Free groups and automorphism groups of infinite fields

Philipp Moritz Lücke
(joint work with Saharon Shelah)

Mathematisches Institut
Rheinische Friedrich-Wilhelms-Universität Bonn
http://www.math.uni-bonn.de/people/pluecke/

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Given an infinite cardinal λ and a group G, we consider the following question.

Question

Is G isomorphic to the automorphism group of a field of cardinality λ?

We start with some negative results.

If K is a field of cardinality λ, then the group $\mathrm{Aut}(K)$ has cardinality at most 2^λ. A simple cardinality argument shows that there are groups of cardinality 2^λ that are not isomorphic to the automorphism group of a field of cardinality λ.

The following theorem shows that we can also find such groups with cardinality λ^+.

Theorem (De Bruijn, 1957)

If λ is an infinite cardinal, then the group $\mathrm{Fin}(\lambda^+)$ consisting of all finite permutations of λ^+ cannot be embedded into the group $\mathrm{Sym}(\lambda)$ consisting of all permutations of λ.

The following result shows that a wide variety of groups of cardinality at most 2^λ is isomorphic to the automorphism group of a field of cardinality λ.

Theorem (Fried & Kollár, 1982; Kaplan & Shelah, 2012)

Let λ be an infinite cardinal. If $\mathcal{L}$ is a first-order language of cardinality at most λ, $\mathcal{M}$ is an $\mathcal{L}$-model of cardinality at most λ and p is either 0 or a prime number, then there is a field K of characteristic p and cardinality λ such that the groups $\mathrm{Aut}(\mathcal{M})$ and $\mathrm{Aut}(K)$ are isomorphic.

In particular, every group of cardinality at most λ is isomorphic to the automorphism group of a field of cardinality λ.

In the remainder of this talk, we focus on free groups and the following question.

Question

Is there a field K whose automorphism group is a free group of cardinality greater than the cardinality of K?

The following theorems show that the cardinality of such a field cannot be countable or a singular strong limit cardinal of countable cofinality.

Theorem (Shelah, 2003)

Let $\mathcal{L}$ be a countable first-order language and $\mathcal{M}$ be a countable $\mathcal{L}$-model. Then $\mathrm{Aut}(\mathcal{M})$ is not an uncountable free group.

Theorem (Shelah, 2003)

Let $\langle \lambda_n \mid n < \omega \rangle$ be a sequence of infinite cardinals with $2^{\lambda_n} < 2^{\lambda n+1}$ for all $n < \omega$, $\lambda = \sum_{n<\omega} \lambda_n$ and $\mu = \sum_{n<\omega} 2^{\lambda_n}$.
If $\mathcal{L}$ is a first-order language of cardinality λ and $\mathcal{M}$ is an $\mathcal{L}$-model of cardinality λ such that $\mathrm{Aut}(\mathcal{M})$ has cardinality greater than μ, then $\mathrm{Aut}(\mathcal{M})$ is not a free group.

In contrast, Winfried Just, Saharon Shelah and Simon Thomas showed that the existence of a field of uncountable regular cardinality providing a positive answer to the above question is consistent.

Theorem (Just, Shelah & Thomas, 1999)

Let λ be an uncountable cardinal with $\lambda = \lambda^{<\lambda}$ and $\nu > \lambda$ be a cardinal. Then there is a partial order $\mathbb{P}$ with the following properties.

- *$\mathbb{P}$ is $<\lambda$-closed and satisfies the λ^+-chain condition.*
- *If G is $\mathbb{P}$-generic over V, then $\mathrm{V}[G]$ contains a field of cardinality λ whose automorhism group is a free group of cardinality ν.*

The following result shows that the existence of such fields already follows from the axioms of set theory for a larger class of cardinals.

Theorem (L. & Shelah)

Let λ be a cardinal with $\lambda = \lambda^{\aleph_0}$ and p be either 0 or a prime number. Then there is a field K of characteristic p and cardinality λ whose automorphism group is a free group of cardinality 2^λ.

In particular, there always is a field K whose automorphism group is a free group of cardinality greater than the cardinality of K.

Under certain cardinal arithmetic assumptions, we can combine the above results to completely characterize the class of cardinals λ with the property that there exists a field of cardinality λ whose automorphism group is a free group of cardinality greater than λ.

The following corollary is an example of such a characterization.

Corollary

Assume that the Continuum Hypothesis and the Singular Cardinal Hypothesis hold. Then the following statements are equivalent for every infinite cardinal λ.

- *There is a field of cardinality λ whose automorphism group is a free group of cardinality greater than λ.*
- *There is a cardinal $\kappa \leq \lambda$ with $2^\kappa > \lambda$ and $\mathrm{cof}(\kappa) > \omega$.*

The above results raise the following question.

Question

Is it consistent to have a cardinal λ of uncountable cofinality such that there is no field of cardinality λ whose automorphism group is a free group of cardinality greater than λ?

The methods developed in the proof of the above theorem also allow us to show that a positive answer to this question has consistency strength strictly greater than $\mathbf{con}(\mathrm{ZFC})$.

Theorem (L. & Shelah)

Let λ be a regular uncountable cardinal such that there is no field of cardinality λ whose automorphism group is a free group of cardinality greater than λ. Then λ^+ is an inaccessible cardinal in $\mathrm{L}[x]$ for every subset x of κ.

Theorem (L. & Shelah)

Let λ be a singular cardinal of uncountable cofinality such that there is no field of cardinality λ whose automorphism group is a free group of cardinality greater than λ. Then there is an inner model with a Woodin cardinal.

Representing inverse limits as automorphism groups of fields

A pair $\mathbb{D} = \langle D, \leq_{\mathbb{D}} \rangle$ is a *directed set* if $\leq_{\mathbb{D}}$ is a reflexive, transitive binary relation on the set D with the property that for all $p, q \in D$ there is a $r \in D$ with $p, q \leq_{\mathbb{D}} r$.

Given a directed set $\mathbb{D} = \langle D, \leq_{\mathbb{D}} \rangle$, we call a pair

$$\mathbb{I} = \langle \langle A_p \mid p \in D \rangle, \langle f_{p,q} \mid p, q \in D, \ p \leq_{\mathbb{D}} q \rangle \rangle$$

an *inverse system of sets over* $\mathbb{D}$ if the following statements hold for all $p, q, r \in D$ with $p \leq_{\mathbb{D}} q \leq_{\mathbb{D}} r$.

- A_p is a non-empty set and $f_{p,q} : A_q \longrightarrow A_p$ is a function.
- $f_{p,p} = \mathrm{id}_{A_p}$ and $f_{p,q} \circ f_{q,r} = f_{p,r}$.

Given such an inverse system $\mathbb{I}$, we call the set

$$A_{\mathbb{I}} = \{(a_p)_{p \in D} \mid f_{p,q}(a_q) = a_p \textit{ for all } p, q \in D \textit{ with } p \leq_{\mathbb{D}} q\}$$

the *inverse limit of* $\mathbb{I}$.

Example

Let λ be an infinite cardinal and $[\lambda]^{\aleph_0}$ be the set of all countable subsets of λ. Then $\mathbb{D} = \langle [\lambda]^{\aleph_0}, \subseteq \rangle$ is a directed set.

Given $p \in [\lambda]^{\aleph_0}$, let ${}^{p}2$ denote the set consisting of all functions $h : p \longrightarrow 2$. If $p \subseteq q \in [\lambda]^{\aleph_0}$, then we define

$$f_{p,q} : {}^{q}2 \longrightarrow {}^{p}2;\ h \longmapsto h \restriction p.$$

Then

$$\mathbb{I}(\lambda) = \langle \langle {}^{p}2 \mid p \in [\lambda]^{\aleph_0} \rangle, \langle f_{p,q} \mid p \subseteq q \in [\lambda]^{\aleph_0} \rangle \rangle$$

is an inverse system of sets over $\mathbb{D}$ and it is easy to see that the function

$$b : A_{\mathbb{I}(\lambda)} \longrightarrow {}^{\lambda}2;\ (h_p)_{p \in [\lambda]^{\aleph_0}} \longmapsto \bigcup \{ h_p \mid p \in [\lambda]^{\aleph_0} \}$$

is a bijection.

Example

Let $\mathbb{T}$ be a tree of height α. Given $\beta < \alpha$, we let $\mathbb{T}(\beta)$ denote the β-th level of $\mathbb{T}$. If $\gamma \leq \beta < \alpha$, then we let $f_{\gamma,\beta} : \mathbb{T}(\beta) \longrightarrow \mathbb{T}(\gamma)$ denote the map that sends an element t of $\mathbb{T}(\beta)$ to the unique element s of $\mathbb{T}(\gamma)$ with $s \leq_{\mathbb{T}} t$.

Then

$$\mathbb{I}(\mathbb{T}) = \langle\langle\mathbb{T}(\beta) \mid \beta < \alpha\rangle, \langle f_{\gamma,\beta} \mid \gamma \leq \beta < \alpha\rangle\rangle$$

is an inverse system of sets over $\langle\alpha, \leq\rangle$ and the function

$$b : A_{\mathbb{I}(\mathbb{T})} \longrightarrow [\mathbb{T}];\ (t_\beta)_{\beta<\alpha} \longmapsto \{t_\beta \mid \beta < \alpha\}$$

is a bijection between $A_{\mathbb{I}(\mathbb{T})}$ and the set $[\mathbb{T}]$ of all cofinal branches through $\mathbb{T}$.

Given a directed set $\mathbb{D} = \langle D, \leq_{\mathbb{D}} \rangle$, a pair

$$\mathbb{I} = \langle \langle G_p \mid p \in D \rangle, \langle h_{p,q} \mid p, q \in D,\ p \leq_{\mathbb{D}} q \rangle \rangle$$

an *inverse system of groups over* $\mathbb{D}$ if the following statements hold for all $p, q, r \in D$ with $p \leq_{\mathbb{D}} q \leq_{\mathbb{D}} r$.

- G_p is a group and $h_{p,q} : G_q \longrightarrow G_p$ is a homomorphism of groups.
- $h_{p,p} = \mathrm{id}_{G_p}$ and $h_{p,q} \circ h_{q,r} = h_{p,r}$.

Given such an inverse system $\mathbb{I}$, we call the subgroup

$$G_{\mathbb{I}} = \{(g_p)_{p \in D} \mid h_{p,q}(g_q) = g_p \textit{ for all } p, q \in D \textit{ with } p \leq_{\mathbb{D}} q\}$$

of the product of the G_p's the *inverse limit of* $\mathbb{I}$.

Example

Let

$$\mathbb{I} = \langle \langle A_p \mid p \in D \rangle, \langle f_{p,q} \mid p, q \in D,\ p \leq_{\mathbb{D}} q \rangle \rangle$$

be an inverse system of sets over some directed set $\mathbb{D} = \langle D, \leq_{\mathbb{D}} \rangle$.

Given $p \in D$, we let G_p denote a free group with generators $\{x_{p,a} \mid a \in A_p\}$. If $p, q \in D$ with $p \leq_{\mathbb{D}} q$, then we let $h_{p,q} : G_q \longrightarrow G_p$ denote the unique homomorphism of groups with

$$h_{p,q}(x_{q,a}) = x_{p,f_{p,q}(a)}$$

for all $a \in A_q$.

Then

$$\mathbb{I}_{fg} = \langle \langle G_p \mid p \in D \rangle, \langle h_{p,q} \mid p, q \in D,\ p \leq_{\mathbb{D}} q \rangle \rangle$$

is an inverse system of groups over $\mathbb{D}$.

The following result is the first step in the proof of the above theorem.

Theorem

Let

$$\mathbb{I} = \langle \langle G_q \mid q \in D \rangle, \langle h_{q,r} \mid q, r \in D,\ q \leq_{\mathbb{D}} r \rangle \rangle$$

be an inverse system of groups over a directed set $\mathbb{D} = \langle D, \leq_{\mathbb{D}} \rangle$ *and* p *be either* 0 *or a prime number. Then there is a field* K *of characteristic* p *with the following properties.*

- *The groups* $\mathrm{Aut}(K)$ *and* $G_{\mathbb{I}}$ *are isomorphic.*
- $|K| \leq \max\{\aleph_0, \sum_{q \in D} |G_q|\}$.

Proof.

Define a model $\mathcal{M}_{\mathbb{I}}$ by the following clauses.

- *The domain of $\mathcal{M}_{\mathbb{I}}$ is the set*

$$M_{\mathbb{I}} = \{\langle g, q, i\rangle \mid q \in D,\ g \in G_q,\ i < 2\}.$$

- *The element $\langle g, q, 1\rangle$ is a constant for all $q \in D$ and $g \in G_q$.*
- *The set $P_q = \{\langle g, q, 0\rangle \mid g \in G_q\}$ is a predicate for all $q \in D$.*
- *The set*

$$H_{q,r} = \{\langle\langle g, r, 0\rangle, \langle h_{q,r}(g), q, 0\rangle\rangle \mid g \in G_r\}$$

  is a predicate for all $q, r \in D$ with $q \leq_{\mathbb{D}} r$.
- *The set*

$$F_q = \{\langle\langle g, q, 0\rangle, \langle h, q, 1\rangle, \langle g \cdot h, q, 0\rangle\rangle \mid g, h \in G_q\}$$

  is a predicate for all $q \in D$.

Proof (cont.).

Pick $\sigma \in \mathrm{Aut}(\mathcal{M}_{\mathbb{I}})$ *and* $q \in D$.

Claim

$\sigma \restriction P_q : P_q \longrightarrow P_q$ and $\sigma(\langle g, q, 1\rangle) = \langle g, q, 1\rangle$ for all $g \in G_q$. □

Claim

There is a unique $c_{\sigma,q} \in G_q$ with $\sigma(\langle g, q, 0\rangle) = \langle c_{\sigma,q} \cdot g, q, 0\rangle$ for all $g \in G_q$.

Proof of the Claim.

There is a unique $c_{\sigma,q} \in G_q$ *such that* $\sigma(\langle \mathbb{1}_{G_q}, q, 0\rangle) = \langle c_{\sigma,q}, q, 0\rangle$. *Then*

$$F_q(\langle c_{\sigma,q}, q, 0\rangle, \langle g, q, 1\rangle, \sigma(\langle g, q, 0\rangle))$$

for all $g \in G_q$. *This implies* $\sigma(\langle g, q, 0\rangle) = \langle c_{\sigma,q} \cdot g, q, 0\rangle$. □

Proof (cont.).

Claim

The sequence $(c_{\sigma,q})_{q \in D}$ is an element of $G_{\mathbb{I}}$ for every $\sigma \in \mathrm{Aut}(\mathcal{M}_{\mathbb{I}})$.

Proof of the Claim.

The definition of $\mathcal{M}_{\mathbb{I}}$ *yields* $H_{q,r}(\langle \mathbb{1}_{G_r}, r, 0\rangle, \langle \mathbb{1}_{G_q}, q, 0\rangle)$. *We can conclude* $H_{q,r}(\langle c_{\sigma,r}, r, 0\rangle, \langle c_{\sigma,q}, q, 0\rangle)$ *and* $h_{q,r}(c_{\sigma,r}) = c_{\sigma,q}$. □

Claim

The map

$$\Phi : \mathrm{Aut}(\mathcal{M}_{\mathbb{I}}) \longrightarrow G_{\mathbb{I}};\ \sigma \longmapsto (c_{\sigma,q})_q \in D$$

is an isomorphism of groups. □

Since there is a field K *of cardinality* $\max\{\aleph_0, \sum_{q \in D} |G_q|\}$ *and characteristic* p *with* $\mathrm{Aut}(K) \cong \mathrm{Aut}(\mathcal{M}_{\mathbb{I}})$, *this completes the proof of the theorem.* □

Free groups as inverse limits

The following result shows how free groups can be represented as inverse limits of systems of groups assuming the existence of certain *suitable* inverse systems of sets.

Theorem

Let $\mathbb{D} = \langle D, \leq_{\mathbb{D}} \rangle$ *be a directed set with the property that for every countable subset* Q *of* D *there is a* $q_* \in D$ *with* $q_* \leq_{\mathbb{D}} q$ *for all* $q \in Q$. *If*

$$\mathbb{I} = \langle \langle A_p \mid p \in D \rangle, \langle f_{p,q} \mid p, q \in D,\ p \leq_{\mathbb{D}} q \rangle \rangle$$

is an inverse system of sets over $\mathbb{D}$ *with inverse limit* $A_{\mathbb{I}_0} \neq \emptyset$ *and*

$$\mathbb{I}_{fg} = \langle \langle G_p \mid p \in D \rangle, \langle h_{p,q} \mid p, q \in D,\ p \leq_{\mathbb{D}} q \rangle \rangle$$

is the corresponding inverse system of free groups, then the inverse limit $G_{\mathbb{I}_{fg}}$ *is a free group of cardinality* $\max\{\aleph_0, |A_{\mathbb{I}_0}|\}$.

Proof.

Given $\vec{a} = (a_p)_{p \in D} \in A_{\mathbb{I}}$, *we define*

$$\vec{g}_{\vec{a}} = (x_{p,a_p})_{p \in D} \in \prod_{p \in D} G_p.$$

It is easy to see that $\vec{g}_{\vec{a}}$ *is an element of* $G_{\mathbb{I}_{fg}}$.

Given an element $\vec{g} = (g_p)_{p \in D} \in G_{\mathbb{I}_{fg}}$ *and* $p \in D$, *we let*

- $n(\vec{g}, p) < \omega$,
- $k(\vec{g}, p, 1), \ \ldots \ , k(\vec{g}, p, n(\vec{g}, p)) \in \mathbb{Z} \setminus \{0\}$,
- $a(\vec{g}, p, 1), \ \ldots \ , a(\vec{g}, p, n(\vec{g}, p)) \in A_p$

denote the uniquely determined objects with the property that the word

$$w_{\vec{g},p} \ = \ x_{p,a(\vec{g},p,1)}^{k(\vec{g},p,1)} \ \ldots \ x_{p,a(\vec{g},p,n(\vec{g},p))}^{k(\vec{g},p,n(\vec{g},p))}$$

is the unique reduced word representing g_p, *i.e.* $w_{\vec{g},p}$ *represents* g_p *and* $a(\vec{g}, p, i) \neq a(\vec{g}, p, i+1)$ *for all* $1 \leq i < n(\vec{g}, p)$.

Proof (cont.).

Claim

If $\vec{g} = (g_p)_{p \in D} \in G_{\mathbb{I}_{fg}}$ and $p, q \in D$ with $p \leq_{\mathbb{D}} q$, then $n(\vec{g}, p) \leq n(\vec{g}, q)$.

Proof of the Claim.

Since $h_{p,q}(g_q) = g_p$, we know that the word

$$w \ = \ x_{p, f_{p,q}(a(\vec{g},q,1))}^{k(\vec{g},q,1)} \ \ldots \ x_{p, f_{p,q}(a(\vec{g},q,n(\vec{g},q)))}^{k(\vec{g},q,n(\vec{g},q))}$$

also represents g_p. Hence $w_{\vec{g},p}$ can be obtained from w by a finite number of reductions. This implies $n(\vec{g}, p) \leq n(\vec{g}, q)$. □

Proof (cont.).

Claim

If $\vec{g} \in G_{\mathbb{I}_{fg}}$, then there are $p_{\vec{g}} \in D$ and $n_{\vec{g}} < \omega$ such that $n_{\vec{g}} = n(\vec{g}, p)$ for all $p \in D$ with $p_{\vec{g}} \leq_{\mathbb{D}} p$.

Proof of the Claim.

Let $\vec{g} = (g_p)_{p \in D}$ and assume, toward a contradiction, that for every $p \in D$ there is a $q \in D$ with $p \leq_{\mathbb{D}} q$ and $n(\vec{g}, p) < n(\vec{g}, q)$.
Then we can construct a sequence $\langle p_i \mid i < \omega \rangle$ such that $p_i \leq_{\mathbb{D}} p_{i+1}$ and $n(\vec{g}, p_i) < n(\vec{g}, p_{i+1})$ for all $i < \omega$. By our assumptions, there is a $p_* \in D$ with $p_i \leq_{\mathbb{D}} p_*$ for all $i < \omega$. By the above claim, we have $n(\vec{g}, p_*) > i$ for all $i < \omega$, a contradiction. $\square$

Proof (cont.).

Claim

If $\vec{g} = (g_p)_{p \in D} \in G_{\mathbb{I}_{fg}}$ and $p, q \in D$ with $p_{\vec{g}} \leq_{\mathbb{D}} p \leq_{\mathbb{D}} q$, then $a(\vec{g}, p, i) = f_{p,q}(a(\vec{g}, q, i))$ and $k(\vec{g}, p, i) = k(\vec{g}, q, i)$ for all $1 \leq i < n_{\vec{g}}$.

Proof of the Claim.

As above, let

$$w \;=\; x_{p,f_{p,q}(a(\vec{g},q,1))}^{k(\vec{g},q,1)} \;\cdots\; x_{p,f_{p,q}(a(\vec{g},q,n(\vec{g},q)))}^{k(\vec{g},q,n(\vec{g},q))}.$$

Then w is reduced, because otherwise there would be a reduced word $x_{p,a_1}^{k_0} \ldots x_{p,a_l}^{k_{l-1}}$ with $l < n(\vec{g}, q) = n(\vec{g}, p)$ representing g_p and this would contradict the choice of $w_{\vec{g},p}$. We can conclude $w = w_{\vec{g},p}$ and, again by the uniqueness of $w_{\vec{g},p}$, this yields the statements of the claim. □

Proof (cont.).

Claim

The group $G_{\mathbb{I}_{fg}}$ is generated by the set $\{\vec{g}_{\vec{a}} \mid \vec{a} \in A_{\mathbb{I}}\}$.

Proof of the Claim.

Let $\vec{g} = (g_p)_{p \in D} \in G_{\mathbb{I}}$. Set $n = n_{\vec{g}}$ and $k_i = k(\vec{g}, p_{\vec{g}}, i)$ for all $1 \leq i \leq n$. For each $p \in D$, we fix an element $\bar{p}$ of D with $p, p_{\vec{g}} \leq_{\mathbb{D}} \bar{p}$. Given $p \in D$ and $1 \leq i \leq n$, define

$$a_{\vec{g},p,i} = f_{p,\bar{p}}(a(\vec{g}, \bar{p}, i)) \in A_p.$$

An easy computation shows

$$\vec{a}_{\vec{g},i} = (a_{\vec{g},p,i})_{p \in D} \in A_{\mathbb{I}}.$$

We know that

$$g_p = h_{p,\bar{p}}(g_{\bar{p}}) = h_{p,\bar{p}}\left(x^{k_1}_{\bar{p},a(\vec{g},\bar{p},1)} \cdot \ \ldots \ \cdot x^{k_n}_{\bar{p},a(\vec{g},\bar{p},n)}\right) = x^{k_1}_{p,a_{\vec{g},p,1}} \cdot \ \ldots \ \cdot x^{k_n}_{p,a_{\vec{g},p,n}}$$

holds for all $p \in D$ and this shows $\vec{g} = \vec{g}^{\,k_1}_{\vec{a}_{\vec{g},1}} \cdot \ \ldots \ \cdot \vec{g}^{\,k_n}_{\vec{a}_{\vec{g},n}}$. $\square$

Proof (cont.).

Claim

The group $G_{\mathbb{I}_{fg}}$ is freely generated by the set $\{\vec{g}_{\vec{a}} \mid \vec{a} \in A_{\mathbb{I}}\}$.

Proof of the Claim.

Assume, toward a contradiction, that we can find $1 \leq n < \omega$, $\vec{a}_1, \ldots, \vec{a}_n \in A_{\mathbb{I}}$ *and* $k_1, \ldots, k_n \in \mathbb{Z} \setminus \{0\}$ *with*

$$\mathbb{1}_{G_{\mathbb{I}_{fg}}} \;=\; \vec{g}_{\vec{a}_1}^{\,k_1} \cdot \;\ldots\; \cdot \vec{g}_{\vec{a}_n}^{\,k_n}$$

and $\vec{a}_i \neq \vec{a}_{i+1}$ *for all* $1 \leq i < n$. *Let* $\vec{a}_i = (a_{p,i})_{p \in D}$. *Then there are* $p_1, \ldots, p_{n-1} \in D$ *with* $a_{p_i,i} \neq a_{p_i,i+1}$ *for all* $1 \leq i < n$ *and we can find a* $p \in D$ *with* $p_1, \ldots, p_{n-1} \leq_{\mathbb{D}} p$. *This implies* $a_{p,i} \neq a_{p,i+1}$. *By our assumption, the word*

$$w = x_{p,a_1}^{k_1} \cdot \;\ldots\; \cdot x_{p,a_n}^{k_n}$$

is equivalent to the trivial word. But this yields a contradiction, because w *is reduced and not trivial.* □

This completes the proof of the theorem. □

Proof of the Main Theorem.

Let λ be a cardinal with $\lambda = \lambda^{\aleph_0}$ and $\mathbb{D} = \langle [\lambda]^{\aleph_0}, \subseteq \rangle$. Then $\mathbb{D}$ satisfies the assumptions of the above theorem. Let

$$\mathbb{I}(\lambda) \; = \; \langle \langle {}^{p}2 \mid p \in [\lambda]^{\aleph_0} \rangle, \langle f_{p,q} \mid p \subseteq q \in [\lambda]^{\aleph_0} \rangle \rangle$$

denote the corresponding inverse system of sets and $\mathbb{I}(\lambda)_{fg}$ be the corresponding inverse system of free groups. Then

$$\sum_{p \in [\lambda]^{\aleph_0}} |G_{{}^{p}2}| \; = \; \lambda^{\aleph_0} \cdot 2^{\aleph_0} \; = \; \lambda$$

and the above result shows that $G_{\mathbb{I}(\lambda)_{fg}}$ is a free group of cardinality 2^{λ}.

The results of the last section now allow us to find a field K of characteristic p and cardinality λ such that the groups $\mathrm{Aut}(K)$ and $G_{\mathbb{I}(\lambda)_{fg}}$ are isomorphic. $\square$

Our statements about the consistency strength of the non-existence of certain fields are a direct consequence of the following observation.

Lemma

Let λ be a cardinal of uncountable cofinality and M be an inner model with $\lambda^+ = (\lambda^+)^{\mathrm{M}}$ and $(\lambda^{<\lambda})^{\mathrm{M}} = \lambda$. Then there is a field K of cardinality λ with the property that $\mathrm{Aut}(K)$ is a free group of cardinality greater than λ.

Proof.

Set $\mathbb{T} = \langle ({}^{<\lambda}2)^{\mathrm{M}}, \subseteq \rangle$. Then $\mathbb{T}$ is a tree of cardinality and height λ with at least λ^+-many cofinal branches. Let

$$\mathbb{I}(\mathbb{T}) = \langle \langle \mathbb{T}(\alpha) \mid \alpha < \lambda \rangle, \langle f_{\beta,\alpha} \mid \beta \leq \alpha < \lambda \rangle \rangle$$

denote the corresponding inverse system of sets over $\langle \lambda, \leq \rangle$ and $\mathbb{I}(\mathbb{T})_{fg}$ denote the corresponding inverse system of free groups. Then

$$\sum_{\alpha<\lambda} |G_{\mathbb{T}(\alpha)}| \ \leq \ \lambda \cdot \lambda \ = \ \lambda$$

and the above theorem implies that $G_{\mathbb{I}(\mathbb{T})_{fg}}$ is a free group of cardinality greater than λ. As above, this allows us to find a field K with the desired properties. $\square$

Proof of the Theorem.

Let λ be a regular uncountable cardinal and assume that there is an $x \subseteq \lambda$ such that λ^+ is not inaccessible in $\mathrm{L}[x]$. Then there is a $y \subseteq \lambda$ with $\lambda^+ = (\lambda^+)^{\mathrm{L}[y]}$. Since our assumptions imply $(\lambda^{<\lambda})^{\mathrm{L}[y]} = \lambda$, we can apply the above lemma to construct a field of cardinality λ whose automorphism group is a free group of cardinality greater than λ. □

Proof of the Theorem.

Now, let λ be a singular cardinal of uncountable cofinality and assume that there is no inner model with a Woodin cardinal. Then we can construct the *core model* K *below one Woodin cardinal*. It satisfies the *Generalized Continuum Hypothesis* and has the *covering property*. In particular, we have $\lambda^+ = (\lambda^+)^{\mathrm{K}}$ and $(\lambda^{<\lambda})^{\mathrm{K}} = \lambda$. Another application of the above lemma yields a field of cardinality λ whose automorphism group is a free group of cardinality greater than λ. □

Thank you for listening!