Descriptive set theory at uncountable cardinals

$\mathbf{\Delta}^1_1$-subsets of ${}^\kappa\kappa$

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Motivation

Let κ be an uncountable regular cardinal with $\kappa = \kappa^{<\kappa}$. The space of all functions $f : \kappa \longrightarrow \kappa$ is called generalized Baire Space for κ. We want to study the definable subsets of this space and their regularity properties.

Notation

- If X is a non-empty set, $n < \omega$ and $A \subseteq X^{n+1}$, then we define $\exists^x A = \{\langle x_0, \ldots, x_{n-1}\rangle \in X^n \mid (\exists x_n)\ \langle x_0, \ldots, x_n\rangle \in A\}$.
- If $\lambda \in \mathrm{On}$, then we let ${}^\lambda X$ denote the set of all functions f with $\mathrm{dom}(f) = \lambda$ and $\mathrm{ran}(f) \subseteq X$. Set ${}^{<\lambda}X = \bigcup_{\alpha<\lambda} {}^\alpha X$.
- If λ is closed under Gödel-Pairing $\prec\cdot,\cdot\succ$, $f \in {}^\lambda X$ and $\alpha < \lambda$, then we define $(f)_\alpha$ to be the unique function $g \in {}^\lambda X$ with $g(\beta) = f(\prec\alpha, \beta\succ)$ for all $\beta < \lambda$.

Notation (cont.)

- Given a non-empty set X, we call a set T a *tree on* X^n if $T \subseteq ({}^{<\gamma}X)^n$ for some $\gamma \in \mathrm{On}$ and the following statements hold.
  - For all $\langle s_0, \dots, s_{n-1} \rangle \in T$, $\mathrm{lh}(s_0) = \cdots = \mathrm{lh}(s_{n-1})$.
  - If $\langle s_0, \dots, s_{n-1} \rangle \in T$ and $\alpha < \mathrm{lh}(s_0)$, then $\langle s_0 \restriction \alpha, \dots, s_{n-1} \restriction \alpha \rangle \in T$.
- We call $|T| = \sup\{\mathrm{dom}(t) + 1 \mid t \in T\}$ the *height of* T.
- A tuple $\langle x_0, \dots, x_{n-1} \rangle \in \left({}^{|T|}X\right)^n$ is a *cofinal branch through* T if $\langle x_0 \restriction \alpha, \dots, x_{n-1} \restriction \alpha \rangle \in T$ for all $\alpha < |T|$.
- We let $[T]$ denote the set of all cofinal branches through T.
- If T is a tree on X^{n+1} of height λ, then we define

$$\rho[T] = \exists^x [T] \subseteq ({}^\lambda X)^n.$$

Projective subsets of ${}^\kappa\kappa$

Definition

Let κ be an infinite cardinal.

- A subset A of $({}^\kappa\kappa)^n$ is a $\mathbf{\Sigma}^1_1$-subset if there is a tree T on κ^{n+1} of height κ with $A = \rho[T]$.
- A subset A of $({}^\kappa\kappa)^n$ is a $\mathbf{\Pi}^1_k$-subset if $({}^\kappa\kappa)^n \setminus A$ is a $\mathbf{\Sigma}^1_k$-subset.
- A subset A of $({}^\kappa\kappa)^n$ is a $\mathbf{\Sigma}^1_{k+1}$-subset if there is a $\mathbf{\Pi}^1_k$-subset B of $({}^\kappa\kappa)^{n+1}$ with $A = \exists^x B$.
- A subset A of $({}^\kappa\kappa)^n$ is a $\mathbf{\Delta}^1_k$-subset if it is both a $\mathbf{\Sigma}^1_k$-subset and a $\mathbf{\Pi}^1_k$-subset.

Proposition

Let κ be an uncountable regular cardinal with $\kappa = \kappa^{<\kappa}$ and $A \subseteq {}^\kappa\kappa$. The following statements are equivalent.

- *A is a $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa$.*
- *A is $\Sigma_1(H_{\kappa^+}, \{x\})$ for some $x \in {}^\kappa\kappa$.*

Harrington's Theorem

The motivation of this work is to find a version of the following result for uncountable regular cardinals κ with $\kappa^{<\kappa} = \kappa$.

Theorem (Harrington, 1977)

Assume $\omega_1 = \omega_1^{\mathrm{L}}$. For every subset A of ${}^\omega\omega$, there is a partial order $\mathbb{P}$ with the following properties.

- *$\mathbb{P}$ satisfies the countable chain condition.*
- *If G then is $\mathbb{P}$-generic over V, then A is a $\mathbf{\Pi}^1_2$-subset of ${}^\omega\omega$ in $\mathrm{V}[G]$.*

It turns out that assumptions like "$\omega_1 = \omega_1^{\mathrm{L}}$" are not needed in the uncountable version. We state the main result.

Main Result

Theorem

Let κ be a regular uncountable cardinal with $\kappa = \kappa^{<\kappa}$. If A is a subset of ${}^\kappa\kappa$, then there is a partial order $\mathbb{P}$ with the following properties.

- *$\mathbb{P}$ is $< \kappa$-closed, satisfies the κ^+-chain condition and has cardinality at most 2^κ.*
- *If G is $\mathbb{P}$-generic over V, then A is a $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa$ in $\mathrm{V}[G]$.*

Absoluteness

The proof of this result shows that this coding has certain absoluteness properties in $\mathrm{V}[G]$: There are trees $T_0, T_1 \in \mathrm{V}[G]$ on $\kappa \times \kappa$ of height κ and a non-trivial class Γ of $< \kappa$-closed forcings that satisfy the κ^+-chain condition such that $\rho[T_0]^{\mathrm{V}[G][G']} = A$ and $\rho[T_1]^{\mathrm{V}[G][G']} = ({}^\kappa\kappa)^{\mathrm{V}[G][G']} \setminus A$ holds whenever G' is $\mathbb{Q}$-generic over $\mathrm{V}[G]$ with $\mathbb{Q} \in \Gamma$.

The Cohen-Forcing $\mathrm{Fn}(\kappa, 2, \kappa)$ is contained in this class Γ and this allows us to analyze certain regularity properties of A in $\mathrm{V}[G]$.

Moreover, this absoluteness allows us to prove the following result.

$\mathbf{\Delta}^1_2$-well-orderings of ${}^\kappa\kappa$

Theorem

If κ is a regular uncountable cardinal with $\kappa = \kappa^{<\kappa}$, then there is a partial order $\mathbb{P}$ with the following properties.

- *$\mathbb{P}$ is $<\kappa$-closed, satisfies the κ^+-chain condition and has cardinality at most 2^κ.*
- *If G is $\mathbb{P}$-generic over V, then there is a well-ordering of $({}^\kappa\kappa)^{\mathrm{V}[G]}$ whose graph is a $\mathbf{\Delta}^1_2$-subset of ${}^\kappa\kappa \times {}^\kappa\kappa$ in $\mathrm{V}[G]$.*

$\mathbf{\Delta}^1_1$-well-orderings of ${}^\kappa\kappa$

It is natural to ask whether the above result is optimal. If $2^\kappa = \kappa^+$ holds, then it is possible to modify the proof of the last result to force a $\mathbf{\Delta}^1_1$-well-ordering of ${}^\kappa\kappa$.

Theorem

If κ is a regular uncountable cardinal with $\kappa = \kappa^{<\kappa}$ and $2^\kappa = \kappa^+$, then there is a partial order $\mathbb{P}$ with the following properties.

- *$\mathbb{P}$ is $< \kappa$-closed, satisfies the κ^+-chain condition and has cardinality at most 2^κ.*
- *If G is $\mathbb{P}$-generic over V, then there is a well-ordering of $({}^\kappa\kappa)^{\mathrm{V}[G]}$ whose graph is a $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa \times {}^\kappa\kappa$ in $\mathrm{V}[G]$.*

In this talk, I would like to present …

- … a forcing that codes subsets of ${}^\kappa\kappa$ into the cofinal branches of a κ-tree and the strong absoluteness properties of this forcing. This absoluteness will replace assumptions like "$\omega_1 = \omega_1^{\mathrm{L}}$".
- … the idea behind the construction of the forcing that produces a $\mathbf{\Delta}^1_2$-well-ordering of ${}^\kappa\kappa$. Using the above forcing, I will outline the proof of a *baby-version* of the well-ordering result that forces a $\mathbf{\Delta}^1_3$-well-ordering of ${}^\kappa\kappa$.

Kurepa Tree Coding

Theorem

Let κ be a regular uncountable cardinal with $\kappa^{<\kappa} = \kappa$. For every subset A of ${}^\kappa\kappa$, there is a partial order $\mathbb{P}$ with the following properties.

- *$\mathbb{P}$ is $< \kappa$-closed, satisfies the κ^+-chain condition and has cardinality at most 2^κ.*
- *If G is $\mathbb{P}$-generic over V, then there is a tree $T \in \mathrm{V}[G]$ on $\kappa \times \kappa$ of height κ such that*

$$\mathrm{V}[G] \models \left[\mathbb{1}_{\mathbb{Q}} \Vdash \check{A} = \rho[\check{T}]\right]$$

holds for all $< \kappa$-closed partial orders $\mathbb{Q} \in \mathrm{V}[G]$.

The forcing $\mathbb{P}(A)$

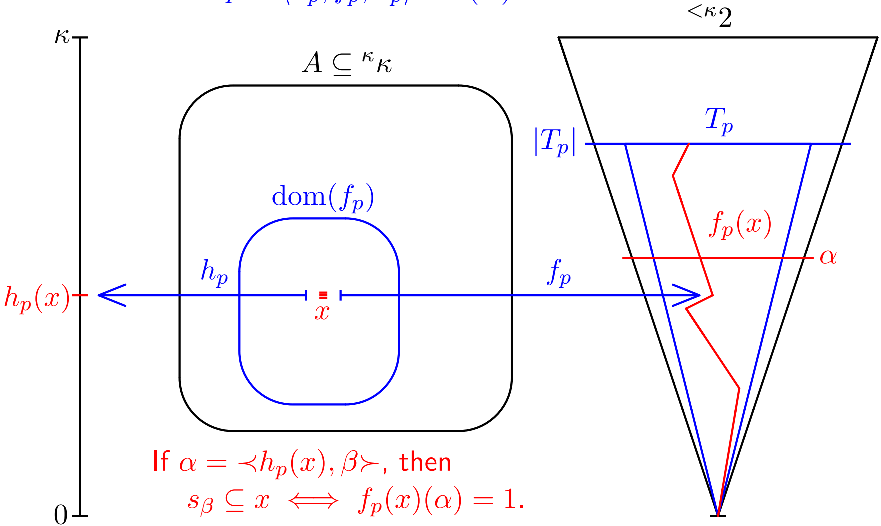
Remember that a tree T on X^n of height κ is a κ-tree if the α-th level $T_\alpha = \{t \in T \mid \mathrm{lh}(t) = \alpha\}$ has cardinality less than κ for all $\alpha < \kappa$. A κ-Kurepa tree is a κ-tree that has at least κ^+-many distinct cofinal branches. Given $A \subseteq {}^\kappa\kappa$, we will construct a forcing $\mathbb{P}(A)$ that adds a binary κ-tree whose cofinal branches code the individual elements of A. This forcing is a modification of the standard forcing that adds a κ-Kurepa tree.

From now on, we fix a regular uncountable cardinal κ with $\kappa = \kappa^{<\kappa}$ and an enumeration $\langle s_\alpha \mid \alpha < \kappa\rangle$ of ${}^{<\kappa}\kappa$ with $\mathrm{lh}(s_\alpha) \leq \alpha$ for all $\alpha < \kappa$ and $\{\alpha < \kappa \mid s = s_\alpha\}$ unbounded in κ for all $s \in {}^{<\kappa}\kappa$.

Given $A \subseteq {}^\kappa\kappa$, we define $\mathbb{P}(A)$ to be the partial order consisting of conditions $p = \langle T_p, f_p, h_p \rangle$ with the following properties.

- T_p is a subtree of ${}^{<\kappa}2$ that satisfies the following statements.
  - T_p has cardinality less than κ.
  - Each $t \in T_p$ with $\mathrm{dom}(t) + 1 < |T_p|$ has two immediate successors in T_p.
  - Each $t \in T_p$ is contained in a cofinal branch through T_p.
- $f_p : A \xrightarrow{part} [T_p]$ is a partial function such that $\mathrm{dom}(f_p)$ has cardinality less than κ.
- $h_p : A \xrightarrow{part} \kappa$ is a partial function with the following properties.
  - $\mathrm{dom}(h_p) = \mathrm{dom}(f_p)$.
  - For all $x \in \mathrm{dom}(h_p)$ and $\alpha, \beta < |T_p|$ with $\alpha = \prec h_p(x), \beta \succ$, we have
$$s_\beta \subseteq x \iff f_p(x)(\alpha) = 1.$$

$p = \langle T_p, f_p, h_p \rangle \in \mathbb{P}(A)$



If $\alpha = \prec h_p(x), \beta \succ$, then
$s_\beta \subseteq x \iff f_p(x)(\alpha) = 1.$

We define $p \leq_{\mathbb{P}(A)} q$ to hold if the following statements are satisfied.

- T_p is an end-extension of T_q.
- For all $x \in \mathrm{dom}(f_q)$, $x \in \mathrm{dom}(f_p)$ and $f_q(x)$ is an initial segment of $f_p(x)$.
- $h_q = h_p \restriction \mathrm{dom}(h_q)$.

Lemma

$\mathbb{P}(A)$ is $< \kappa$-closed, satisfies the κ^+-chain condition and has cardinality at most 2^κ. $\square$

The next lemma will allow us to show that various subset of $\mathbb{P}(A)$ are dense.

Lemma

Fix a condition $p \in \mathbb{P}(A)$ and sequences $\vec{c}$ and $\vec{t}$ as follows.

- $\vec{c} = \langle c_x \in {}^\kappa 2 \mid x \in \operatorname{dom}(f_p)\rangle$.
- $\vec{t} = \langle t_\lambda \in {}^\lambda 2 \mid \lambda \in \operatorname{Lim}, |T_p| \leq \lambda < \kappa\rangle$.

There is a $\leq_{\mathbb{P}(A)}$-descending sequence $\langle p_\mu \in \mathbb{P}(A) \mid |T_p| \leq \mu < \kappa\rangle$ such that $p = p_{|T_p|}$ and the following statements hold for all $|T_p| \leq \mu < \kappa$.

- $\operatorname{dom}(f_{p_\mu}) = \operatorname{dom}(f_p)$ *and* $|T_{p_\mu}| = \mu$.
- *If* $x \in \operatorname{dom}(f_p)$ *and* $\mu \neq \prec h_p(x), \beta \succ$ *for all* $\beta < \kappa$, *then* $f_{p_{\mu+1}}(x)(\mu) = c_x(\mu)$.
- *If* $\mu \in \operatorname{Lim}$ *and* $t_\mu \in T_{p_{\mu+1}}$, *then* $t_\mu \in \operatorname{ran}(f_{p_\mu})$.

Proof.

We construct the sequences inductively. If $\mu \in \mathrm{Lim}$, then we define $T_{p_\mu} = \bigcup\{T_{p_{\bar\mu}} \mid |T_p| \leq \bar\mu < \mu\}$. Given $x \in \mathrm{dom}(f_p)$, we define $f_{p_\mu}(x) = \bigcup\{f_{p_{\bar\mu}}(x) \mid |T_p| \leq \bar\mu < \mu\}$.
If $\mu = \bar\mu + 1$ with $\bar\mu \notin \mathrm{Lim}$, then $T_{p_{\bar\mu}}$ has a maximal level and there is only one suitable tree T_{p_μ} of height μ end-extending it. For all $x \in \mathrm{dom}(f_p)$, we define $f_{p_\mu}(x)$ to be the unique element s of ${}^\mu 2$ with $f_{p_{\bar\mu}}(x) \subseteq s$ and

$$s(\bar\mu) = \begin{cases} 1, & \text{if } \bar\mu = \prec h_p(x), \beta\succ \text{ and } s_\beta \subseteq x, \\ 0, & \text{if } \bar\mu = \prec h_p(x), \beta\succ \text{ and } s_\beta \nsubseteq x, \\ c_x(\bar\mu), & \text{otherwise.} \end{cases}$$

Finally, if $\mu = \bar\mu + 1$ with $\bar\mu \in \mathrm{Lim}$, then we define B to be the set of all branches through $T_{p_{\bar\mu}}$. By the definition of $\mathbb{P}(A)$, for each $t \in T_{p_{\bar\mu}}$ there is a $b_t \in B$ with $t \subseteq b_t$ and $b_t \neq t_{\bar\mu}$. We define

$$T_{p_\mu} = T_{p_{\bar\mu}} \cup \{f_{p_{\bar\mu}}(x) \mid x \in \mathrm{dom}(f_p)\} \cup \{b_t \mid t \in T_{p_{\bar\mu}}\}.$$

The partial function f_{p_μ} is defined as in the first successor case. □

Now it is easy to check that the following subsets are dense in $\mathbb{P}(A)$.

- $C_\mu = \{p \in \mathbb{P}(A) \mid |T_p| > \mu\}$ for all $\mu < \kappa$.
- $D_x = \{p \in \mathbb{P}(A) \mid x \in \mathrm{dom}(f_p)\}$ for all $x \in A$.
- $E_{x,y} = \{p \in \mathbb{P}(A) \mid x, y \in \mathrm{dom}(f_p), f_p(x) \neq f_p(y)\}$ for all $x, y \in A$.
- $F_z = \{p \in \mathbb{P}(A) \mid |T_p| = \mu + 1,\ z \restriction \mu \notin T_p\}$ for all $z \in {}^\kappa 2$.

Corollary

Let G be $\mathbb{P}(A)$-generic over V. The following statements hold in $\mathrm{V}[G]$.

- $T_G = \bigcup_{p \in G} T_p$ *is a binary κ-tree with* $[T_G] \cap \mathrm{V} = \emptyset$.
- *If we define* $F_G(x) = \bigcup \{ f_p(x) \mid p \in G, x \in \mathrm{dom}(f_p) \}$ *for all* $x \in A$, *then* $F_G : A \longrightarrow [T_G]$ *is an injection.*
- *Let* $H_G = \bigcup_{p \in G} h_g$. *Then* $H_G : A \longrightarrow \kappa$ *and*

$$s_\beta \subseteq x \iff F_G(x)(\prec H_G(x), \beta \succ) = 1$$

for all $x \in A$ *and* $\beta < \kappa$. □

Next, we show that the branches of T_G correspond to elements of A in an absolute way.

Lemma

Let G_0 be $\mathbb{P}(A)$-generic over V, $\mathbb{Q} \in \mathrm{V}[G_0]$ be a $< \kappa$-closed forcing and G_1 be $\mathbb{Q}$-generic over $\mathrm{V}[G_0]$. Then

$$\mathrm{V}[G_0][G_1] \models \text{“} F_{G_0} : A \longrightarrow [T_{G_0}]^{\mathrm{V}[G_0][G_1]} \textit{ is surjective”}.$$

Proof.

There is $\dot{\mathbb{Q}} \in \mathrm{V}^{\mathbb{P}(A)}$ with $\mathbb{1}_{\mathbb{P}(A)} \Vdash$ " $\dot{\mathbb{Q}}$ *is a* $<\check{\kappa}$*-closed forcing*" and $\mathbb{Q} = \dot{\mathbb{Q}}^{G_0}$. Let $\dot{T} \in \mathrm{V}^{\mathbb{P}*\dot{\mathbb{Q}}}$ be the canonical name for T_{G_0} and $\dot{F} \in \mathrm{V}^{\mathbb{P}*\dot{\mathbb{Q}}}$ be the canonical name for F_{G_0}. Assume, toward a contradiction, that there is an $x \in [T_{G_0}]^{\mathrm{V}[G_0][G_1]} \setminus \mathrm{ran}(F_{G_0})$. Let $\tau \in \mathrm{V}^{\mathbb{P}*\dot{\mathbb{Q}}}$ be a name for x. By the above corollary, $x \notin \mathrm{V}$ and there is a $\langle p_0, \dot{q}_0\rangle \in G_0 * G_1$ with

$$\langle p_0, \dot{q}_0\rangle \Vdash [\tau \in [\dot{T}] \wedge \tau \notin \check{\mathrm{V}} \wedge \tau \notin \mathrm{ran}(\dot{F})].$$

Since $\mathbb{P} * \dot{\mathbb{Q}}$ is $<\kappa$-closed, we can apply the above assumptions to construct a $\langle p, \dot{q}\rangle \leq_{\mathbb{P}*\dot{\mathbb{Q}}} \langle p_0, \dot{q}_0\rangle$ and $t \in [T_p]$ with $|T_p| \in \mathrm{Lim}$, $t \notin \mathrm{ran}(f_p)$ and $\langle p, \dot{q}\rangle \Vdash \check{t} \subseteq \tau$. In particular, $\langle p, \dot{q}\rangle \Vdash \check{t} \in \dot{T}$.
Using the Extension-Lemma, we can find a $p_* \leq_{\mathbb{P}(A)} p$ with $|T_{p_*}| = |T_p| + 1$ and $t \notin T_{p_*}$. But this means $\langle p_*, \dot{q}\rangle \Vdash \check{t} \notin \dot{T}$, a contradiction. □

Corollary

Let G_0 be $\mathbb{P}(A)$-generic over V, $\mathbb{Q} \in \mathrm{V}[G_0]$ be a $<\kappa$-closed forcing and G_1 be $\mathbb{Q}$-generic over $\mathrm{V}[G_0]$. The following statements are equivalent for $y \in ({}^\kappa\kappa)^{\mathrm{V}[G_0][G_1]}$.

- *$y \in A$.*
- *There is $z \in [T_{G_0}]^{\mathrm{V}[G_0][G_1]}$ and $\gamma < \kappa$ such that*

$$s_\beta \subseteq y \iff z(\prec\gamma, \beta\succ) = 1 \tag{1}$$

  holds for all $\beta < \kappa$

Proof.

We prove the non-trivial implication. Let $z \in [T_{G_0}]^{\mathrm{V}[G_0][G_1]}$ and $\gamma < \kappa$ witness that (1) holds for $y \in ({}^\kappa\kappa)^{\mathrm{V}[G_0][G_1]}$. By the above lemma, we have $z = F_{G_0}(x) \in \mathrm{V}[G_0]$ for some $x \in A$. Pick $p \in G_0$ with $x \in \mathrm{dom}(f_p)$.

Proof (cont.).

Assume, toward a contradiction, that $\gamma \neq h_p(x) = H_{G_0}(x)$. By the Extension-Lemma, this implies that the set

$$D_s = \{q \leq_{\mathbb{P}(A)} p \mid |T_q| = \mu + 1,\ \mu = \prec\gamma, \beta\succ,\ f_q(x)(\mu) = 0,\ s_\beta = s\}$$

is dense below p for all $s \subseteq y$. Hence $z(\prec\gamma, \beta\succ) = F_{G_0}(x)(\prec\gamma, \beta\succ) = 0$ for some $\beta < \kappa$ with $s_\beta = s \subseteq y$ and this means $s \nsubseteq y$, a contradiction. Therefore $\gamma = H_{G_0}(x)$ and we can conclude that

$$\begin{aligned} s_\beta \subseteq y \iff & z(\prec\gamma, \beta\succ) = 1 \\ \iff & F_{G_0}(x)(\prec H_{G_0}(x), \beta\succ) = 1 \iff s_\beta \subseteq x \end{aligned}$$

holds for all $\beta < \kappa$. This proves $y = x \in A$. □

Proof of the Σ^1_1-Coding Theorem.

Let G_0 *be* $\mathbb{P}(A)$*-generic over* V*. In* $\mathrm{V}[G_0]$*, we define* T *to be the tree on* $\kappa \times \kappa$ *consisting of elements* $\langle s, t\rangle \in {}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$ *such there are* $\gamma < \kappa$ *and* $t_0 \in T_G$ *with the following properties.*

- $\mathrm{lh}(s) = \mathrm{lh}(t) = \mathrm{lh}(t_0)$ *and* $t(\alpha) = \prec t_0(\alpha), \gamma \succ$ *for all* $\alpha \in \mathrm{dom}(s)$.
- *For all* $\alpha \in \mathrm{dom}(s)$ *and* $\beta < \kappa$ *with* $\alpha = \prec \gamma, \beta \succ$*, we have*

$$s_\beta \subseteq s \iff t_0(\alpha) = 1.$$

Fix $x \in A$ *and define* $z \in {}^\kappa\kappa$ *by* $z(\alpha) = \prec F_{G_0}(x)(\alpha), H_{G_0}(x)\succ$ *for all* $\alpha < \kappa$*. Given* $\alpha < \mu < \kappa$ *with* $\alpha = \prec H_{G_0}(x), \beta\succ$*, we have*

$$s_\beta \subseteq x \restriction \mu \iff s_\beta \subseteq x \iff (F_{G_0}(x) \restriction \mu)(\alpha) = 1.$$

This shows that $F_{G_0}(x) \restriction \mu$ *and* $H_{G_0}(x)$ *witness that* $\langle x \restriction \mu,\ z \restriction \mu\rangle \in T$ *for all* $\mu < \kappa$ *and therefore* $x \in \rho[T]$*.*

Proof (cont.).

Next, let $\mathbb{Q} \in \mathrm{V}[G_0]$ *be a* $<\kappa$*-closed forcing and* G_1 *be* $\mathbb{Q}$*-generic over* $\mathrm{V}[G_0]$*. For all* $\langle y, z\rangle \in [T]^{\mathrm{V}[G_0][G_1]}$*, there is* $\gamma < \kappa$ *and* $z_0 \in [T_{G_0}]^{\mathrm{V}[G_0][G_1]}$ *with* $z(\alpha) = \prec z_0(\alpha), \gamma \succ$ *for all* $\alpha < \kappa$*. Let* $\alpha, \beta, \mu < \kappa$ *with* $\alpha = \prec\gamma, \beta\succ$ *and* $\alpha < \mu$*. We have*

$$s_\beta \subseteq y \iff s_\beta \subseteq y \restriction \mu \iff z_0(\alpha) = 1$$

and, by the above Corollary, this implies $y \in A$*. This shows*

$$A = \rho[T]^{\mathrm{V}[G_0][G_1]}.$$

□

Generic F_σ-Covers

Given $A \subseteq {}^\kappa\kappa$, we define $\mathbb{Q}(A)$ to be the partial order consisting of conditions $p = \langle t_p, a_p \rangle$ with $t_p \in {}^{<\kappa}2$ and $a_p \in [A]^{<\kappa}$.
The ordering $p \leq_{\mathbb{Q}(A)} q$ is defined by the following clauses.

- $t_q \subseteq t_p$ and $a_q \subseteq a_p$.
- $(\forall x \in a_q)(\forall \alpha \in \mathrm{dom}(t_p) \setminus \mathrm{dom}(t_q))\ [s_\alpha \subseteq x \rightarrow t_p(\alpha) = 0]$.

It is easy to check that this is in fact a partial order. In addition, it is easy to see that two conditions p and q are compatible if and only if t_p and t_q are compatible as elements of ${}^{<\kappa}2$ and $\langle t_p \cup t_q, a_p \cup a_q \rangle \leq_{\mathbb{Q}(A)} p, q$.

Lemma

$\mathbb{Q}(A)$ is $< \kappa$-closed, satisfies the κ^+-chain condition and has cardinality at most 2^κ. □

Theorem

Let G be $\mathbb{Q}(A)$-generic over V. If we define $t_G = \bigcup\{t_p \mid p \in G\}$, then $t_G \in {}^{\kappa}2$ and

$$x \in A \iff (\exists \beta < \kappa)(\forall \beta \leq \alpha < \kappa)\ [s_\alpha \subseteq x \rightarrow t_G(\alpha) = 0]$$

for all $x \in ({}^{\kappa}\kappa)^{\mathrm{V}}$. Moreover,

$$G = \{p \in \mathbb{Q}(A) \mid t_p \subseteq t_G \wedge (\forall \alpha \in \kappa \setminus \mathrm{dom}(t_p))(\forall x \in a_p)\\ [s_\alpha \subseteq x \rightarrow t_G(\alpha) = 0]\}.$$

□

Δ^1_2-well-orderings of ${}^\kappa\kappa$

In this section, we will construct a partial order that forces the existence of a well-ordering of ${}^\kappa\kappa$ whose graph is a $\mathbf{\Delta}^1_3$-subset of ${}^\kappa\kappa \times {}^\kappa\kappa$.

We sketch the idea behind this construction.

- Use the above result to find partial order $\mathbb{P}$ that makes a well-ordering $<_{\mathrm{V}}$ of $({}^\kappa\kappa)^{\mathrm{V}}$ $\mathbf{\Sigma}^1_1$-definable in any further forcing extension by a $<\kappa$-closed partial order. Since $\mathbb{P}$ has cardinality at most 2^κ, we may assume $\mathbb{P} \subseteq {}^\kappa\kappa$. Let G_0 be $\mathbb{P}$-generic over V.
- In $\mathrm{V}[G_0]$, find a partial order $\mathbb{Q} = \dot{\mathbb{Q}}^{G_0} = \mathbb{Q}(\dot{A}^{G_0})$ that makes $G_0 \subseteq ({}^\kappa\kappa)^{\mathrm{V}}$ definable. Let G_1 be $\mathbb{Q}$-generic over $\mathrm{V}[G_0]$.
- In $\mathrm{V}[G_0 * G_1]$, there is a definable function

$$\eta^* : ({}^\kappa\kappa)^{\mathrm{V}[G_0 * G_1]} \longrightarrow ({}^\kappa\kappa)^{\mathrm{V}}$$

  that sends each $x \in ({}^\kappa\kappa)^{\mathrm{V}[G_0 * G_1]}$ to the $<_{\mathrm{V}}$-least $x_0 \in ({}^\kappa\kappa)^{\mathrm{V}}$ that codes a $(\mathbb{P} * \dot{\mathbb{Q}})$-nice name for x. This function induces a definable well-ordering of $({}^\kappa\kappa)^{\mathrm{V}[G_0 * G_1]}$.

Let $<_{\mathrm{V}}$ be a well-ordering of ${}^\kappa\kappa$ and $\mathbb{P}$ be the partial order that codes the set

$$W = \{x \in {}^\kappa\kappa \mid (x)_0 <_{\mathrm{V}} (x)_1,\ (x)_\alpha = \mathrm{id}_\kappa \textit{ for all } 1 < \alpha < \kappa\}$$

as a $\mathbf{\Sigma}^1_1$-subset using the above result. Since $\mathbb{P}$ has cardinality at most 2^κ and satisfies the κ^+-chain condition, there is a surjection

$$\xi : {}^\kappa\kappa \longrightarrow \{\mathcal{A} \subseteq \mathbb{P} \mid \mathcal{A} \textit{ is an anti-chain in } \mathbb{P}\}.$$

There are canonical names $\dot{A}, \dot{\mathbb{Q}} \in \mathrm{V}^{\mathbb{P}}$ *with*

$$\dot{A}^{G_0} = \{x \in ({}^\kappa\kappa)^{\mathrm{V}} \mid \xi(x) \cap G_0 \neq \emptyset\},$$

and $\dot{\mathbb{Q}}^{G_0} = \mathbb{Q}(\dot{A}^{G_0})$ *whenever* G_0 *is* $\mathbb{P}$*-generic over* V*.*

Let $G = G_0 * G_1$ *be* $(\mathbb{P} * \dot{\mathbb{Q}})$*-generic over* V*. From now on, we work in* $\mathrm{V}[G]$*.*

Claim.
$({}^\kappa\kappa)^{\mathrm{V}}$ *and* $\dot{A}^{G_0}$ *are* $\mathbf{\Sigma}^1_1$*-subsets of* ${}^\kappa\kappa$ *and* $<_{\mathrm{V}}$ *is a* $\mathbf{\Sigma}^1_1$*-subset of* ${}^\kappa\kappa \times {}^\kappa\kappa$ *in* $\mathrm{V}[G]$*.* $\square$

Given $x \in ({}^\kappa\kappa)^{\mathrm{V}}$, we define

$$\sigma_x = \{\langle\check{\alpha}, p\rangle \in \mathrm{V}^{\mathbb{P}} \times \mathbb{P} \mid \alpha < \kappa,\ p \in \xi((x)_\alpha)\} \in \mathrm{V}^{\mathbb{P}}.$$

σ_x is clearly a $\mathbb{P}$-nice name for a subset of κ and for every nice name $\sigma \in \mathrm{V}^{\mathbb{P}}$ for a subset of κ there is an $x \in ({}^\kappa\kappa)^{\mathrm{V}}$ with $\sigma = \sigma_x$.

We define E^{G_0} to be the set of all pairs $\langle x, y\rangle$ with the following properties.

- $x \in ({}^\kappa\kappa)^{\mathrm{V}[G]}$ *and* $y \in ({}^\kappa\kappa)^{\mathrm{V}}$.
- *For all $\alpha, \beta < \kappa$, $\sigma_y^{G_0} = \{\prec\beta, \alpha\succ \mid x(\alpha) = \beta\}$.*

Claim.
E^{G_0} is a $\mathbf{\Delta}^1_2$-subset of ${}^\kappa\kappa \times {}^\kappa\kappa$ and $({}^\kappa\kappa)^{\mathrm{V}[G_0]} = \exists^x E^{G_0}$. In particular, $({}^\kappa\kappa)^{\mathrm{V}[G_0]}$ is a $\mathbf{\Sigma}^1_2$-subset of ${}^\kappa\kappa$ in $\mathrm{V}[G]$. □

Given $x \in {}^{\kappa}\kappa$, we define a condition $p_x \in \mathbb{Q}({}^{\kappa}\kappa)$ by setting

$$s_{p_x} = (x)_1 \restriction ((x)_0(0))$$

and

$$a_{p_x} = \{(x)_\alpha \in {}^{\kappa}\kappa \mid 1 < \alpha \leq (x)_0(1)\}.$$

Clearly, for every $p \in \mathbb{Q}({}^{\kappa}\kappa)$, there is an $x \in {}^{\kappa}\kappa$ with $p = p_x$.

Claim.

The set $\bar{G}_1 = \{x \in ({}^{\kappa}\kappa)^{\mathrm{V}[G]} \mid p_x \in G_1\}$ is a $\mathbf{\Delta}^1_2$-subset of ${}^{\kappa}\kappa$ in $\mathrm{V}[G]$. □

Next, we define N to be the set consisting of all $x \in {}^\kappa\kappa$ such that

$$\{p_{(x)_{\prec\alpha,\beta\succ}} \in \mathbb{Q}({}^\kappa\kappa) \mid \beta < \kappa\}$$

is an anti-chain in $\mathbb{Q}({}^\kappa\kappa)$ for all $\alpha < \kappa$.

Claim.
N is a $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa$. □

Given $x \in N$, we define

$$\tau_x = \{\langle\check{\alpha}, p_{(x)_{\prec\alpha,\beta\succ}}\rangle \in \mathrm{V}^{\mathbb{Q}({}^\kappa\kappa)} \times \mathbb{Q}({}^\kappa\kappa) \mid \alpha, \beta < \kappa\}.$$

τ_x is a $\mathbb{Q}({}^\kappa\kappa)$-nice name for a subset of κ and for every $\mathbb{Q}({}^\kappa\kappa)$-nice name τ for a subset of κ there is an $x \in {}^\kappa\kappa$ with $\tau = \tau_x$.

We define E^{G_1} to be the set of all pairs $\langle x, y\rangle$ with the following properties.

- $x \in ({}^\kappa\kappa)^{\mathrm{V}[G]}$, $y \in N \cap \mathrm{V}[G_0]$ *and* $\tau_y \in \mathrm{V}[G_0]^{\dot{\mathbb{Q}}^{G_0}}$.
- *For all* $\alpha, \beta < \kappa$, $\tau_y^{G_1} = \{\prec\beta, \alpha\succ \mid x(\alpha) = \beta\}$.

Claim.
E^{G_1} *is a* $\mathbf{\Sigma}^1_2$*-subset of* ${}^\kappa\kappa \times {}^\kappa\kappa$ *in* $\mathrm{V}[G]$ *and* $({}^\kappa\kappa)^{\mathrm{V}[G]} = \exists^x E^{G_1}$. $\square$

Define a function $\eta : {}^\kappa\kappa \longrightarrow \mathcal{P}\left(({}^\kappa\kappa)^{\mathrm{V}}\right)$ *by setting*

$$\eta(x) = \{x_0 \in ({}^\kappa\kappa)^{\mathrm{V}} \mid (\exists x_1 \in ({}^\kappa\kappa)^{\mathrm{V}[G_0]})\ [E^{G_0}(x_1, x_0) \wedge E^{G_1}(x, x_1)]\}.$$

The above claims show that $\eta(x) \neq \emptyset$ *and* $\eta(x) \cap \eta(y) = \emptyset$ *for all* $x, y \in ({}^\kappa\kappa)^{\mathrm{V}[G]}$ *with* $x \neq y$. *This shows that the induced function*

$$\eta^* : {}^\kappa\kappa \longrightarrow ({}^\kappa\kappa)^{\mathrm{V}};\ x \longmapsto \min{}_{<_{\mathrm{V}}}(\eta(x))$$

is injective. Define a binary relation $<^*$ *on* ${}^\kappa\kappa$ *by setting*

$$x <^* y \iff \eta^*(x) <_{\mathrm{V}} \eta^*(y)$$

for all $x, y \in {}^\kappa\kappa$.

Claim.

The relation $<^*$ *is a well-ordering of* ${}^\kappa\kappa$ *and the graph of* $<^*$ *is a* $\mathbf{\Delta}^1_3$*-subset of* ${}^\kappa\kappa$ *in* $\mathrm{V}[G]$. □

□

Some Questions

Definition

- A subset $U \subseteq {}^\kappa\kappa$ is open if, for every $x \in U$, there is an $\alpha < \kappa$ with $\{y \in {}^\kappa\kappa \mid x \upharpoonright \alpha \subseteq y\} \subseteq U$.
- A subset $B \subseteq {}^\kappa\kappa$ is κ^+-Borel if it is an element of the smallest κ^+-algebra on ${}^\kappa\kappa$ containing all open subsets of ${}^\kappa\kappa$.

Remark

Every κ^+-Borel subset is a $\mathbf{\Delta}^1_1$-subset. There are $\mathbf{\Delta}^1_1$-subsets that are not κ^+-Borel.

Some Questions

Question

Given $A \subseteq {}^\kappa\kappa$, is there a $< \kappa$-closed partial order $\mathbb{P}$ that satisfies the κ^+-chain condition and forces A to be a κ^+-Borel subset in every $\mathbb{P}$-generic extension of V?

Remark

To obtain a positive answer, it suffices to constructed a partial order $\mathbb{P}$ with the above properties that forces $({}^\kappa\kappa)^{\mathrm{V}}$ to be a κ^+-Borel subset without a perfect subset in every $\mathbb{P}$-generic extension of V.

Some Questions

Question

Does $2^\kappa > \kappa^+$ imply that there is no well-ordering of ${}^\kappa\kappa$ whose graph is $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa \times {}^\kappa\kappa$?

If the answer to this question is *no*, then it is natural to ask the following question.

Question

Can we always find a $< \kappa$-closed partial order that satisfies the κ^+-chain condition and forces the existence of a well-ordering of ${}^\kappa\kappa$ whose graph is a $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa \times {}^\kappa\kappa$?

Thank you for listening!