THE AUTOMORPHISM TOWER PROBLEM

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Bonn, 11/02/2011

If G is an infinite group, then the group $\mathrm{Aut(G)}$ of all automorphisms of G can be a very complex object, not only from the point of view of algebra, but also in a set-theoretical sense.

I will introduce an algebraic construction that illustrates this phenomenon. This construction is called the *automorphism tower*.

Let G be a group. If g is an element of G, then the map

$$\iota_g : G \longrightarrow G;\ h \longmapsto g \circ h \circ g^{-1}.$$

is an automorphism of G and we call ι_g the *inner automorphism of* G corresponding to g. We let $\mathrm{Inn(G)}$ denote the group of all inner automorphisms of G.

The map

$$\iota_G : G \longrightarrow \mathrm{Aut(G)};\ g \longmapsto \iota_g$$

is a homomorphism of groups with $\ker(\iota_G) = \mathrm{C}(G)$.

Given $g \in G$ and $\pi \in \mathrm{Aut(G)}$, an easy computation shows that

$$\iota_{\pi(g)} = \pi \circ \iota_g \circ \pi^{-1}$$

holds and this implies that $\mathrm{Inn(G)}$ is a normal subgroup of $\mathrm{Aut(G)}$.

If G is a group with trivial centre, then ι_G is an embedding of groups and the above equality implies that

$$\mathrm{C_{Aut(G)}(Inn(G))} = \{\mathrm{id_G}\}$$

holds. In particular, $\mathrm{Aut(G)}$ is a group with trivial centre in this case.

By iterating this process, we construct the *automorphism tower of a centreless group G*.

Definition

Let G be a group with trivial centre. We call a sequence $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ of groups *an automorphism tower of* G if the following statements hold.

- $G = G_0$.
- If $\alpha \in \mathrm{On}$, then G_α is a normal subgroup of $G_{\alpha+1}$ and the induced homomorphism

$$\varphi_\alpha : G_{\alpha+1} \longrightarrow \mathrm{Aut(G_\alpha)};\ g \mapsto \iota_g \restriction G_\alpha$$

  is an isomorphism.
- If $\alpha \in \mathrm{Lim}$, then $G_\alpha = \bigcup\{G_\beta \mid \beta < \alpha\}$.

In this definition, we replaced $\mathrm{Aut(G_\alpha)}$ by an isomorphic copy $G_{\alpha+1}$ that contains G_α as a normal subgroup. This allows us to take unions at limit stages. Without this isomorphic correction, we would have to take direct limits at limit stages.

By induction, we can construct such a tower for each centreless group and it is easy to show that each group G_α in such a tower is uniquely determined up to an isomorphism which is the identity on G. We can therefore speak of *the* α-th group G_α in the automorphism tower of a centreless group G.

It is natural to ask whether the automorphism tower of every centreless group eventually *terminates* in the sense that there is an ordinal α with $G_\alpha = G_{\alpha+1}$ and therefore $G_\alpha = G_\beta$ for all $\beta \geq \alpha$.

A classical result due to Helmut Wielandt shows that the automorphism tower of every finite centreless group terminates.

Theorem (H. Wielandt, 1939)

If G is a finite group with trivial centre, then there is an $n < \omega$ with $G_n = G_{n+1}$.

Simon Thomas showed that the automorphism tower of every centreless group eventually terminates by proving the following result. An application of Fodor's Lemma (and hence of the *Axiom of Choice*) lies at the heart of this argument.

Theorem (S. Thomas, 1985 & 1998)

If G is an infinite centreless group of cardinality κ, then there is an $\alpha < (2^\kappa)^+$ with $G_\alpha = G_{\alpha+1}$.

This result allows us to make the following definitions.

Definition

Given a centreless group G, we let $\tau(G)$ denote the least ordinal α with $G_\alpha = G_{\alpha+1}$. We call this ordinal the *height of the automorphism tower of* G. If κ is an infinite cardinal, then we define

$$\tau_\kappa = \mathrm{lub}\{\tau(G) \mid G \textit{ is a centreless group of cardinality } \kappa\}.$$

A result of Simon Thomas shows that for every infinite cardinal κ and every ordinal $\alpha < \kappa^+$ there is a group G of cardinality κ such that $\tau(G) = \alpha$. Since there are only 2^κ-many centreless groups of cardinality κ and $(2^\kappa)^+$ is a regular cardinal, we can combine the above results to see that

$$\kappa^+ \leq \tau_\kappa < (2^\kappa)^+$$

holds for every infinite cardinal κ.

The following open problem is the motivation for my work on this topic.

Problem (The automorphism tower problem)

Find a model $\mathcal{M}$ of ZFC and an infinite cardinal κ in $\mathcal{M}$ such that it is possible to compute the exact value of τ_κ in $\mathcal{M}$.

In the above statement, the phrase "*compute the value of τ_κ*" should be interpreted as "*give a set-theoretic characterizations of τ_κ*". Examples of such characterizations would be $\mathcal{M} \models$ "$\tau_\kappa = \kappa^+$" or $\mathcal{M} \models$ "$\tau_\kappa = 2^\kappa$".

A PROOF OF THE AUTOMORPHISM TOWER THEOREM

I will present an easy proof of Simon Thomas' *automorphism tower theorem* due to Itay Kaplan and Saharon Shelah.

Let G be a group and A be a subset of the domain of G. We define $\equiv_{G,A}$ to be the relation on G consisting of all pairs $\langle g, h \rangle$ such that there is a monomorphism

$$\varphi : \langle A \cup \{g\} \rangle \longrightarrow G$$

with $\varphi \upharpoonright A = \mathrm{id}_A$ and $\varphi(g) = h$.

Proposition

$\equiv_{G,A}$ is an equivalence relation on G. $\square$

Definition

Given a group G and a subset A of the domain of G, we call the pair $\langle G, A \rangle$ *special* if $\equiv_{G,A}$ is the identity on G.

Theorem (I. Kaplan & S. Shelah, 2009)

If $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ *is the automorphism tower of a centreless group, then the following statements hold for all* $\alpha \in \mathrm{On}$.

- $\mathrm{C}_{G_\alpha}(G_0) = \{\mathbb{1}_{G_0}\}$.
- $\langle G_\alpha, G_0\rangle$ *is a special pair.*

Proof.

Proof by induction. The cases $\alpha = 0$ *and* $\alpha \in \mathrm{Lim}$ *are trivial.*

Let $g \in \mathrm{C}_{G_{\alpha+1}}(G_0)$. *If* $h \in G_\alpha$ *and* $\varphi = \iota_g \restriction \langle G_0 \cup \{h\}\rangle$, *then* φ *is a monomorphism with* $\varphi \restriction G_0 = \mathrm{id}_{G_0}$ *and this means*

$$h \equiv_{G_\alpha, G_0} \iota_g(h).$$

By the induction hypothesis, this is equivalent to $h = \iota_g(h)$. *We can conclude* $g \in \mathrm{C}_{G_{\alpha+1}}(G_\alpha) = \{\mathbb{1}_{G_0}\}$.

Proof (cont.).

Let $g \in G_{\alpha+1}$ *and* $\varphi : \langle G_0 \cup \{g\} \rangle \longrightarrow G_{\alpha+1}$ *be a monomorphism with* $\varphi \upharpoonright G_0 = \mathrm{id}_{G_0}$.

Pick $h \in G_0$. *Then* $\iota_g(h) \in G_\alpha \cap \mathrm{dom}(\varphi)$,

$$\varphi(\iota_g(h)) = \iota_{\varphi(g)}(h) \in G_\alpha$$

and, if we define $\psi = \varphi \upharpoonright \langle G_0 \cup \{\iota_g(h)\} \rangle$, *then* $\psi : \langle G_0 \cup \{\iota_g(h)\} \rangle \longrightarrow G_\alpha$ *is a monomorphism with* $\psi \upharpoonright G_0 = \mathrm{id}_{G_0}$. *Hence*

$$\iota_g(h) \equiv_{G_\alpha, G_0} \varphi(\iota_g(h))$$

and this implies

$$\iota_g(h) = \varphi(\iota_g(h)) = \iota_{\varphi(g)}(h).$$

We can conclude $g^{-1} \circ \varphi(g) \in \mathrm{C}_{G_{\alpha+1}}(G_0) = \{\mathbb{1}_{G_0}\}$. $\square$

Let G be a group, A be a subset of the domain of G and $\mathcal{L}_A$ be the language of group theory expanded by a constant symbol $\dot{g}$ for every $g \in A$. We let $\mathcal{T}_A$ denote the set of all $\mathcal{L}_A$-terms $t \equiv t(v)$ with exactly one free variable. We interpret G as an $\mathcal{L}_A$-structure in the natural way. If $g \in G$, then we define

$$\mathsf{qft}_{G,A}(g) = \{t(v) \in \mathcal{T}_A \mid G \models \text{“}t(g) = \mathbb{1}\text{”}\}.$$

Proposition

If G is a group and A is a subset of the domain of G, then the following statements are equivalent for all $g, h \in G$.

- $\mathsf{qft}_{G,A}(g) = \mathsf{qft}_{G,A}(h)$.
- $g \equiv_{G,A} h$.

In particular, if $\langle G, A\rangle$ is a special pair, then $[g \mapsto \mathsf{qft}_{G,A}(g)]$ is an injection of G into $\mathcal{P}(\mathcal{T}_A)$.

Proof of the automorphism tower theorem.

Let G be an infinite centreless group of cardinality κ and assume, toward a contradiction, that there is a sequence $\langle g_\alpha \mid \alpha < (2^\kappa)^+ \rangle$ with $g_\alpha \in G_{\alpha+1} \setminus G_\alpha$ for every $\alpha < (2^\kappa)^+$.

It is easy to see that

$$\mathsf{qft}_{G_{\alpha+1},G_0}(g_\alpha) = \mathsf{qft}_{G_{\beta+1},G_0}(g_\alpha)$$

holds for all $\alpha \leq \beta < (2^\kappa)^+$. This shows that the function

$$i : (2^\kappa)^+ \longrightarrow \mathcal{P}(\mathcal{T}_{G_0});\ \alpha \longmapsto \mathsf{qft}_{G_{\alpha+1},G_0}(g_\alpha)$$

is an injection. Since $\mathcal{P}(\mathcal{T}_{G_0})$ has cardinality 2^κ, we have reached a contradiction. $\square$

BETTER UPPER BOUNDS FOR THE HEIGHTS OF AUTOMORPHISM TOWERS

Remember that

$$\kappa^+ \ \leq \ \tau_\kappa \ < \ (2^\kappa)^+$$

holds for every infinite cardinal κ.

The aim of my work was to find better upper bounds for τ_κ that are uniformly definable in parameter κ.

The following consistency result shows that there is no better estimate for τ_κ by cardinals.

Theorem (W. Just, S. Shelah & S. Thomas, 1999)

Assume that the (GCH) *holds in the ground model* V*. Let* κ *be an uncountable regular cardinal with* $\kappa = \kappa^{<\kappa}$ *and* ν *be a cardinal with* $\kappa < \mathrm{cof}(\nu)$*. If* $\alpha < \nu^+$*, then there is a partial order* $\mathbb{P}$ *with the following properties.*

- $\mathbb{P}$ *is* $<\kappa$*-closed and satisfies the* κ^+*-chain condition.*
- *If* F *is* $\mathbb{P}$*-generic over* V*, then* $(2^\kappa) = \nu$ *and there is a centreless group* $G \in \mathrm{V}[F]$ *such that* $\tau(G) = \alpha$ *holds in* $\mathrm{V}[F]$*.*

Corollary

It is consistent with the axioms of set theory that there is a cardinal κ *with* $\tau_\kappa > 2^\kappa$*.*

In another direction, Simon Thomas showed that the cardinality of τ_κ can consistently be smaller than 2^κ.

Theorem (S. Thomas, 1998)

Let κ, λ and ν be regular cardinals with $\kappa = \kappa^{<\kappa}$, $\kappa \leq \lambda$, $2^\lambda = \lambda^+$, $\nu \geq \lambda^{++}$ and $\nu = \nu^\lambda$. If G is $\mathrm{Add}(\kappa, \nu)$-generic over V, then $\tau_\lambda \leq \lambda^{++}$ and $2^\lambda = \nu$ hold in $\mathrm{V}[G]$.

These result show that a better upper bound for τ_κ should be a uniformly definable *ordinal* in the interval $[\kappa^+, (2^\kappa)^+]$ that is consistently smaller than 2^κ for regular κ.

Results of Itay Kaplan and Saharon Shelah provide an example of such a bound.

Definition

Given a set X, we let $\mathrm{L(X)}$ define the least inner model containing X and define

$$\theta_X = \mathrm{lub}\{\alpha \in \mathrm{On} \mid (\exists f \in \mathrm{L}(X))\ f : X \longrightarrow \alpha \textit{ is a surjection}\}.$$

Theorem (I. Kaplan & S. Shelah, 2009)

If κ is an infinite cardinal, then $\tau_\kappa < \theta_{\mathcal{P}(\kappa)}$.

By refining methods developed in the proof of this result, it is possible to find a better upper bound for τ_κ with the help of *abstract recursion theory*, i.e. the *theory of admissible sets*.

To state this result, we need to introduce some concepts from this theory.

Definition

A set $\mathbb{A}$ is *admissible* if it has the following properties.

- $\mathbb{A}$ is nonempty, transitive and closed under pairing and union.
- $\langle \mathbb{A}, \in \rangle$ satisfies Δ_0-*Seperation*, i.e.

  $$(\forall x_0, \ldots, x_n)(\exists y)(\forall z)\ [z \in y \leftrightarrow [y \in x_0 \wedge \varphi(x_0, \ldots, x_n, z)]]$$

  holds in $\langle \mathbb{A}, \in \rangle$ for every Δ_0-formula $\varphi(u_0, \ldots, u_n, v)$.
- $\langle \mathbb{A}, \in \rangle$ satisfies Δ_0-*Collection*, i.e.

  $$\begin{aligned}(\forall x_0, \ldots, x_n)[(\forall y \in x_0)(\exists z)\ \varphi(x_0, \ldots, x_n, y, z) \\ \rightarrow (\exists w)(\forall y \in x_0)(\exists z \in w)\ \varphi(x_0, \ldots, x_n, y, z)]\end{aligned}$$

  holds in $\langle \mathbb{A}, \in \rangle$ for every Δ_0-formula $\varphi(u_0, \ldots, u_n, v_0, v_1)$.

Definition

Given a set x, an ordinal α is x-admissible if there is an admissible set $\mathbb{A}$ with $x \in \mathbb{A}$ and $\alpha = \mathbb{A} \cap \mathrm{On}$.

We are now ready to state the result.

Theorem (P.L.)

Let κ be an infinite cardinal and α be $\mathcal{P}(\kappa)$-admissible. If $\tau_\kappa \neq \alpha + 1$, then $\tau_\kappa < \alpha$.

Since there are many $\mathcal{P}(\kappa)$-admissible ordinals below $\theta_{\mathcal{P}(\kappa)}$, this result improves the Kaplan/Shelah-bound.

In the following, I sketch a proof of this result. The following statement is the central idea behind this proof.

Proposition

Let $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ be the automorphism tower of a centreless group, λ be a limit ordinal of uncountable cofinality and $g \in G_{\lambda+1}$. If the set

$$B^g_{\lambda,\alpha} = \{\min\{\beta < \lambda \mid \iota_g(h), \iota_{g^{-1}}(h) \in G_{\beta+1}\} \mid h \in G_\alpha\}$$

is bounded in λ for every $\alpha < \lambda$, then $g \in G_\lambda$.

Proof.

We can construct a strictly increasing sequence $\langle \alpha_n < \lambda \mid n < \omega\rangle$ with $B^g_{\lambda,\alpha_n} \subseteq \alpha_{n+1}$ for all $n < \omega$.

If $\alpha = \sup_{n<\omega} \alpha_n$, then $\alpha \in \mathrm{Lim} \cap \lambda$ and $\iota_g \restriction G_\alpha \in \mathrm{Aut}(\mathrm{G}_\alpha)$. There is a $g_* \in G_{\alpha+1}$ with $\iota_{g_*} \restriction G_\alpha = \iota_g \restriction G_\alpha$. This implies

$$g^{-1} \circ g_* \in \mathrm{C}_{G_{\lambda+1}}(G_0) = \{\mathbb{1}_{G_0}\}$$

and $g = g_* \in G_{\alpha+1} \subseteq G_\lambda$. $\square$

This proposition directly implies the following corollary.

Corollary

Let $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ be the automorphism tower of a centreless group G and κ be an infinite regular cardinal. If $|G_\alpha| < \kappa$ holds for every $\alpha < \kappa$, then $\tau(G) \leq \kappa$. $\square$

The idea behind the proof of the new bound is to take the statement of this proposition and replace *cardinality* by *complexity* and *regularity* by *admissibility*. The first replacement is achieved by the following theorem.

Theorem

Let $\mathbb{A}$ be an admissible set and $\alpha = \mathbb{A} \cap \mathrm{On}$. If κ is an infinite cardinal with $\mathcal{P}(\kappa) \in \mathbb{A}$ and G is a centreless group with domain κ, then there is an automorphism tower $\langle G_\beta \mid \beta \in \mathrm{On}\rangle$ such that $G_{\bar{\alpha}} \in \mathbb{A}$ for every $\bar{\alpha} < \alpha$ and the following statements hold.

- *G_α is definable in $\langle \mathbb{A}, \in\rangle$ by a Σ_1-formula with parameters.*
- *The map*

  $$t : \alpha \longrightarrow \mathbb{A};\ \bar{\alpha} \longmapsto G_{\bar{\alpha}}$$

  is definable in $\langle \mathbb{A}, \in\rangle$ by a Σ_1-formula with parameters.
- *Given $g \in G_{\alpha+1}$, the maps $\iota_g \restriction G_\alpha$ and $\iota_{g^{-1}} \restriction G_\alpha$ are definable in $\langle \mathbb{A}, \in\rangle$ by Σ_1-formulae with parameters.*

The following basic result from admissible set theory will make the second replacement possible.

Theorem

Let $\mathbb{A}$ be an admissible set and $f : \mathbb{A} \longrightarrow \mathbb{A}$ be a function that is definable in $\langle \mathbb{A}, \in \rangle$ by a Σ_1-formula with parameters. If $x \in \mathbb{A}$, then there is an $y \in \mathbb{A}$ with $f\text{"}x \subseteq y$.

I outline how the new bound can be derived from these theorems.

Sketch of the proof.

Let κ be an infinite cardinal, α be the least $\mathcal{P}(\kappa)$-admissible ordinal and $\mathbb{A}$ be an admissible set with $\mathcal{P}(\kappa) \in \mathbb{A}$ and $\alpha = \mathbb{A} \cap \mathrm{On}$. Then $\mathrm{cof}(\alpha) > \kappa$.

Fix a centreless group G with domain κ and let $\langle G_\beta \mid \beta \in \mathrm{On} \rangle$ be the automorphism tower produced by the above theorem with respect to $\mathbb{A}$.

Fix $g \in G_{\alpha+1}$. If $\bar{\alpha} < \alpha$ and

$$c^g_{\bar{\alpha}} : G_{\bar{\alpha}} \longrightarrow \alpha;\ h \longmapsto \min\{\beta < \alpha \mid \iota_g(h), \iota_{g^{-1}}(h) \in G_\beta\},$$

then the function $c^g_{\bar{\alpha}}$ is definable in $\langle \mathbb{A}, \in \rangle$ by a Σ_1-formula with parameters. Since $G_{\bar{\alpha}} \in \mathbb{A}$, the second theorem implies that the set $B^g_{\alpha,\bar{\alpha}} = \{c^g_{\bar{\alpha}}(h) \mid h \in G_{\bar{\alpha}}\}$ is bounded in α for all $\bar{\alpha} < \alpha$. By the above Proposition, this shows $g \in G_\alpha$. We can conclude $\tau_\kappa \leq \alpha + 1$.

Assume $\tau_\kappa \leq \alpha$. Then the set $\mathcal{G}_\kappa$ consisting of all centreless groups with domain κ is an element of $\mathbb{A}$ and the function

$$\tau : \mathcal{G}_\kappa \longrightarrow \alpha;\ G \longmapsto \tau(G)$$

is definable in $\langle \mathbb{A}, \in \rangle$ by a Σ_1-formula with parameters. Another application of the second theorem yields $\tau_\kappa < \alpha$. □

CHANGING THE HEIGHTS OF AUTOMORPHISM TOWERS

One of the reasons why it is so difficult to compute the value of τ_κ is that there are groups whose automorphism tower heights depend on the model of set theory in which they are computed.

Theorem (S. Thomas, 1998)

- *There is a partial order $\mathbb{P}$ satisfying the countable chain condition and a centreless group G with "$\tau(G) = 0$" and $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\tau(\check{G}) \geq 1$".*
- *There is a centreless group H with "$\tau(H) = 2$" and $\mathbb{1}_{\mathbb{Q}} \Vdash$ "$\tau(\check{H}) = 1$" for every notion of forcing $\mathbb{Q}$ that adds a new real.*

The following result suggests that there is no nontrivial correlation between the heights of automorphism towers of a centreless group computed in different models of set theory.

Theorem (J. Hamkins & S. Thomas, 2000)

It is consistent with the axioms of set theory that for every infinite cardinal κ and every ordinal $\alpha < \kappa$, there exists a centreless group G with the following properties.

- $\tau(G) = \alpha$.
- *Given $0 < \beta < \kappa$, there exists a partial order $\mathbb{P}$ preserving cofinalities and cardinalities with $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\tau(\check{G}) = \check{\beta}$".*

Gunter Fuchs and I strengthened this result by construction groups whose automorphism tower can be iteratively changed by forcing.

Theorem (G. Fuchs & P.L.)

It is consistent with the axioms of set theory that for every infinite cardinal κ there is a centreless group G with "$\tau(G) = 0$" and the property that for every function $s : \kappa \longrightarrow (\kappa \setminus \{0\})$ there is a sequence $\langle \mathbb{P}_\alpha \mid 0 < \alpha < \kappa \rangle$ of partial orders such that the following statements hold.

- *For all $0 < \alpha < \kappa$, $\mathbb{P}_\alpha$ preserves cardinalities and cofinalities.*
- *For all $0 < \alpha < \beta < \kappa$, there is a partial order $\mathbb{Q}$ with $\mathbb{P}_\beta = \mathbb{P}_\alpha \times \mathbb{Q}$.*
- *For all $\alpha < \kappa$, we have $\mathbb{1}_{\mathbb{P}_{\alpha+1}} \Vdash$ "$\tau(\check{G}) = \check{s}(\check{\alpha})$".*
- *If $0 < \alpha < \kappa$ is a limit ordinal, then $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash$ "$\tau(\check{G}) = 1$".*

In another direction, it is also possible to construct models of set theory that contain groups with *unbounded potential automorphism tower height*.

Theorem (P.L.)

It is consistent with the axioms of set theory that there exists a centreless group G with the property that for every ordinal α there is a partial order $\mathbb{P}$ preserving cofinalities and cardinalities with $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\tau(\check{G}) \geq \check{\alpha}$".

Sketch of the proof.

Assume that the (GCH) *holds. If* α *is an arbitrary ordinal, then the Just/Shelah/Thomas result shows that there is a* σ*-closed partial order* $\mathbb{P}_\alpha$ *satisfying the* $\aleph_2$*-chain condition with*

$$\mathbb{1}_{\mathbb{P}_\alpha} \Vdash (\exists G) \left[G \text{ is a centreless group of cardinality } \aleph_1 \text{ with } \tau(G) = \check{\alpha}\right].$$

There is a partial order $\mathbb{Q}_\alpha$ *of cardinality* $\aleph_2$ *and* $\mathbb{Q}_\alpha$*-names* $\dot{G}_\alpha$, $\dot{x}_\alpha$ *and* $\dot{\mathbb{Q}}_\alpha$ *with the following properties.*

- $\dot{x}_\alpha$ *is a* $\mathbb{Q}_\alpha$*-nice name for a subset of* ω_1*.*
- $\mathbb{1}_{\mathbb{Q}_\alpha} \Vdash$ *"*$\dot{\mathbb{Q}}_\alpha$ *is a partial order" and the partial orders* $\mathbb{P}_\alpha$ *and* $\mathbb{Q}_\alpha * \dot{\mathbb{Q}}_\alpha$ *are forcing equivalent.*
- *If* F *is* $\mathbb{Q}_\alpha$*-generic over* V *and* $G = \dot{G}_\alpha^F$*, then* G *is a centreless group with domain* ω_1*,* $\dot{x}_\alpha^F$ *codes the group operation of* G *and* $\mathbb{1}_{\dot{\mathbb{Q}}_\alpha^F} \Vdash$ *"*$\tau(\check{G}) = \check{\alpha}$*" holds in* $\mathrm{V}[F]$*.*

Up to isomorphisms, there are only $\aleph_3$-many partial orders of size $\aleph_2$ and there is an $\alpha \in \mathrm{On}$ such that $\mathcal{C} = \{\beta \in \mathrm{On} \mid \mathbb{Q}_\alpha \cong \mathbb{Q}_\beta\}$ is a proper class. For each $\beta \in \mathcal{C}$, we let $\dot{y}_\beta$ denote the $\mathbb{Q}_\alpha$-nice name corresponding to $\dot{x}_\beta$. Again, there are only set-many $\mathbb{Q}_\alpha$-nice names for subsets of ω_1 and there is a $\beta \in \mathcal{C}$ such that $\mathcal{D} = \{\gamma \in \mathcal{C} \mid \dot{y}_\beta = \dot{y}_\gamma\}$ is a proper class.

Let F be $\mathbb{Q}_\beta$-generic over V, $G = \dot{G}_\beta^F$ and α be an arbitrary ordinal. There is a $\gamma \in \mathcal{D}$ with $\gamma \geq \alpha$. We can find an $F' \in \mathrm{V}[F]$ such that F' is $\mathbb{Q}_\gamma$-generic over V, $\mathrm{V}[F] = \mathrm{V}[F']$ and $\dot{x}_\beta^F = \dot{x}_\gamma^{F'}$. In particular, $\dot{G}_\gamma^{\bar{F}} = \dot{G}_\beta^F = G$ and $\mathbb{1}_{\dot{\mathbb{Q}}_\gamma^{F'}} \Vdash$ "$\tau(\check{G}) = \check{\gamma} \geq \check{\alpha}$" holds in $\mathrm{V}[F]$. $\square$

AUTOMORPHISM TOWERS OF COUNTABLE GROUPS

All groups appearing in the above non-absoluteness results are uncountable.

The following result shows that it is not possible to have similar results for countable groups.

Theorem (P.L.)

Let $\mathcal{M}$ be a transitive model of ZFC and G be a centreless group contained in $\mathcal{M}$. If G is countable in $\mathcal{M}$ and $\tau(G)^{\mathcal{M}} > 1$, then $\tau(G) > 1$.

The proof of this result uses a technique from the theory of *Polish groups* called *automatic continuity*.

Let G be a countable centreless group. We equip $\mathrm{Aut(G)}$ with the topology induced by basic open sets of the form

$$U_{X,\pi} = \{\sigma \in \mathrm{Aut(G)} \mid \pi \restriction X = \sigma \restriction X\}$$

with $\pi \in \mathrm{Aut(G)}$ and $X \subseteq_{fin} G$. This topology is a Polish group topology. It is called the *function topology on* $\mathrm{Aut(G)}$. We will show that every automorphism of $\mathrm{Aut(G)}$ is continuous with respect to this topology. The following classical theorem is the starting point of this argument.

Theorem (G. Mackey, 1957)

The following statements are equivalent for a Polish group G.

- *G has a unique Polish group topology.*
- *There is a countable point-separating family of subsets of the domain of G whose members are Borel with respect to any Polish group topology on G.*

Corollary

If G is a countable centreless group, then $\mathrm{Aut(G)}$ has a unique Polish group topology.

Proof.

Given a term $t \equiv t(u) \in \mathcal{T}_{\mathrm{Inn(G)}}$, we let C_t denote the set of all $\pi \in \mathrm{Aut(G)}$ with $\mathrm{Aut(G)} \models$ "$\mathrm{t}(\pi) = \mathbb{1}$". An easy induction shows that sets of the form C_t are closed with respect to any Hausdorff group topology on G. Moreover, the family $\{C_t \mid t \in \mathcal{T}_{\mathrm{Inn(G)}}\}$ separates points, because $\langle \mathrm{Aut(G)}, \mathrm{Inn(G)} \rangle$ is a special pair. By Mackey's theorem, the function topology is the unique Polish group topology on G. □

Corollary

Every automorphism of $\mathrm{Aut(G)}$ is continuous with respect to the function topology on $\mathrm{Aut(G)}$. □

Sketch of the proof.

Let $\mathcal{M}$ be a transitive ZFC*-model and* $G \in \mathcal{M}$ *be a centreless group that is countable in* $\mathcal{M}$*. By* $\mathbf{\Sigma}^1_1$*-absoluteness,* $\mathrm{Aut(G)}^{\mathcal{M}}$ *is dense in* $\mathrm{Aut(G)}$ *with respect to the function topology. Moreover, if* $\xi \in \mathrm{Aut(Aut(G))}^{\mathcal{M}}$*, then*

$$\xi : \mathrm{Aut(G)}^{\mathcal{M}} \longrightarrow \mathrm{Aut(G)}$$

is continuous with respect to the function topology and the induced subspace topology, because ξ *is continous with respect to the function topology from* $\mathcal{M}$ *and this topology is contained in the subspace topology.*

Then there is a homomorphism

$$\xi^* : \mathrm{Aut(G)} \longrightarrow \mathrm{Aut(G)}$$

such that $\xi^*(\lim_{n\to\infty} g_n) = \lim_{n\to\infty} \xi(g_n)$ *holds for every Cauchy-sequence* $(g_n)_n$ *in* $\mathrm{Aut(G)}^{\mathcal{M}}$*.*

The assignment $[\xi \mapsto \xi^*]$ *induces a monomorphism*

$$\eta : G_2^{\mathcal{M}} \longrightarrow G_2$$

with $\eta \restriction G_0 = \mathrm{id}_{G_0}$.

Assume, toward a contradiction, that $\tau(G) \leq 1$ *and there is a* $\xi \in G_2^{\mathcal{M}} \setminus G_1^{\mathcal{M}}$. *If* $g \in G$, *then there is an* $h \in G$ *with* $\iota_{\eta(\xi)}(g) = h$, *because* $\eta(\xi) \in G_1$. *This means*

$$\eta(h) = h = \iota_{\eta(\xi)}(g) = \eta(\iota_\xi(g))$$

and hence $h = \iota_\xi(g)$. *This argument shows that* $\iota_\xi \restriction G \in \mathrm{Aut(G)}^{\mathcal{M}}$ *and there is a* $g \in G_1^{\mathcal{M}}$ *with* $\iota_g \restriction G = \iota_\xi \restriction G$. *We can conclude* $g^{-1} \circ \xi \in \mathrm{C}_{G_2^{\mathcal{M}}}(G) = \{\mathbb{1}_G\}$ *and* $\xi \in G_1^{\mathcal{M}}$, *a contradiction.* $\square$

Thank you for listening!