Simple definitions of complicated sets

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Introduction

Set-theoretic objects whose construction requires the Axiom of Choice are frequently referred to as *pathological sets*.

We list some prominent examples of such sets:

- Non-Lebesgue measurable sets of real numbers.
- Hamel bases of the real numbers over the rational numbers.
- Well-orderings of power sets of infinite cardinals.
- Bi-stationary (i.e. stationary and co-stationary) subsets of uncountable regular cardinals.
- Colourings witnessing failures of weak compactness at accessible cardinals.

It is natural to ask whether there are set-theoretical properties that can be used to distinguish pathological sets from objects that are explicitly constructed.

Results from descriptive set theory show that pathological sets of real numbers cannot be defined by simple formulas in second-order arithmetic.

Moreover, both strong large cardinal assumptions and forcing axioms imply that this implication can be extended to arbitrary formulas.

In this talk, I want to present results dealing with the *set-theoretic definability* of pathological objects, i.e. with the question whether objects usually obtained from the Axiom of Choice can be defined in the structure $\langle \mathrm{V}, \in \rangle$ using simple formulas.

I will focus on the definability of well-orderings of the reals and bi-stationary subsets of uncountable regular cardinals.

We start by making the notion of *simple formulas* more precise.

First, we will restrict ourselves to formulas that only use cardinals and sets of small hereditary cardinality as parameters.

Next, we measure the complexity of formulas using the *Levy hierarchy*.

Remember that a formula in the language $\mathcal{L}_\in = \{\in\}$ of set theory is a Σ_0*-formula* if it is contained in the smallest collection of $\mathcal{L}_\in$-formulas that contains all atomic formulas and is closed under negations, conjunctions and bounded quantification.

Moreover, a $\mathcal{L}_\in$-formula is a Σ_{n+1}*-formula* for some $n < \omega$ if it is of the form $\exists x \, \neg\varphi$ for some Σ_n-formula φ.

Note that the class of all formulas that are $\mathbf{ZFC}$-provably equivalent to a Σ_{n+1}-formula is closed under existential quantification, bounded quantification, conjunctions and disjunctions.

The following observation shows that the we only have to consider Σ_n-formulas for $n < 3$ when we study the definability of the pathological objects listed above.

Proposition

Let X be a class of sets of ordinals with the property that both X and $\mathrm{V} \setminus X$ are definable by Σ_n-formulas using some parameter y.

If z is a set with the property that $\mathrm{HOD}_z \cap X \neq \emptyset$, then there is an $A \in \mathrm{HOD}_z \cap X$ such that the set $\{A\}$ is $\Sigma_n(y, z)$-definable.

Moreover, this proposition shows that the assumption $\mathrm{V} = \mathrm{HOD}$ implies the existence of pathological sets of various types that are definable by Σ_2-formulas that either use no parameters or just the relevant cardinal as a parameter.

Since standard arguments show that pathological sets cannot be defined by Σ_0-formulas with simple parameters, we will focus on Σ_1-definitions.

Well-orderings of the reals

We start by combining classical results to study well-orderings of $\mathbb{R}$ that are definable by Σ_1-formulas that only use reals as parameters.

Proposition

The following statements are equivalent for every subset X of $\mathbb{R}$:

- *X is $\mathbf{\Sigma}^1_2$-definable.*
- *X is definable by a Σ_1-formula with parameters in $\mathrm{H}(\omega_1)$.*

A classical result of Mansfield can now be phrased in the following way:

Theorem (Mansfield)

The following statements are equivalent:

- *There is a well-ordering of* $\mathbb{R}$ *that is definable by a* Σ_1*-formula with parameters in* $\mathrm{H}(\omega_1)$.
- *There exists an* $x \in \mathbb{R}$ *with* $\mathbb{R} \subseteq \mathrm{L}[x]$.

Corollary

If $x^{\#}$ *exists for every* $x \in \mathbb{R}$*, then no well-ordering of* $\mathbb{R}$ *is definable by a* Σ_1*-formula with parameters in* $\mathrm{H}(\omega_1)$.

The next result extends the above characterization of $\mathbf{\Sigma}^1_2$-sets to the next level of the projective hierarchy.

Theorem (L.–Schindler–Schlicht)

Assume that one of the following statements holds:

- *There is a measurable cardinal above a Woodin cardinal.*
- *Bounded Martin's Maximum* $\mathbf{BMM}$ *holds and the non-stationary ideal on* ω_1 *is precipitous.*
- *There is a precipitous ideal on* ω_1 *and a measurable cardinal.*

Then the following statements are equivalent for every subset X *of* $\mathbb{R}$*:*

- X *is* $\mathbf{\Sigma}^1_3$*-definable.*
- X *is definable by a* Σ_1*-formula with parameters in* $\mathrm{H}(\omega_1) \cup \{\omega_1\}$*.*

Sketch of the Proof.

Assume that either there is a measurable cardinal above a Woodin cardinal or that $\mathbf{BMM}$ holds and the non-stationary ideal on ω_1 is precipitous.

Fix a Σ_1-formula $\varphi(v_0, v_1, v_2)$ and $z \in \mathrm{H}(\omega_1)$ with the property that

$$X = \{x \in \mathbb{R} \mid \varphi(\omega_1, x, z)\}.$$

Define Y to be the set of all $y \in \mathbb{R}$ with the property that there is a countable transitive model M of $\mathbf{ZFC}^-$ and $\delta \in M$ such that $y, z \in M$ and the following statements hold:

- δ is a Woodin cardinal in M and M ω_1-iterable with respect to the *countable stationary tower forcing* $\mathbb{Q}^M_{<\delta}$ and its images.
- $\varphi(\omega_1^M, y, z)$ holds.

Then M is a $\mathbf{\Sigma}^1_3$-subset of $\mathbb{R}$.

Moreover, it can be shown that $X = Y$. □

Corollary (L.–Schindler–Schlicht)

Assume that one of the following statements holds:

- *There is a measurable cardinal above a Woodin cardinal.*
- $\mathbf{BMM}$ *holds and the non-stationary ideal on* ω_1 *is precipitous.*
- *There is a precipitous ideal on* ω_1 *and a measurable cardinal.*

Then no well-ordering of $\mathbb{R}$ *is definable by a* Σ_1*-formula with parameters in* $\mathrm{H}(\omega_1) \cup \{\omega_1\}$.

The following result shows that the above large cardinal assumptions is almost optimal.

Moreover, it shows that the above characterization of $\mathbf{\Sigma}^1_3$-subsets of $\mathbb{R}$ does not follow from substantially weaker large cardinal assumptions.

Theorem (L.–Schindler–Schlicht)

Suppose that M_1 exists. Then the following statements hold in M_1:

- *The canonical well-ordering of M_1 restricted to $\mathrm{H}(\omega_2)$ is definable by a Σ_1-formula with parameter ω_1.*
- *The set $\{\mathbb{R}\}$ is definable by a Σ_1-formula with parameter ω_1.*

Under stronger large cardinal assumptions, the above results can be extended to Σ_1-definitions with larger classes of parameters.

Theorem (L.–Schindler–Schlicht)

Suppose that there is a proper class of Woodin cardinals.

If B is a universally Baire set of reals, then no well-ordering of $\mathbb{R}$ is definable over the structure $\langle \mathrm{H}(\omega_2), \in, B, \mathrm{NS}_{\omega_1} \rangle$ by a Σ_1-formula with parameters in $\mathrm{H}(\omega_1) \cup \{\omega_1\}$.

By replacing generic iterations with standard iterations of set-sized models, we can generalize the above results to Σ_1-definitions that use large cardinals as parameters.

Definition

Let κ be an uncountable cardinal.

- A *weak κ-model* is a transitive model M of $\mathbf{ZFC}^{-}$ of size κ with $\kappa \in M$.
- (Sharpe–Welch) The cardinal κ is *ω_1-iterable* if for every subset A of κ there is a weak κ-model M and a weakly amenable M-ultrafilter U on κ such that $A \in M$ and the structure $\langle M, \in, U \rangle$ is ω_1-iterable.

Theorem (L.–Schindler–Schlicht)

If κ is either an ω_1-iterable cardinal or a regular cardinal that is a stationary limit of ω_1-iterable cardinals, then the following statements are equivalent for every subset X of $\mathbb{R}$:

- *X is $\boldsymbol{\Sigma}^1_3$-definable.*
- *X is definable by a Σ_1-formula with parameters in $\mathrm{H}(\omega_1) \cup \{\kappa\}$.*

Sketch of the Proof.

Let κ be an uncountable regular cardinal that is either ω_1-iterable or a stationary limit of ω_1-iterable cardinals.

Assume that $\varphi(v_0, v_1, v_2)$ is a Σ_1-formula and $z \in \mathrm{H}(\omega_1)$ with

$$X = \{x \in \mathbb{R} \mid \varphi(x, z, \kappa)\}.$$

Define Y to be the set of all $y \in \mathbb{R}$ with the property that there is a countable transitive model M of $\mathbf{ZFC}^-$ with $y, z \in M$ and a cardinal δ of M such that the following statements hold:

- $\varphi(y, z, \delta)$ holds in M.
- There is a weakly amenable M-ultrafilter F on δ with the property that the structure $\langle M, \in, F\rangle$ is ω_1-iterable.

Then Y is a $\mathbf{\Sigma}^1_3$-subset of $\mathbb{R}$.

Moreover, it can be shown that $X = Y$. □

Using the fact that measurable cardinals are ω_1-iterable and therefore Woodin cardinal are stationary limits of ω_1-iterable cardinals, we obtain the following corollary:

Corollary

If κ is either a measurable cardinal above a Woodin cardinal or a Woodin cardinal below a measurable cardinal, then no well-ordering of the reals is definable by a Σ_1-formula with parameters in $\mathrm{H}(\omega_1) \cup \{\kappa\}$.

In contrast, it turns out that the existence of a well-ordering of $\mathbb{R}$ that is definable by a Σ_1-formula with parameter ω_2 is compatible with all large cardinal assumptions.

This statement relies on the following observation:

Lemma

If the Bounded Proper Forcing Axioms $\mathbf{BPFA}$ *holds, then the set* $\{\mathrm{H}(\omega_2)\}$ *is definable by a* Σ_1*-formula that only uses the cardinal* ω_2 *as a parameter.*

The lemma is proven using techniques developed by Caicedo and Veličković that yield a finite fragment F of $\mathbf{ZFC}^- + \mathbf{BPFA} +$ "ω_2 *exists*" with the property that $\mathbf{ZFC} + \mathbf{BPFA}$ proves that every transitive model M of F with $\omega_2 = \omega_2^M$ contains all reals.

Corollary

Assume that $\mathbf{BPFA}$ *holds. If there is a well-ordering* $\lhd$ *of* $\mathbb{R}$ *that is definable over the structure* $\mathrm{H}(\omega_2), \in\rangle$ *by a formula with parameter* $z \in \mathrm{H}(\omega_2)$*, then the well-ordering* $\lhd$ *is definable by a* Σ_1*-formula that only uses* ω_2 *and* z *as parameters.*

- (Asperó) If κ is supercompact, then there is a semi-proper partial order $\mathbb{P} \subseteq \mathrm{H}(\kappa)$ with the property that whenever G is $\mathbb{P}$-generic over V, then $\mathbf{PFA}^{++}$ holds in $\mathrm{V}[G]$ and there is a well-ordering of $\mathrm{H}(\omega_2)^{\mathrm{V}[G]}$ that is definable over $\langle \mathrm{H}(\omega_2), \in\rangle$ by a formula without parameters.
- (Larson) If κ is a supercompact limit of supercompact cardinals, then there is a semi-proper partial order $\mathbb{P}$ with the property that whenever G is $\mathbb{P}$-generic over V, then $\mathbf{MM}^{+\omega}$ holds in $\mathrm{V}[G]$ and there is a well-ordering of $\mathrm{H}(\omega_2)^{\mathrm{V}[G]}$ that is definable over $\langle \mathrm{H}(\omega_2), \in\rangle$ by a formula without parameters.

Bi-stationary subsets

We now want to study the definability of *bi-stationary* subsets of uncountable regular cardinals, focussing on the property defined below.

Definition

Given $n < \omega$, an uncountable regular cardinal κ has the $\mathbf{\Sigma}_n$*-club property* if for all $A \subseteq \kappa$ with the property that the set $\{A\}$ is definable by a Σ_1-formula with parameters in $\mathrm{H}(\kappa) \cup \{\kappa\}$, then A either contains a club subset of κ or is disjoint from such a set.

Proposition

Every uncountable regular cardinal has the $\mathbf{\Sigma}_0$*-club property.*

For $n > 0$, the $\mathbf{\Sigma}_n$-club property can be shown to be equivalent to a strong partition property for definable colourings:

Lemma

The following statements are equivalent for every uncountable regular cardinal κ and every $0 < n < \omega$:

- *κ has the $\mathbf{\Sigma}_n$-club property.*
- *For every colouring $c : [\kappa]^{<\omega} \longrightarrow \alpha$ with $\alpha < \kappa$ that is definable by a Σ_n-formula with parameters in $\mathrm{H}(\kappa) \cup \{\kappa\}$, there is a c-homogeneous closed unbounded subset of κ.*

Using techniques developed in the proof of the above results, it is possible to prove the following theorem.

Theorem

Assume that one of the following assumptions holds:

- *There is a measurable cardinal above a Woodin cardinal.*
- *There is a measurable cardinal and a precipitous ideal on* ω_1.
- $\mathbf{BMM}$ *holds and the non-stationary ideal on* ω_1 *is precipitous.*
- *Woodin's Axiom* $(*)$ *holds.*

Then ω_1 *has the* $\mathbf{\Sigma}_1$*-club property.*

Remember that Woodin's Axiom $(*)$ states that $\mathbf{AD}$ holds in $\mathrm{L}(\mathbb{R})$ and there is some G that is $\mathbb{P}_{max}$-generic over $\mathrm{L}(\mathbb{R})$ with $\mathcal{P}(\omega_1) \subseteq \mathrm{L}(\mathbb{R})[G]$.

Proposition

Assume that $\mathbf{AD}$ *holds in* $\mathrm{L}(\mathbb{R})$ *and* G *is* $\mathbb{P}_{max}$*-generic over* $\mathrm{L}(\mathbb{R})$*. Then* ω_1 *has the* $\mathbf{\Sigma}_2$*-club property in* $\mathrm{L}(\mathbb{R})[G]$*.*

Now, assume that $(*)$ holds and G that is $\mathbb{P}_{max}$-generic over $\mathrm{L}(\mathbb{R})$ with $\mathcal{P}(\omega_1) \subseteq \mathrm{L}(\mathbb{R})[G]$.

Fix $z \in \mathrm{H}(\omega_1)$ and a subset A of ω_1 with the property that the set $\{A\}$ is definable by a Σ_1-formula with parameters ω_1 and z.

Since $\mathrm{H}(\omega_2) \subseteq \mathrm{L}(\mathbb{R})[G]$, the same formula defines $\{A\}$ in $\mathrm{L}(\mathbb{R})[G]$ and therefore the above proposition implies that A either contains a club or is disjoint from such a subset.

Theorem (L.–Schindler–Schlicht)

Assume that one of the following assumptions holds:

- *There is a measurable cardinal above a Woodin cardinal.*
- *There is a measurable cardinal and a precipitous ideal on ω_1.*
- $\mathbf{BMM}$ *holds and the non-stationary ideal on ω_1 is precipitous.*

Then the following statements hold for every Σ_1-formula $\varphi(v_0, v_1, v_2)$ and all $z \in \mathrm{H}(\omega_1)$:

- *If there is $A \subseteq \omega_1$ stationary with $\varphi(A, \omega_1, z)$, then there is an element B of the club filter on ω_1 with $\varphi(B, \omega_1, z)$.*
- *If there is $A \subseteq \omega_1$ co-stationary with $\varphi(A, \omega_1, z)$, then there is an element B of the non-stationary ideal on ω_1 with $\varphi(B, \omega_1, z)$.*

The proof of this theorem again uses iterated generic ultrapowers and Woodin's countable stationary tower forcing.

Similar variations of our earlier results also allow us to show that certain large cardinals possess the $\mathbf{\Sigma}_1$-club property.

Lemma

Let κ be an uncountable regular cardinal, let $z \in \mathrm{H}(\kappa)$ and let $\varphi(v_0, v_1, v_2)$ be a Σ_1-formula. Assume that there is a unique subset A of κ with the property that $\varphi(A, \kappa, z)$ holds. If there exist a weak κ-model M with

$$A, \mathrm{tc}(\{z\}) \in M \models \varphi(A, \kappa, z)$$

and a weakly amenable M-ultrafilter U on κ such that $\langle M, \in, U\rangle$ is ω_1-iterable, then A either contains a club subset of κ or is disjoint from such a set.

Corollary

Both ω_1-iterable cardinals and regular cardinals that are stationary limits of ω_1-iterable cardinals have the $\mathbf{\Sigma}_1$-club property.

We now consider the question whether other types of cardinals can possess the $\mathbf{\Sigma}_n$-club property for $n > 0$.

The following proposition shows that a cardinal with the $\mathbf{\Sigma}_1$-club property is either equal to ω_1 or a limit cardinal.

Moreover, it shows that ω_1 is the only cardinal that can possess the $\mathbf{\Sigma}_2$-club property.

Proposition

- *If ν is an uncountable cardinal, then ν^+ does not have the $\mathbf{\Sigma}_1$-club property.*
- *Regular cardinals greater than ω_1 do not have the $\mathbf{\Sigma}_2$-club property.*

Proof.

First, fix an infinite regular cardinal μ and $\gamma \in \mathrm{Lim} \cap \mu^+$.

If there is a strictly increasing cofinal function $s : \mu \longrightarrow \gamma$, then $\mathrm{cof}(\gamma) = \mu$. In the other case, if there is a limit ordinal $\lambda < \mu$ and a strictly increasing cofinal function $s : \lambda \longrightarrow \gamma$, then $\mathrm{cof}(\gamma) < \mu$.

Hence $\{S^{\mu^+}_{\mu}\}$ is definable by a Σ_1-formula with parameters μ^+ and μ.

Now, assume that there is an uncountable cardinal ν such that the cardinal ν^+ has the $\boldsymbol{\Sigma}_1$-club property. By the above computations, ν is singular.

Let z denote the set of all uncountable regular cardinals less than ν.

Then the set $S^{\nu^+}_{\omega}$ consists of all limit ordinals $\lambda < \nu^+$ with the property that there is no strictly increasing cofinal function $s : \mu \longrightarrow \lambda$ with $\mu \in z$.

Hence $\{S^{\nu^+}_{\omega}\}$ is definable by a Σ_1-formula with parameters ν^+ and z.

Finally, observare that if κ is an uncountable regular cardinal, then the set S^{κ}_{ω} is definable by a Σ_2-formula with parameter κ. $\square$

The following results show that all other constellations are consistent:

Lemma

Let κ be an inaccessible cardinal, let $\mathbb{P} \in \{\mathrm{Add}(\omega, \kappa), \mathrm{Col}(\omega, {<}\kappa)\}$ and let G be $\mathbb{P}$-generic over V. If κ has the $\mathbf{\Sigma}_1$-club property in V, then κ has the $\mathbf{\Sigma}_1$-club property in $\mathrm{V}[G]$.

Lemma

If κ is a measurable cardinal, then there is a generic extension $\mathrm{V}[G]$ of V with the property that $\kappa = \omega_1^{\mathrm{V}[G]}$ and no bi-stationary subset of ω_1 in $\mathrm{V}[G]$ is contained in $\mathrm{HOD}(\mathbb{R})^{\mathrm{V}[G]}$.

In particular, in $\mathrm{V}[G]$, the cardinal ω_1 has the $\mathbf{\Sigma}_2$-club property.

We will now show that the above results actually yield the correct consistency strength of the validity of the $\mathbf{\Sigma}_n$-club properties.

Proposition

If ω_1 has the $\mathbf{\Sigma}_2$-club property, then ω_1 is a measurable cardinal in HOD.

Theorem (L.)

If κ is an uncountable regular cardinal with the $\mathbf{\Sigma}_1$-club property, then κ is an inaccessible cardinal with the $\mathbf{\Sigma}_1$-club property in the Dodd-Jensen core model K^{DJ}.

Welch recently provided an exact characterization of inaccessible cardinals with the $\mathbf{\Sigma}_1$-club property in K^{DJ}, using his notion of Σ_1-*stably measurable cardinals*.

Remember that, if κ is a cardinal and A is a subset of κ, then $I \subseteq \kappa$ is a *good set of indiscernibles for* $\langle \mathrm{L}_\kappa[A], \in, A\rangle$ if the following statements hold for all $\gamma \in I$:

- $\langle \mathrm{L}_\gamma[A \cap \gamma], \in, A \cap \gamma\rangle$ is an elementary substructure of $\langle \mathrm{L}_\kappa[A], \in, A\rangle$.
- $I \setminus \gamma$ is a set of indiscernibles for the structure $\langle \mathrm{L}_\kappa[A], \in, A, \xi\rangle_{\xi<\gamma}$.

Proposition

Let κ be an uncountable regular cardinal with the $\mathbf{\Sigma}_n$-club property.

If A is a subset of κ such that the set $\{A\}$ is definable by a Σ_n-formula with parameters in $\mathrm{H}(\kappa) \cup \{\kappa\}$, then there is a closed unbounded subset of κ that is a good set of indiscernibles for $\langle \mathrm{L}_\kappa[A], \in, A\rangle$.

Corollary

If there exists an uncountable regular cardinal with the $\mathbf{\Sigma}_1$-club property, then $x^{\#}$ exists for every real x.

Lemma

Let κ be an uncountable regular cardinal with the $\mathbf{\Sigma}_n$-club property. If A is a subset of κ with the property that the set $\{A\}$ is definable by a Σ_n-formula with parameters in $\mathrm{H}(\kappa) \cup \{\kappa\}$, then κ is inaccessible in $\mathrm{L}[A]$.

Lemma

Let κ be an uncountable regular cardinal and let A be a subset of κ with the property that κ is an inaccessible cardinal in $\mathrm{L}[A]$.

If there is a good set of indiscernibles for $\langle \mathrm{L}_\kappa[A], \in, A \rangle$ of cardinality κ, then there is a weak κ-model M with $A \in M$ and a weakly amenable countably complete M-ultrafilter on κ.

By combining the above results, we see that the $\boldsymbol{\Sigma}_n$-club property implies Ramseyness with respect to Σ_n-definable subsets of the given cardinal.

Corollary

Let κ be an uncountable regular cardinal with the $\boldsymbol{\Sigma}_n$-club property.

If A is a subset of κ with the property that the set $\{A\}$ is definable by a Σ_n-formula with parameters in $\mathrm{H}(\kappa) \cup \{\kappa\}$, then there is a weak κ-model M with $A \in M$ and a weakly amenable countably complete M-ultrafilter on κ. $\square$

The next lemma shows that the above restriction of Ramseyness characterizes the $\mathbf{\Sigma}_1$-club property in the Dodd–Jensen core model.

Lemma

Assume that $\mathrm{V} = \mathrm{K}^{DJ}$. *Then the following statements are equivalent for every uncountable regular cardinal* κ*:*

- κ *has the* $\mathbf{\Sigma}_1$*-club property.*
- *For all* $A \subseteq \kappa$ *with the property that the set* $\{A\}$ *is definable by* Σ_1*-formula with parameters in* $\mathrm{H}(\kappa) \cup \{\kappa\}$*, there is a weak* κ*-model* M *with* $A \in M$ *and a weakly amenable countably complete* M*-ultrafilter on* κ*.*

Proof of the Theorem.

Fix $z \in \mathrm{H}(\kappa)^{\mathrm{K}^{DJ}}$ *and* $A \in \mathcal{P}(\kappa)^{\mathrm{K}^{DJ}}$ *such that the set* $\{A\}$ *is definable by a* Σ_1*-formula with parameters* κ *and* z *in* K^{DJ}*.*

Our assumption implies the existence of $0^{\#}$ *and therefore results of Dodd and Jensen show that* K^{DJ} *is equal to the union of all lower parts of iterable premice.*

Since the class of all iterable premice is $\Sigma_1(\kappa)$*-definable, this shows that the class* K^{DJ} *is also* $\Sigma_1(\kappa)$*-definable and we can conclude that the set* $\{A\}$ *is definable by a* Σ_1*-formula with parameters* κ *and* z *in* V*.*

Therefore, we can apply an earlier proposition to find a club subset of κ *that is a good set of indiscernibles for* $\langle \mathrm{L}_\kappa[A], \in, A \rangle$*.*

In this situation, we can apply the Jensen Indiscernibles Lemma to find a good set of indiscernibles for $\langle \mathrm{L}_\kappa[A], \in, A \rangle$ *of cardinality* κ *that is an element of* K^{DJ}*.*

Proof (cont.).

Since we showed that κ is inaccessible in $\mathrm{L}[A]$, one of our lemmas now shows that in K^{DJ}, there is a weak κ-model M with $A \in M$ and a weakly amenable countably complete M-ultrafilter on κ.

But now, the above characterization of the $\mathbf{\Sigma}_1$-club property in K^{DJ} allows us to conclude that κ has the $\mathbf{\Sigma}_1$-club property in this model.

Finally, variations of earlier results show that κ is inaccessible in K^{DJ}. $\square$

Open Questions

Question

Do very strong large cardinal assumptions imply that no well-ordering of $\mathrm{H}(\omega_3)$ is definable by a Σ_1-formula with parameter ω_2?

Question

Do very strong forcing axioms, like $\mathbf{MM}^{++}$ or Viale's $\mathbf{MM}^{+++}$, imply that no well-ordering of the reals is definable by a Σ_1-formula with parameter ω_2?

Question

Is it consistent that the set $\{\omega_1\}$ is not definable by a Σ_1-formula with parameter ω_ω?

Note that if ω_ω is Rowbottom, then the set $\{\omega_1\}$ is not definable by a Σ_1-formula with parameter ω_ω.

Thank you for listening!