $\tau(G)^{\mathrm{V}[H_\beta]} = \beta$. Using results from [FH08], Gunter Fuchs and I extended this result in form of the following theorem.

Theorem ([FL10])**.** *It is consistent that there is an infinite centerless group G and an uncountable cardinal κ with the property that for every function $s : \kappa \longrightarrow (\kappa \setminus \{0\})$, there is an ascending sequence $\langle \mathrm{V}[H_\alpha] \mid 0 < \alpha < \kappa\rangle$ of cardinality and cofinality preserving forcing-extensions of* V *such that the following statements hold for all ordinals $0 < \alpha < \kappa$.*

(i) $\mathrm{V} = \mathrm{V}[H_0]$ *and* $\tau(G)^{\mathrm{V}} = 0$.

(ii) $\tau(G)^{\mathrm{V}[H_{\alpha+1}]} = s(\alpha)$.

(iii) If α is a limit ordinal, then $\tau(G)^{\mathrm{V}[H_\alpha]} = 1$.

In particular, the conclusion of the above theorem holds in Gödel's constructible universe L. We also prove variants of the above theorem that show that it is consistent to have an infinite centerless group whose automorphism tower can be changed again and again by passing to smaller and smaller inner models of V. In another direction, I apply results from [JST99] to prove the following.

Theorem. *It is consistent that there is an infinite centerless group G with the property that for every ordinal α, there is a cardinality- and cofinality-preserving forcing-extension* $\mathrm{V}[H]$ *of* V *with* $\tau(G)^{\mathrm{V}[H]} > \alpha$.

The above results suggest that it can be very difficult to compute the automorphism tower height of a given group G, because you have to consider the set-theoretic background in which this computation takes place. The following problem remains open.

Problem. *Find a model M of ZFC and an infinite cardinal $\kappa \in M$ such that it is possible to compute the exact value of*

$$\tau_\kappa^M = \mathrm{lub}\{\tau(G)^M \mid G \textit{ is a centerless group of cardinality } \kappa \textit{ in } M\}.$$

Simon Thomas' result shows that $\tau_\kappa < (2^\kappa)^+$ holds. I can improve this bound to $\tau_\kappa \leq \alpha + 1$, where α is the least $\mathcal{P}(\kappa)$-admissible ordinal, i.e. the least α such that $\mathrm{L}_\alpha(\mathcal{P}(\kappa))$ is a model of the axioms of Kripke-Platek set theory.

References

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