Specializing Aronszajn trees and square sequences by forcing

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Specializing $\kappa^+$-Aronszajn trees
We start by recalling some basic definitions concerning trees of uncountable height.

**Definition**

Let $\theta$ be an uncountable regular cardinal. A tree $T$ of height $\theta$ is a $\theta$-Aronszajn tree if $T$ has no cofinal branches and every level of $T$ has cardinality less than $\theta$.

**Definition**

Let $\kappa$ be an infinite cardinal and $T$ be a tree. We say that $T$ is $\kappa$-special if there is a function $f : T \rightarrow \kappa$ that is injective on chains in $T$. 
Given an infinite cardinal $\kappa$ and a $\kappa^+$-Aronszajn tree $T$, there is a canonical $<\kappa$-closed forcing $\mathbb{P}_T$ that specializes $T$. This partial order consists of partial specializing functions $q : T \xrightarrow{\text{part}} \kappa$ of cardinality less than $\kappa$ ordered by reverse inclusion.

In the case “$\kappa = \omega$”, this forcing can be used to show that Martin’s Axiom implies that all Aronszajn trees are special.

**Theorem (Baumgartner-Malitz-Reinhardt)**

*If $T$ is an Aronszajn tree, then $\mathbb{P}_T$ satisfies the countable chain condition.*
In contrast, it is consistent that forcings of the form $\mathbb{P}_T$ can collapse cardinals. This can be shown using a notion introduced by Laver.

**Definition (Laver)**

Let $\theta$ be an uncountable regular cardinal and $T$ be a tree of cardinality and height $\theta$. A sequence

$$\langle x_\gamma : \omega \rightarrow T(\gamma) \mid \gamma \in A \rangle$$

is an $\omega$-ascent path through $T$ if the following statements hold.

- $A$ is an unbounded subset of $\theta$.
- If $\gamma, \delta \in A$ with $\gamma < \delta$, then there is an $N < \omega$ such that
  $$x_\gamma(n) <_T x_\delta(n)$$

for all $N \leq n < \omega$. 
$A \subseteq \theta$

$T(\gamma)$

$x_\gamma(n)$

$n < \omega$

$\gamma \in A$

$\theta$

$0$
$A \subseteq \theta$

$T(\gamma)$

$T(\delta)$

$\exists N < \omega$

$\forall N \leq m < \omega$

$x_\delta(m) <_T x_\gamma(m)$
Theorem (Shelah)

Let \( \kappa \) be a cardinal of uncountable cofinality and \( T \) be a \( \kappa^+ \)-Aronszajn tree. If there is an \( \omega \)-ascent paths through \( T \), then \( T \) is not \( \kappa \)-special.

Theorem (Shelah-Stanley/Todorčević)

Let \( \kappa \) be a cardinal of uncountable cofinality. If \( \Box_\kappa \) holds, then there is a \( \kappa^+ \)-Aronszajn tree with an \( \omega \)-ascent path.

Corollary

If \( \Box_\kappa \) holds, then there is a \( \kappa^+ \)-Aronszajn tree \( T \) with the property that forcing with \( \mathbb{P}_T \) collapses \( \kappa^+ \).
Given an infinite cardinal $\kappa$ with $\kappa = \kappa^{<\kappa}$, we want to characterize the class of all $\kappa^+$-Aronszajn trees $T$ such that the partial order $\mathbb{P}_T$ satisfies the $\kappa^+$-chain condition.

This characterization uses the following variation of the above concept.

**Definition**

Let $\kappa$ be an infinite cardinal and $T$ be a tree of height $\kappa^+$. Given $\lambda < \kappa$, we call a sequence

$$\langle x_\gamma : \lambda \rightarrow T(\gamma) \mid \gamma \in A \rangle$$

of injections a $\lambda$-path through $T$ if the following statements hold.

- $A$ is an unbounded subset of $\kappa^+$.
- If $\gamma, \delta \in A$ with $\gamma < \delta$, then there is an $\alpha < \lambda$ with

  $$x_\gamma(\alpha) <_T x_\delta(\alpha).$$
$T$ \\

$A \subseteq \theta$ \\

$T(\gamma)$ \\

$x_\gamma(\alpha)$ \\

$\alpha < \lambda$ \\

$\gamma \in A$
$A \subseteq \theta$

$\exists \alpha < \lambda$

$x_\delta(\alpha) \prec_T x_\gamma(\alpha)$
It is easy to see that the existence of a $\lambda$-path through $T$ implies the existence of an antichain of cardinality $\kappa^+$ in $\mathbb{P}_T$:

Fix such a $\lambda$-path $\langle x_\gamma : \lambda \longrightarrow T(\gamma) \mid \gamma \in A \rangle$. Given $\gamma \in A$, define $p_\gamma$ to be the unique condition in $\mathbb{P}_T$ with $\text{dom}(p_\gamma) = \text{ran}(x_\gamma)$ and

$$p_\gamma(x_\gamma(\alpha)) = \alpha$$

for every $\alpha < \lambda$.

Fix $\gamma, \delta \in A$ with $\gamma < \delta$. Then there is an $\alpha < \lambda$ with $x_\gamma(\alpha) <_T x_\delta(\alpha)$ and

$$p_\gamma(x_\gamma(\alpha)) = p_\delta(x_\delta(\alpha)).$$

This shows that the conditions $p_\gamma$ and $p_\delta$ are incompatible in $\mathbb{P}_T$. 
It turns out that the converse of the above implication is also true.

**Theorem**

Let $\kappa$ be an infinite cardinal with $\kappa = \kappa^{<\kappa}$. The following statements are equivalent for every $\kappa^+$-Aronszajn tree $T$.

- $\mathbb{P}_T$ does not satisfy the $\kappa^+$-chain condition.
- There is a $\lambda$-path through $T$ for some $\lambda < \kappa$.

This characterization also shows that the forcing $\mathbb{P}_T$ is the canonical way to obtain an outer model in which $T$ is $\kappa$-special, the cardinals $\kappa$ and $\kappa^+$ are preserved and the assumption $\kappa = \kappa^{<\kappa}$ still holds.
Corollary

In the situation of the above theorem, the following statements are equivalent.

- $\mathbb{P}_T$ satisfies the $\kappa^+$-chain condition.
- There is an outer model $W$ of $V$ such that $\kappa$ is a cardinal with $\kappa = \kappa^{<\kappa}$ in $W$, $(\kappa^+)^V = (\kappa^+)^W$ and $T$ is $\kappa$-special in $W$.

Proof of “$\Leftarrow$”.

Assume, towards a contradiction, that there is a $\lambda$-path $\langle x_\gamma : \lambda \rightarrow T(\gamma) \mid \gamma \in A \rangle$ through $T$ for some $\lambda < \kappa$ and let $f : T \rightarrow \kappa$ denote the specializing function in $W$.

Since $\kappa = \kappa^\lambda$ in $W$, there are $\gamma, \delta \in A$ with $\gamma \neq \delta$ and $f(x_\gamma(\alpha)) = f(x_\delta(\alpha))$ for all $\alpha < \lambda$. Then there is an $\alpha < \lambda$ with $x_\gamma(\alpha) <_T x_\delta(\alpha)$, a contradiction.
Specializing □(κ⁺)-sequences
We are interested in examples of $\kappa^+$-Aronszajn trees without $\lambda$-paths. These examples will be provided by $\Box(\kappa^+)$-sequences.

**Definition**

Given an uncountable regular cardinal $\theta$, we call a sequence $\vec{C} = \langle C_\alpha \mid \alpha < \theta \rangle$ a $\Box(\theta)$-sequence if the following statements hold for all $\alpha < \theta$.

- $C_\alpha$ is a club subset of $\alpha$ and $C_{\alpha+1} = \{\alpha\}$.
- If $\bar{\alpha} \in \text{Lim}(C_\alpha)$, then $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$.
- If $C$ is a club subset of $\theta$, then there is a $\beta \in \text{Lim}(C)$ with $C_\beta \neq C \cap \beta$.

Given such a $\Box(\theta)$-sequence $\vec{C}$, we define $T(\vec{C})$ to be the tree $\langle \theta, <_{\vec{=}}) \rangle$ with

$$\alpha <_{\vec{=}} \beta \iff \alpha \in \text{Lim}(C_\beta).$$

If $\theta = \kappa^+$, then we say that the sequence $\vec{C}$ is special if the tree $T(\vec{C})$ is $\kappa$-special.
Let $\kappa$ be an infinite cardinal and $\vec{C}$ be a $\square(\kappa^+)$-sequence. Todorčević constructed a canonical $\kappa^+$-Aronszajn tree $T(\rho_0\vec{C})$ from $\vec{C}$ using minimal walks through $\vec{C}$.

It can be shown that there is no $\lambda$-path through a tree of the form $T(\rho_0\vec{C})$. Since the tree $T_{\vec{C}}$ is $\kappa$-special if and only if the tree $T(\rho_0\vec{C})$ is $\kappa$-special, this gives rise to the following result.

**Theorem**

Let $\kappa$ be an infinite cardinal with $\kappa = \kappa^{<\kappa}$. If $\vec{C}$ is a $\square(\kappa^+)$-sequence, then the partial order $\mathbb{P}_{T_{\vec{C}}}$ satisfies the $\kappa^+$-chain condition.
An application: Generalizations of Martin’s Axiom to higher cardinalities
Motivation

We consider generalizations of Martin’s Axiom to classes of \(\sigma\)-closed forcings satisfying the \(\aleph_2\)-chain condition.

Definition

Given a partial order \(\mathbb{P}\) and an infinite cardinal \(\kappa\), we let \(\text{FA}_\kappa(\mathbb{P})\) denote the statement that for every collection \(\mathcal{D}\) of \(\kappa\)-many dense subsets of \(\mathbb{P}\), there is a filter \(G\) in \(\mathbb{P}\) that meets all elements of \(\mathcal{D}\).
The following result shows that the obvious generalization of Martin’s Axiom to this class of forcings is inconsistent.

**Theorem (Shelah)**

If $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} > \aleph_2$, then there is a $\sigma$-closed partial order $\mathbb{P}$ satisfying the $\aleph_2$-chain condition such that $\text{FA}_{\aleph_2}(\mathbb{P})$ fails.

Therefore we have to restrict ourselves to certain classes of certain classes of $\sigma$-closed partial orders satisfying the $\aleph_2$-chain condition.
Baumgartner provided a consistent example of such a generalization.

**Definition**

We let $\text{BA}$ denote the statement that $\text{FA}_{\aleph_2}(\mathbb{P})$ holds for every partial order with the following properties.

- $\mathbb{P}$ is well-met ("compatible conditions have an infimum").
- $\mathbb{P}$ is $\sigma$-closed.
- $\mathbb{P}$ is $\aleph_1$-linked ("there is a map $c : \mathbb{P} \to \omega_1$ such that all conditions $p$ and $q$ in $\mathbb{P}$ with $c(p) = c(q)$ are compatible").

**Theorem (Baumgartner)**

*If GCH holds, then there is a $\sigma$-closed partial order satisfying the $\aleph_2$-chain condition that forces $\text{BA}$ to hold.*
Using ideas from Baumgartner’s proof, it is possible to prove the consistency of stronger generalizations using large cardinals.

We let $C$ denote the *Chang Model*, i.e. the smallest inner model closed under countable sequences.

**Definition**

We let $\text{GMA}(C)$ denote the statement that $\text{FA}_{\aleph_2}(\mathbb{P})$ holds for every partial order with the following properties.

- $\mathbb{P}$ is well-met.
- $\mathbb{P}$ is $\sigma$-closed.
- $\mathbb{P}$ is representable in a partial order $\mathbb{Q}$ that satisfies the $\aleph_2$-chain condition and is an element of $C$ ("there is a map $c : \mathbb{P} \rightarrow \mathbb{Q}$ that sends pairs of incompatible conditions to pairs of incompatible conditions").
Theorem

Assume that GCH holds, $\kappa$ is a weakly compact cardinal and $G$ is $\text{Col}(\omega_1, <\kappa)$-generic over $V$. In $V[G]$, there is a $\sigma$-closed partial order satisfying the $\aleph_2$-chain condition that forces $\text{GMA}(C)$ to hold.

This result raises the following questions.

Question

Is the use of large cardinals necessary to prove the consistency of stronger generalizations of Martin’s Axiom? More specifically, if $\omega_2$ is not a large cardinal in $L$, does $\text{FA}_{\aleph_2}(P)$ fail for some $\sigma$-closed partial order $P$ that satisfies the $\aleph_2$-condition and is an element of $C$?
The above results allow us to conclude that this axiom causes certain $\square(\omega_2)$-sequences to be special.

**Corollary**

Assume that $\text{GMA}(C) + \text{CH}$ holds. Then every $\square(\omega_2)$-sequence contained in $C$ is special.

It is possible to derive consistency strength from the above conclusion with the help of classical results of Jensen and methods to construct non-special square sequences developed by Todorčević.
Theorem

Assume that CH holds and every \( \square(\omega_2) \)-sequence contained in \( L[x] \) for some \( x : \omega \rightarrow \text{On} \) is special. Let \( \theta = \omega_2 \).

- \( \theta \) is a Mahlo cardinal in \( L \).

- If \( V \) is a forcing extension of \( L \) by a forcing that either is \( \sigma \)-strategically closed in \( L \) or satisfies the \( \theta \)-chain condition in \( L \), then \( \theta \) is weakly compact in \( L \).
Thank you for listening!