Specializing Aronszajn trees and square sequences by forcing

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Specializing κ^+-Aronszajn trees

We start by recalling some basic definitions concerning trees of uncountable height.

Definition

Let θ be an uncountable regular cardinal. A tree T of height θ is a *θ-Aronszajn tree* if T has no cofinal branches and every level of T has cardinality less than θ.

Definition

Let κ be an infinite cardinal and T be a tree. We say that T is *κ-special* if there is a function $f : T \longrightarrow \kappa$ that is injective on chains in T.

Given an infinite cardinal κ and a κ^+-Aronszajn tree T, there is a canonical $<\kappa$-closed forcing $\mathbb{P}_T$ that specializes T. This partial order consists of partial specializing functions $q : T \xrightarrow{part} \kappa$ of cardinality less than κ ordered by reverse inclusion.

In the case "$\kappa = \omega$", this forcing can be used to show that Martin's Axiom implies that all Aronszajn trees are special.

Theorem (Baumgartner-Malitz-Reinhardt)

If T is an Aronszajn tree, then $\mathbb{P}_T$ satisfies the countable chain condition.

In contrast, it is consistent that forcings of the form $\mathbb{P}_T$ can collapse cardinals. This can be shown using a notion introduced by Laver.

Definition (Laver)

Let θ be an uncountable regular cardinal and T be a tree of cardinality and height θ. A sequence

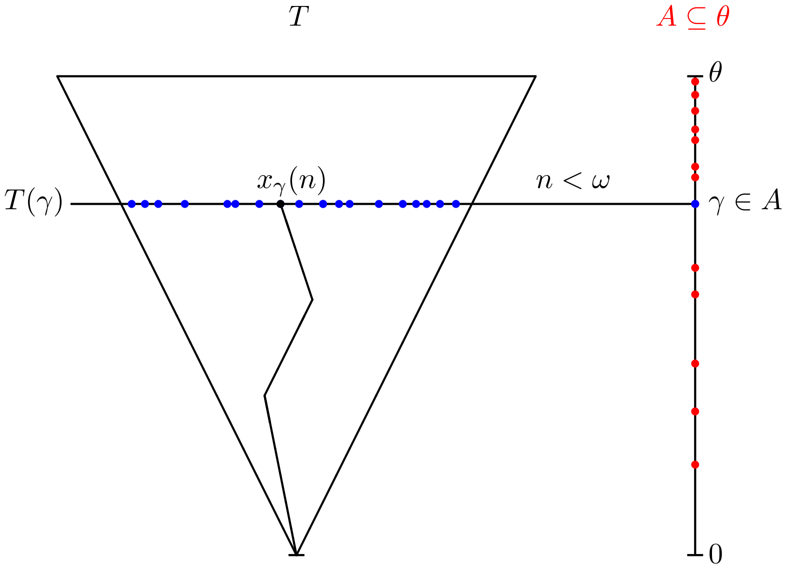
$$\langle x_\gamma : \omega \longrightarrow T(\gamma) \mid \gamma \in A \rangle$$

is an *ω-ascent path through T* if the following statements hold.

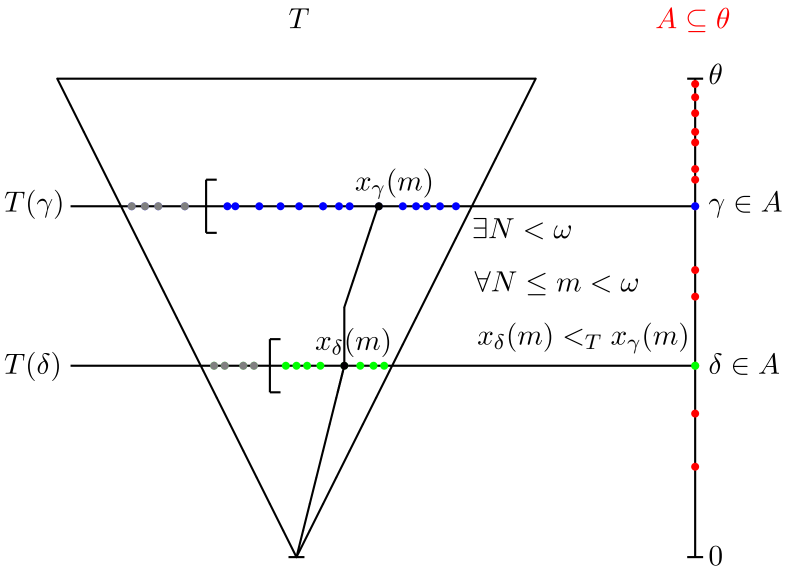
- A is an unbounded subset of θ.
- If $\gamma, \delta \in A$ with $\gamma < \delta$, then there is an $N < \omega$ such that

  $$x_\gamma(n) \ <_T \ x_\delta(n)$$

  for all $N \leq n < \omega$.



T
$A \subseteq \theta$
θ
$x_\gamma(n)$
$n < \omega$
$T(\gamma)$
$\gamma \in A$
0



T
$A \subseteq \theta$
θ
$x_\gamma(m)$
$T(\gamma)$
$\gamma \in A$
$\exists N < \omega$
$\forall N \leq m < \omega$
$x_\delta(m) <_T x_\gamma(m)$
$x_\delta(m)$
$T(\delta)$
$\delta \in A$
0

Theorem (Shelah)

Let κ be a cardinal of uncountable cofinality and T be a κ^+-Aronszajn tree. If there is an ω-ascent paths through T, then T is not κ-special.

Theorem (Shelah-Stanley/Todorčević)

Let κ be a cardinal of uncountable cofinality. If $\square_\kappa$ holds, then there is a κ^+-Aronszajn tree with an ω-ascent path.

Corollary

If $\square_\kappa$ holds, then there is a κ^+-Aronszajn tree T with the property that forcing with $\mathbb{P}_T$ collapses κ^+.

Given an infinite cardinal κ with $\kappa = \kappa^{<\kappa}$, we want to characterize the class of all κ^+-Aronszajn trees T such that the partial order $\mathbb{P}_T$ satisfies the κ^+-chain condition.

This characterization uses the following variation of the above concept.

Definition

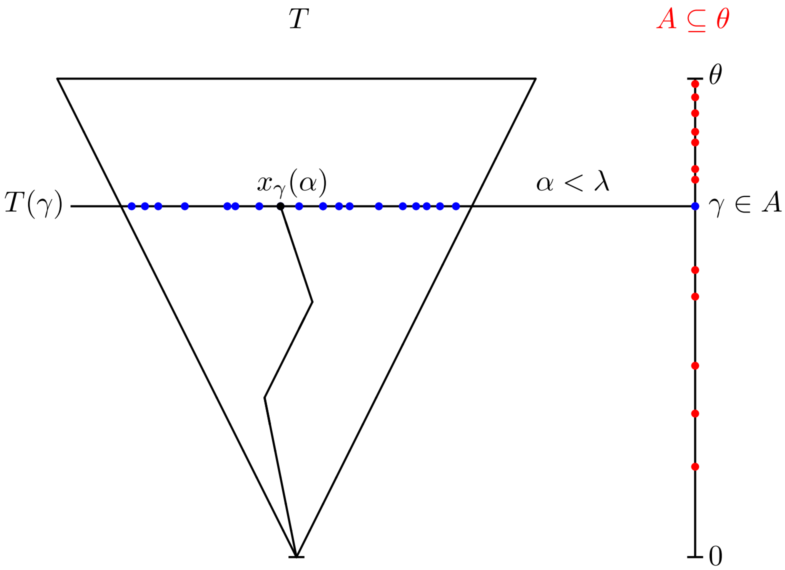
Let κ be an infinite cardinal and T be a tree of height κ^+. Given $\lambda < \kappa$, we call a sequence

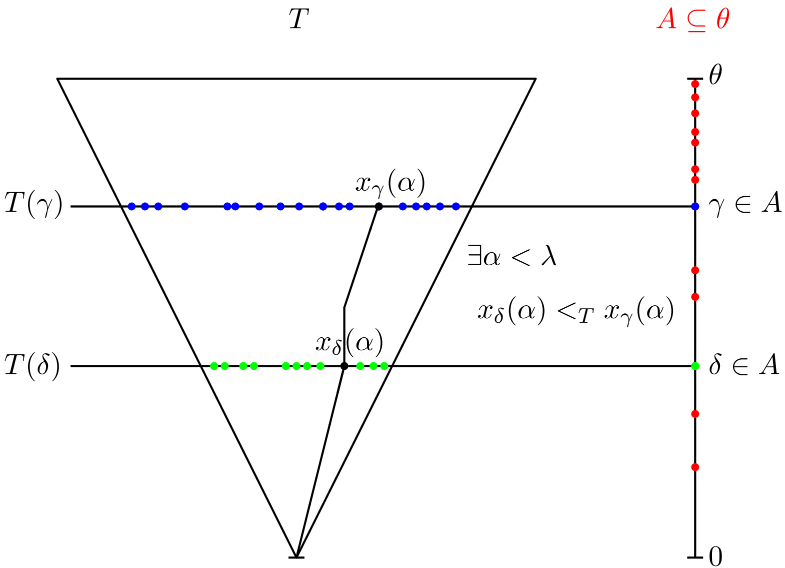
$$\langle x_\gamma : \lambda \longrightarrow T(\gamma) \mid \gamma \in A \rangle$$

of injections a λ-*path through* T if the following statements hold.

- A is an unbounded subset of κ^+.
- If $\gamma, \delta \in A$ with $\gamma < \delta$, then there is an $\alpha < \lambda$ with

$$x_\gamma(\alpha) \ <_T \ x_\delta(\alpha).$$


T
$A \subseteq \theta$
θ
$x_\gamma(\alpha)$
$\alpha < \lambda$
$T(\gamma)$
$\gamma \in A$
0



T
$A \subseteq \theta$
θ
$x_\gamma(\alpha)$
$T(\gamma)$
$\gamma \in A$
$\exists \alpha < \lambda$
$x_\delta(\alpha) <_T x_\gamma(\alpha)$
$x_\delta(\alpha)$
$T(\delta)$
$\delta \in A$
0

It is easy to see that the existence of a λ-path through T implies the existence of an antichain of cardinality κ^+ in $\mathbb{P}_T$:

Fix such a λ-path $\langle x_\gamma : \lambda \longrightarrow T(\gamma) \mid \gamma \in A \rangle$. Given $\gamma \in A$, define p_γ to be the unique condition in $\mathbb{P}_T$ with $\mathrm{dom}(p_\gamma) = \mathrm{ran}(x_\gamma)$ *and*

$$p_\gamma(x_\gamma(\alpha)) \ = \ \alpha$$

for every $\alpha < \lambda$.

Fix $\gamma, \delta \in A$ with $\gamma < \delta$. Then there is an $\alpha < \lambda$ with $x_\gamma(\alpha) <_T x_\delta(\alpha)$ and

$$p_\gamma(x_\gamma(\alpha)) \ = \ p_\delta(x_\delta(\alpha)).$$

This shows that the conditions p_γ and p_δ are incompatible in $\mathbb{P}_T$.

It turns out that the converse of the above implication is also true.

Theorem

Let κ be an infinite cardinal with $\kappa = \kappa^{<\kappa}$. The following statements are equivalent for every κ^+-Aronszajn tree T.

- $\mathbb{P}_T$ *does not satisfy the κ^+-chain condition.*
- *There is a λ-path through T for some $\lambda < \kappa$.*

This characterization also shows that the forcing $\mathbb{P}_T$ is the canonical way to obtain an outer model in which T is κ-special, the cardinals κ and κ^+ are preserved and the assumption $\kappa = \kappa^{<\kappa}$ still holds.

Corollary

In the situation of the above theorem, the following statements are equivalent.

- $\mathbb{P}_T$ *satisfies the* κ^+*-chain condition.*
- *There in an outer model* W *of* V *such that* κ *is a cardinal with* $\kappa = \kappa^{<\kappa}$ *in* W, $(\kappa^+)^{\mathrm{V}} = (\kappa^+)^{\mathrm{W}}$ *and* T *is* κ*-special in* W.

Proof of "$\Longleftarrow$".

Assume, towards a contradiction, that there is a λ-path $\langle x_\gamma : \lambda \longrightarrow T(\gamma) \mid \gamma \in A\rangle$ through T for some $\lambda < \kappa$ and let $f : T \longrightarrow \kappa$ denote the specializing function in W.

Since $\kappa = \kappa^\lambda$ in W, there are $\gamma, \delta \in A$ with $\gamma \neq \delta$ and $f(x_\gamma(\alpha)) = f(x_\delta(\alpha))$ for all $\alpha < \lambda$. Then there is an $\alpha < \lambda$ with $x_\gamma(\alpha) <_T x_\delta(\alpha)$, a contradiction. $\square$

Specializing $\square(\kappa^+)$-sequences

We are interested in examples of κ^+-Aronszajn trees without λ-paths. These examples will be provided by $\square(\kappa^+)$-sequences.

Definition

Given an uncountable regular cardinal θ, we call a sequence $\vec{C} = \langle C_\alpha \mid \alpha < \theta \rangle$ a $\square(\theta)$-*sequence* if the following statements hold for all $\alpha < \theta$.

- C_α is a club subset of α and $C_{\alpha+1} = \{\alpha\}$.
- If $\bar{\alpha} \in \mathrm{Lim}(C_\alpha)$, then $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$.
- If C is a club subset of θ, then there is a $\beta \in \mathrm{Lim}(C)$ with $C_\beta \neq C \cap \beta$.

Given such a $\square(\theta)$-sequence $\vec{C}$, we define $T(\vec{C})$ to be the tree $\langle \theta, <_{\vec{C}} \rangle$ with

$$\alpha <_{\vec{C}} \beta \iff \alpha \in \mathrm{Lim}(C_\beta).$$

If $\theta = \kappa^+$, then we say that the sequence $\vec{C}$ is *special* if the tree $T(\vec{C})$ is κ-special.

Let κ be an infinite cardinal and $\vec{C}$ be a $\square(\kappa^+)$-sequence. Todorčević constructed a canonical κ^+-Aronszajn tree $T(\rho_0^{\vec{C}})$ from $\vec{C}$ using *minimal walks through* $\vec{C}$.

It can be shown that there is no λ-path through a tree of the form $T(\rho_0^{\vec{C}})$. Since the tree $T_{\vec{C}}$ is κ-special if and only if the tree $T(\rho_0^{\vec{C}})$ is κ-special, this gives rise to the following result.

Theorem

Let κ be an infinite cardinal with $\kappa = \kappa^{<\kappa}$. If $\vec{C}$ is a $\square(\kappa^+)$-sequence, then the partial order $\mathbb{P}_{T_{\vec{C}}}$ satisfies the κ^+-chain condition.

An application: Generalizations of Martin's Axiom to higher cardinalities

Motivation

We consider generalizations of Martin's Axiom to classes of σ-closed forcings satisfying the $\aleph_2$-chain condition.

Definition

Given a partial order $\mathbb{P}$ and an infinite cardinal κ, we let $\mathsf{FA}_\kappa(\mathbb{P})$ denote the statement that for every collection $\mathcal{D}$ of κ-many dense subsets of $\mathbb{P}$, there is a filter G in $\mathbb{P}$ that meets all elements of $\mathcal{D}$.

The following result shows that the obvious generalization of Martin's Axiom to this class of forcings is inconsistent.

Theorem (Shelah)

If $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} > \aleph_2$, then there is a σ-closed partial order $\mathbb{P}$ satisfying the $\aleph_2$-chain condition such that $\mathsf{FA}_{\aleph_2}(\mathbb{P})$ fails.

Therefore we have to restrict ourselves to certain classes of certain classes of σ-closed partial orders satisfying the $\aleph_2$-chain condition.

Baumgartner provided a consistent example of such a generalization.

Definition

We let BA denote the statement that $\mathsf{FA}_{\aleph_2}(\mathbb{P})$ holds for every partial order with the following properties.

- $\mathbb{P}$ is well-met ("*compatible conditions have an infimum*").
- $\mathbb{P}$ is σ-closed.
- $\mathbb{P}$ is $\aleph_1$-linked ("*there is a map* $c : \mathbb{P} \longrightarrow \omega_1$ *such that all conditions* p *and* q *in* $\mathbb{P}$ *with* $c(p) = c(q)$ *are compatible*").

Theorem (Baumgartner)

If GCH *holds, then there is a* σ*-closed partial order satisfying the* $\aleph_2$*-chain condition that forces* BA *to hold.*

Using ideas from Baumgartner's proof, it is possible to prove the consistency of stronger generalizations using large cardinals.

We let C denote the *Chang Model*, i.e. the smallest inner model closed under countable sequences.

Definition

We let $\mathsf{GMA}(\mathrm{C})$ denote the statement that $\mathsf{FA}_{\aleph_2}(\mathbb{P})$ holds for every partial order with the following properties.

- $\mathbb{P}$ is well-met.
- $\mathbb{P}$ is σ-closed.
- $\mathbb{P}$ is representable in a partial order $\mathbb{Q}$ that satisfies the $\aleph_2$-chain condition and is an element of C ("*there is a map* $c : \mathbb{P} \longrightarrow \mathbb{Q}$ *that sends pairs of incompatible conditions to pairs of incompatible conditions*").

Theorem

Assume that GCH *holds,* κ *is a weakly compact cardinal and* G *is* $\mathrm{Col}(\omega_1, {<}\kappa)$*-generic over* V*. In* $\mathrm{V}[G]$*, there is a* σ*-closed partial order satisfying the* $\aleph_2$*-chain condition that forces* $\mathsf{GMA}(\mathrm{C})$ *to hold.*

This result raises the following questions.

Question

Is the use of large cardinals necessary to prove the consistency of stronger generalizations of Martin's Axiom ?
More specifically, if ω_2 *is not a large cardinal in* L*, does* $\mathsf{FA}_{\aleph_2}(\mathbb{P})$ *fail for some* σ*-closed partial order* $\mathbb{P}$ *that satisfies the* $\aleph_2$*-condition and is an element of* C *?*

The above results allow us to conclude that this axiom causes certain $\square(\omega_2)$-sequences to be special.

Corollary

Assume that $\mathsf{GMA}(\mathrm{C}) + \mathrm{CH}$ *holds. Then every* $\square(\omega_2)$*-sequence contained in* C *is special.*

It is possible to derive consistency strength from the above conclusion with the help of classical results of Jensen and methods to construct non-special square sequences developed by Todorčević.

Theorem

Assume that CH *holds and every* $\square(\omega_2)$*-sequence contained in* $\mathrm{L}[x]$ *for some* $x : \omega \longrightarrow \mathrm{On}$ *is special. Let* $\theta = \omega_2$*.*

- θ *is a Mahlo cardinal in* L*.*
- *If* V *is a forcing extension of* L *by a forcing that either is* σ*-strategically closed in* L *or satisfies the* θ*-chain condition in* L*, then* θ *is weakly compact in* L*.*

Thank you for listening!