Ascending paths and large antichains in products

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Introduction

The work presented in this talk studies combinatorial properties of trees of uncountable regular heights that cause these trees to be non-special in a very absolute way.

- A partial order $\mathbb{T}$ is a *tree* if it has a unique minimal element $\mathrm{root}(\mathbb{T})$ and sets of the form $\mathrm{pred}_{\mathbb{T}}(t) = \{s \in \mathbb{T} \mid s <_{\mathbb{T}} t\}$ are well-ordered by $<_{\mathbb{T}}$ for every $t \in \mathbb{T}$.

- Given a tree $\mathbb{T}$ and $t \in \mathbb{T}$, we define $\mathrm{lh}_{\mathbb{T}}(t)$ to be the order-type of $\langle \mathrm{pred}_{\mathbb{T}}(t), <_{\mathbb{T}} \rangle$ and we define $\mathrm{ht}(\mathbb{T}) = \sup_{t \in \mathbb{T}} \mathrm{lh}_{\mathbb{T}}(t)$ to be the *height* of $\mathbb{T}$.

- Given a tree $\mathbb{T}$ and $\gamma < \mathrm{ht}(\mathbb{T})$, we define $\mathbb{T}(\gamma) = \{t \in \mathbb{T} \mid \mathrm{lh}_{\mathbb{T}}(t) = \gamma\}$.

Cofinal branches

One of the most basic questions about a tree $\mathbb{T}$ of infinite height is the question whether $\mathbb{T}$ has a *cofinal branch*, i.e. the question whether there is a subset B of $\mathbb{T}$ with the property that B is linearly ordered by $<_{\mathbb{T}}$ and the set $\{\mathrm{lh}_{\mathbb{T}}(t) \mid t \in B\}$ is unbounded in $\mathrm{ht}(\mathbb{T})$.

It is well-known that the non-existence of cofinal branches through trees of uncountable heights is not upwards-absolute between models of set theory.

It turns out that a large class of trees of uncountable regular heights have no cofinal branches for *very special* (and absolute) reasons.

Special trees

Definition (Todorčević)

Let θ be an uncountable regular cardinal, let S be a subset of θ and let $\mathbb{T}$ be a tree of height θ.

- A map $r : \mathbb{T} \upharpoonright S \longrightarrow \mathbb{T}$ is *regressive* if $r(t) <_{\mathbb{T}} t$ holds for $t \in \mathbb{T} \upharpoonright S$ that is not minimal in $\mathbb{T}$.
- We say that S *is nonstationary with respect to* $\mathbb{T}$ if there is a regressive map $r : \mathbb{T} \upharpoonright S \longrightarrow \mathbb{T}$ with the property that for every $t \in \mathbb{T}$ there is a function $c_t : r^{-1}\{t\} \longrightarrow \theta_t$ such that θ_t is a cardinal smaller than κ and c_t is injective on $\leq_{\mathbb{T}}$-chains.
- The tree $\mathbb{T}$ is *special* if the set θ is nonstationary with respect to $\mathbb{T}$.

Todorčević showed that the above definition generalizes the classical definition of special trees of successor height.

Theorem (Todorčević)

If κ is an infinite cardinal, then the following statements are equivalent for every tree $\mathbb{T}$ of height κ^+:

- $\mathbb{T}$ *is special.*
- *The set $S^{\kappa^+}_{\mathrm{cof}(\kappa)}$ is nonstationary with respect to $\mathbb{T}$.*
- $\mathbb{T}$ *is the union of κ-many antichains.*

An easy argument shows that special trees do not contain cofinal branches.

Proposition

If θ is an uncountable regular cardinal, $\mathbb{T}$ is a tree of height θ and S is a stationary subset of θ that is nonstationary with respect to $\mathbb{T}$, then there are no cofinal branches through $\mathbb{T}$.

Corollary

If $\mathbb{T}$ is a special tree of uncountable regular height, then there are no cofinal branches through $\mathbb{T}$.

Moreover, the above non-existence of cofinal branches through special trees is absolute in a strong sense: If $\mathbb{T}$ is a special tree of uncountable regular height and W is an outer model of V such that $\mathrm{ht}(\mathbb{T})$ is uncountable regular cardinal in W, then there are no cofinal branches through $\mathbb{T}$ in W.

Specializable trees

The above observations raise the question whether there are also trees without cofinal branches that are non-special for very absolute reasons.

A classical result of Baumgartner, Malitz and Reinhardt shows that there are no trees of height ω_1 with this property.

Theorem (Baumgartner-Malitz-Reinhardt)

If $\mathbb{T}$ is a tree of height ω_1 without cofinal branches, then there is a partial order $\mathbb{P}$ satisfying the countable chain condition with $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\check{\mathbb{T}}$ is special".

In contrast, Baumgartner showed that there can be trees of greater heights that are hard to specialize.

Theorem (Baumgartner)

In L*, there is an* $\aleph_2$*-Souslin tree* $\mathbb{T}$ *with the property that* $\mathbb{T}$ *is non-special in every outer model* W *of* L *with* $\omega_2^{\mathrm{L}} = \omega_2^{\mathrm{W}}$*.*

Examples of trees with similar properties appear in the work of Brodsky, Cummings, Laver, Rinot, Shelah, Stanley, Todorčević, Torres Pérez and others.

In the following, we will present a combinatorial property that causes trees to be non-special in a very absolute way. Variants of this properties are used in all of the construction mentioned above. This property is also closely related to the notion of *narrow systems* introduced by Magidor and Shelah in their work on the tree property at successors of singular cardinals.

Ascending paths

In the following, let θ denote an uncountable regular cardinal and let $\mathbb{T}$ be a tree of height θ.

Definition

Given a cardinal $\lambda > 0$, a sequence

$$\langle b_\gamma : \lambda \longrightarrow \mathbb{T}(\gamma) \mid \gamma < \theta \rangle$$

of functions is an *ascending path of width λ through* $\mathbb{T}$ if for all $\gamma < \delta < \theta$, there are $\alpha, \beta < \lambda$ with $b_\gamma(\alpha) <_{\mathbb{T}} b_\delta(\beta)$.

Then the existence of a cofinal branch through $\mathbb{T}$ is equivalent to the existence of an ascending path of width 1 through $\mathbb{T}$.

The following lemma shows that the same is true for ascending paths of finite width. We will later show how this result is proven.

Lemma

If there is an ascending paths of finite width through $\mathbb{T}$, then $\mathbb{T}$ has a cofinal branch.

We list two more basic observations.

Proposition

- *There is an ascending path of width θ through $\mathbb{T}$.*
- *Assume that $\theta = \nu^+$. Then there is an ascending path of width ν through $\mathbb{T}$ if and only if for every $\gamma < \theta$ there is $t \in \mathbb{T}(\gamma)$ such that for every $\gamma < \delta < \theta$ there is $u \in \mathbb{T}(\delta)$ with $t <_{\mathbb{T}} u$.*

The following lemma generalizes a result of Todorčević and Torres Pérez. It shows how ascending paths cause certain trees to be non-special.

Lemma

Let $\lambda < \theta$ be a cardinal with the property that θ is not a successor of a cardinal of cofinality less than or equal to λ and let $S \subseteq S^{\theta}_{>\lambda}$ be stationary in θ. If S is nonstationary with respect to $\mathbb{T}$, then there is no ascending path of width λ through $\mathbb{T}$.

Corollary

Let $\lambda < \theta$ be a cardinal with the property that θ is not a successor of a cardinal of cofinality less than or equal to λ. If $\mathbb{T}$ contains an ascending path of width λ, then $\mathbb{T}$ is not special.

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The above corollary leaves open the question whether there can be a singular cardinal ν and a special tree of height ν^+ that contains an ascending path of width $\mathrm{cof}(\nu)$.

This question was recently answered by Chris Lambie-Hanson.

Theorem (Lambie-Hanson)

If ν is a singular cardinal and $\square_\nu$ holds, then there is a special ν^+-Aronszajn tree that contains an ascending path of width $\mathrm{cof}(\nu)$.

Corollary

Let $\lambda < \theta$ be a cardinal with the property that θ is not a successor of a cardinal of cofinality less than or equal to λ. If $\mathbb{T}$ contains an ascending path of width λ, then $\mathbb{T}$ is not special.

This result shows that ascending paths cause trees to be non-special in a very absolute way: in the situation of the corollary, the tree $\mathbb{T}$ remains non-special in every outer model in which θ satisfies the assumptions of the corollary.

In the following, we will show that if θ satisfies certain cardinal arithmetic assumptions, then the converse of this implication is also true, i.e. if $\mathbb{T}$ does not contain an ascending path of small width, then $\mathbb{T}$ is special in a cofinality preserving outer model ("$\mathbb{T}$ *is specializable*").

Forcings that specialize trees

The following partial order is the natural candidate for a forcing that specializes a given tree.

Definition

Let κ be an infinite regular cardinal. We define $\mathbb{P}_\kappa(\mathbb{T})$ to be the partial order whose conditions are partial functions from $\mathbb{T}$ to κ of cardinality less than κ that are injective on chains in $\mathbb{T}$ and whose ordering is given by reversed inclusion.

It is easy to see that $\mathbb{P}_\kappa(\mathbb{T})$ is ${<}\kappa$-closed and, if θ is regular in a $\mathbb{P}_\kappa(\mathbb{T})$-generic extension, then $\mathbb{T}$ is special in this extension.

Therefore it is natural to ask under which conditions the regularity of θ is preserved by forcing with $\mathbb{P}_\kappa(\mathbb{T})$.

Theorem

The following statements are equivalent for every infinite regular cardinal $\kappa < \theta$ with $\mu^{<\kappa} < \theta$ for all $\mu < \theta$:

- *There is no ascending path of width less than κ through $\mathbb{T}$.*
- *The partial order $\mathbb{P}_\kappa(\mathbb{T})$ satisfies the θ-chain condition.*

Assuming certain fragments of the GCH, this theorem allows us to characterize specializable trees of successor heights using ascending paths.

Corollary

Assume that $\theta = \kappa^+$ with $\kappa = \kappa^{<\kappa}$. Then the following statements are equivalent for every tree $\mathbb{T}$ of height θ:

- *$\mathbb{T}$ is specializable.*
- *There is no ascending path of width less than κ through $\mathbb{T}$.*

It is not known whether the conclusion of the above corollary also holds without the cardinal arithmetic assumption.

Using results of Viale and Weiss on the existence of guessing models, it is possible to show that PFA implies every tree of height ω_2 that contains an ascending path of width ω has a cofinal branch.

Therefore it is interesting to consider the following question.

Question

Let κ be an infinite regular cardinal and let $\mathbb{T}$ be a tree of height κ^+ that does not contain an ascending path of width less than κ. Is $\mathbb{T}$ specializable?

In order to apply the above results, we consider non-existence results for trees containing ascending paths of small width.

The following result directly generalizes an argument used in the proof of the above result of Baumgartner, Malitz and Reinhardt.

Lemma

Let $\lambda > 0$ be a cardinal with the property that for every collection $\mathcal{S}$ of θ-many subsets of θ there is a ${<}\lambda^+$-closed $\mathcal{S}$-ultrafilter on θ consisting of unbounded subsets. Then every tree of height θ that contains an ascending path of width λ has a cofinal branch.

Theorem

- *Every tree of height θ that contains an ascending path of finite width has a cofinal branch.*
- *If θ is weakly compact, then every tree of height θ that contains an ascending path of width less than θ has a cofinal branch.*
- *If $\kappa \leq \theta$ is a θ-compact cardinal, then every tree of height θ that contains an ascending path of width less than κ has a cofinal branch.*

Corollary

Assume that κ is a regular limit of strongly compact cardinals. Then every tree of height κ^+ without a cofinal branch is specializable.

Using recent results by Lambie-Hanson, it is possible to prove the following strengthening of the above lemma.

Lemma

Let λ be a cardinal with the property that for every collection $\mathcal{S}$ of θ-many subsets of θ there is a $<\!\lambda^+$-closed partial order $\mathbb{P}$ such that forcing with $\mathbb{P}$ adds a $<\!\lambda^+$-closed $\mathcal{S}$-ultrafilter on θ consisting of unbounded subsets. Then every tree of height θ that contains an ascending path of width λ has a cofinal branch.

With the help of this lemma, we can prove analogues of the above statements for successor cardinals.

Theorem

- *If θ is weakly compact, $\kappa < \theta$ is an uncountable regular cardinal and G is $\mathrm{Col}(\kappa, {<}\theta)$-generic over V, then in $\mathrm{V}[G]$ every tree of height θ that contains an ascending path of width less than κ has a cofinal branch.*
- *If $\kappa \leq \theta$ is supercompact, $\nu < \kappa$ is an uncountable regular cardinal and G is $\mathrm{Col}(\nu, {<}\kappa)$-generic over V, then in $\mathrm{V}[G]$ every tree of height θ that contains an ascending path of width less than κ has a cofinal branch.*

Corollary

If θ is a weakly compact cardinal, $\kappa < \theta$ is an uncountable regular cardinal and G is $\mathrm{Col}(\kappa, {<}\theta)$-generic over V, then in $\mathrm{V}[G]$ every tree of height θ without a cofinal branch is specializable.

Trees constructed from walks

We present another class of trees that do not contain ascending paths of small width.

Definition (Todorčević)

A sequence $\vec{C} = \langle C_\gamma \subseteq \gamma \mid \gamma \in \mathrm{Lim} \cap \theta \rangle$ is a $\square(\theta)$-*sequence* if the following statements hold:

- If $\gamma \in \mathrm{Lim} \cap \theta$, then C_γ is a club in γ.
- If $\delta \in \mathrm{Lim} \cap \theta$ and $\gamma \in \mathrm{Lim}(C_\delta)$, then $C_\gamma = C_\delta \cap \gamma$.
- There is no club C in θ with $C_\gamma = C \cap \gamma$ for all $\gamma \in \mathrm{Lim}(C)$.

Theorem (Jensen-Todorčević)

Assume that θ is not weakly compact in L, then there is a $\square(\theta)$-sequence.

Using his method of *walks on ordinals*, Todorčević constructs a canonical θ-Aronszajn tree $\mathbb{T}_{\vec{C}}$ from a $\square(\theta)$-sequence $\vec{C}$ ("*tree of full codes of the walks through* $\vec{C}$").

Theorem

If $\vec{C}$ is $\square(\theta)$-sequence and λ is a cardinal with $\lambda^+ < \theta$, then the tree $\mathbb{T}_{\vec{C}}$ does not contain an ascending path of width λ.

Theorem (Todorčević)

Assume $\theta > \aleph_1$. If there is a $\square(\theta)$-sequence, then there is a $\square(\theta)$-sequence $\vec{C}$ with the property that the tree $\mathbb{T}_{\vec{C}}$ is not special.

Corollary

Assume that $\kappa = \kappa^{<\kappa}$ is uncountable. If there is a $\square(\kappa^+)$-sequence, then there is a $\square(\theta)$-sequence $\vec{C}$ and a collection $\mathcal{D}$ of κ^+-many dense subsets of $\mathbb{P}_\kappa(\mathbb{T}_{\vec{C}})$ with the property that there is no $\mathcal{D}$-generic filter.

Given an uncountable regular cardinal κ and a partial order $\mathbb{P}$, we let $\mathbf{FA}_{\kappa}(\mathbb{P})$ denote the statement that for every collection $\mathcal{D}$ of κ-many dense subsets of $\mathbb{P}$, there is a $\mathcal{D}$-generic filter on $\mathbb{P}$.

Shelah showed that CH implies the existence of a σ-closed partial order satisfying the $\aleph_2$-chain condition with the property that $\mathbf{FA}_{\aleph_2}(\mathbb{P})$ fails.

Since Shelah's example is not *well-met* (there are compatible conditions without a greatest lower bound), it is natural to ask the following question.

Question

Is CH consistent with the assumption that $\mathbf{FA}_{\aleph_2}(\mathbb{P})$ holds for all σ-closed, well-met partial orders satisfying the $\aleph_2$-chain condition?

The above corollary shows that $\square(\aleph_2)$ has to fail in models of this theory and therefore it implies that $\aleph_2$ is weakly compact in L.

The infinite productivity of the Knaster property

We discuss another application of the notion of ascending paths. This application deals with questions on the *productivity* of certain chain conditions and characterizations of weakly compact cardinals.

A well-known question of Todorčević asks whether a regular cardinal $\theta > \aleph_1$ is weakly compact if and only if the θ-chain condition is productive.

This question is motivated by the following observation.

Proposition

Assume that θ is weakly compact. If $\lambda < \theta$, $\langle \mathbb{P}_\alpha \mid \alpha < \lambda \rangle$ is a sequence of partial orders satisfying the θ-chain condition and A is a subset of the full support product $\prod_{\alpha<\lambda} \mathbb{P}_\alpha$ of cardinality θ, then there is a subset B of A of cardinality θ consisting of pairwise compatible conditions.

Remember that a partial order is *θ-Knaster* if every θ-sized collection of conditions can be refined to a θ-sized set of pairwise compatible conditions.

An easy argument shows that the class of θ-Knaster partial orders is closed under finite products.

The above proposition shows that, if θ is weakly compact, then the class of θ-Knaster partial orders is closed under λ-support products for all $\lambda < \theta$.

This observation motivates the following variation of Todorčević's question.

Question

Are the following statements equivalent for every uncountable regular cardinal θ?

- θ is weakly compact.
- The class of θ-Knaster partial orders is closed under λ-support products for every $\lambda < \theta$.

In the following, we will use ascending paths to show that a positive answer to the above question is consistent.

Proposition

Let θ be an uncountable regular cardinal, let $\mathbb{T}$ be a tree of height θ and let λ be an infinite cardinal. If there is an ascending path of width λ through $\mathbb{T}$, then the full support product $\prod_\lambda \mathbb{P}_\omega(\mathbb{T})$ does not satisfy the θ-chain condition.

Proof.

Let $\langle b_\gamma : \lambda \longrightarrow \mathbb{T}(\gamma) \mid \gamma < \theta \rangle$ be an ascending path through $\mathbb{T}$.

Given $\gamma < \theta$, let $\vec{p}_\gamma$ denote the unique condition in the full support product $\prod_{\lambda\times\lambda} \mathbb{P}_\omega(\mathbb{T})$ with the property that $\mathrm{dom}(\vec{p}_\gamma)(\alpha,\beta) = \{b_\gamma(\alpha), b_\gamma(\beta)\}$ and $\mathrm{ran}(\vec{p}_\gamma)(\alpha,\beta) = \{0\}$ for all $\alpha,\beta < \lambda$.

Then $\langle \vec{p}_\gamma \mid \gamma < \theta \rangle$ enumerates an antichain in $\prod_{\lambda\times\lambda} \mathbb{P}_\omega(\mathbb{T})$. $\square$

The following lemma shows that partial orders of the above form can be θ-Knaster.

Lemma (Cox-L.)

Let θ be an uncountable regular cardinal and let $\mathbb{T}$ be a normal θ-Aronszajn tree. If there is a stationary subset S of θ such that S is nonstationary with respect to $\mathbb{T}$, then the partial order $\mathbb{P}_\omega(\mathbb{T})$ is θ-Knaster.

Corollary

Assume that a normal θ-Aronszajn tree $\mathbb{T}$ containing an ascending path of width λ and a stationary subset S of θ such that S is nonstationary with respect to $\mathbb{T}$. Then there is a θ-Knaster partial order $\mathbb{P}$ with the property that the full support product $\prod_\lambda \mathbb{P}$ contains an antichain of size θ.

Using Todorčević's method of walks on ordinals, it is possible to construct such trees from suitable $\square$-sequences.

Definition

A $\square(\theta)$-sequence *avoids* $A \subseteq \theta$ if $A \cap \mathrm{Lim}(C_\gamma) = \emptyset$ holds for all $\gamma \in \mathrm{Lim} \cap \theta$.

Theorem

Let $\lambda < \theta$ be an infinite regular cardinal and let $S \subseteq S^\theta_\lambda$ be stationary in θ. Assume that there is a $\square(\theta)$-sequence that avoids S. Then there is a normal θ-Aronszajn tree $\mathbb{T}$ with the property that $\mathbb{T}$ contains an ascending path of width λ and the set S is nonstationary with respect to $\mathbb{T}$.

Results of Jensen, Schimmerling and Zeman show that such $\square$-sequences exist in canonical inner models for non-weakly compact cardinals. This allows us to show that the above characterization holds in these models.

Theorem

Let $\mathrm{L}[E]$ be a Jensen-style extender model. In $\mathrm{L}[E]$, the following statements are equivalent for every uncountable regular cardinal θ:

- *θ is weakly compact.*
- *The class of θ-Knaster partial orders is closed under λ-support products for all $\lambda < \theta$.*

Moreover, if θ is not the successor of a subcompact cardinal in $\mathrm{L}[E]$, then the above statements are also equivalent to the following statement:

- *The class of θ-Knaster partial orders is closed under countable support products.*

In contrast, the above characterization of weak compactness can also consistently fail.

Theorem (Cox-L.)

If θ is a weakly compact cardinal, then there is a partial order $\mathbb{P}$ such that the following statements hold in $\mathrm{V}[G]$ whenever G is $\mathbb{P}$-generic over V.

- *θ is inaccessible and not weakly compact.*
- *For every $\lambda < \theta$, the class of θ-Knaster partial orders is closed under λ-support products.*

The proof of this result relies on the concept of $\mathcal{U}$-layered partial orders developed in joint work with Sean Cox.

Definition

Let κ be an uncountable regular cardinal, let $\lambda \geq \kappa$ be a cardinal, let $\mathcal{F}$ be a normal filter on $\mathcal{P}_\kappa(\lambda)$ and let $\mathbb{P}$ be a partial order.

- We say $\mathbb{P}$ is $\mathcal{F}$-layered, if it has cardinality at most λ and

$$\{a \in \mathcal{P}_\kappa(\lambda) \mid s[a] \textit{ is a regular suborder of } \mathbb{P}\} \in \mathcal{F}$$

  holds for every surjection $s : \lambda \longrightarrow \mathbb{P}$.
- We say that $\mathbb{P}$ is *completely $\mathcal{F}$-layered* if every subset of $\mathbb{P}$ of cardinality at most λ is contained in a regular suborder of $\mathbb{P}$ of cardinality at most λ and every regular suborder of $\mathbb{P}$ of size at most λ is $\mathcal{F}$-layered.

Lemma (Cox-L.)

In the situation of the above definition, if $\lambda = \lambda^{<\kappa}$ holds, then every completely $\mathcal{F}$-layered partial order is κ-Knaster.

Theorem (Cox-L.)

Let κ be a weakly compact cardinal and let $\mathcal{F}_{wc}$ denote the weakly compact filter on $\mathcal{P}_\kappa(\kappa)$. Then a partial order $\mathbb{P}$ satisfies the κ-chain condition if and only if $\mathbb{P}$ is completely $\mathcal{F}_{wc}$-layered.

Lemma (Cox-L.)

In the situation of the above definition, assume that $\kappa = \lambda$ is inaccessible and $\nu < \kappa$ is a cardinal with

$$\{a \in \mathcal{P}_\kappa(\kappa) \mid \varphi[{}^\nu a] \subseteq a\} \in \mathcal{F}$$

for every function $\varphi : {}^\nu\kappa \longrightarrow \kappa$. Then the class of completely $\mathcal{F}$-layered partial orders is closed under ν-support products.

In combination with a classical result of Kunen, the following result implies the above theorem.

Theorem (Cox-L.)

Let θ be an inaccessible cardinal with the property that there is a θ-Souslin tree $\mathbb{T}$ with $\mathbb{1}_{\mathbb{T}} \Vdash$ "$\check{\theta}$ is weakly compact". Set

$$\mathcal{F} \;=\; \{A \subseteq \mathcal{P}_\theta(\theta) \mid \mathbb{1}_{\mathbb{T}} \Vdash \text{“}\check{A} \in \mathcal{F}_{wc}\text{”}\}.$$

Then:

- *$\mathcal{F}$ is a normal filter on $\mathcal{P}_\theta(\theta)$ with $\{a \in \mathcal{P}_\theta(\theta) \mid \varphi[{}^\nu a] \subseteq a\} \in \mathcal{F}$ for all $\nu < \theta$ and every function $\varphi : {}^\nu\theta \longrightarrow \theta$.*
- *Every θ-Knaster partial order is completely $\mathcal{F}$-layered.*

Open questions

We close this talk with questions raised by the above results.

Question

Let $\kappa > \aleph_1$ be a regular cardinal. Does the existence of a $\square(\theta)$-sequence imply the existence of a θ-Aronszajn tree containing an ascending path of width λ with $\lambda^+ < \theta$?

Question

Assume PFA. Is every tree of height ω_2 specializable?

Question

Is it consistent that the class of $\aleph_2$-Knaster partial orders is closed under countable products?

Thank you for listening!