Locally definable well-orders

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Introduction

The following classical theorem of Mansfield is the starting point of our work.

Theorem (Mansfield)

The following statements are equivalent.

- *Every real is constructible.*
- *There is a Δ^1_2-well-ordering of the reals.*

Corollary

If there is a Δ^1_2-well-ordering of the reals, then CH holds and there are no measurable cardinals.

A set of reals is $\Pi^1_2(z)$-definable if and only if it is definable over the collection $\mathrm{H}(\omega_1)$ of all hereditary countable sets by a Π_1-formula with parameter z. Therefore we can reformulate Mansfield's theorem in the following way.

Theorem (Mansfield)

The following statements are equivalent.

- *Every subset of ω is constructible.*
- *There is a well-ordering of the set ${}^\omega\omega$ of all functions from ω to ω that is definable over $\langle \mathrm{H}(\omega_1), \in \rangle$ by a Π_1-formula without parameters.*

This formulation allows us to consider generalizations of Mansfield's result to higher cardinals.

Obviously, the first implications of Mansfield's theorem still holds if we replace ω by an uncountable cardinal.

Proposition

Let κ be an infinite cardinal such that every subset of κ is constructible. Then there is a well-ordering of the set ${}^{\kappa}\kappa$ of all functions from κ to κ that is definable over the collection $\mathrm{H}(\kappa^{+})$ of all sets of hereditary cardinality at most κ by a Π_1-formula without parameters.

Therefore it is natural to ask whether the other implication or its consequences also hold if we replace ω by an uncountable cardinal.

Question

What are the provable consequences of the existence of a well-ordering of ${}^{\omega_1}\omega_1$ that is definable over $\langle \mathrm{H}(\omega_2), \in \rangle$ by a Π_1-formula without parameters?

Boldface $\mathbf{\Pi}_1$-definitions

We start by considering well-orderings of ${}^{\omega_1}\omega_1$ that are definable over $\langle \mathrm{H}(\omega_2), \in\rangle$ by Π_1-formulas with parameters.

The following result shows that we can force the existence of such well-orders over models of CH while preserving many structural features of the ground model.

Theorem (Holy–L.)

Let κ be an uncountable cardinal such that $\kappa = \kappa^{<\kappa}$ and 2^κ is regular. Then there is a partial order $\mathbb{P}$ with the following properties.

- *$\mathbb{P}$ is $<\kappa$-closed and forcing with $\mathbb{P}$ preserves cofinalities $\leq 2^\kappa$ and the value of 2^κ.*
- *If G is $\mathbb{P}$-generic over the ground model V, then there is a well-ordering of $({}^\kappa\kappa)^{\mathrm{V}[G]}$ that is definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ by a Π_1-formula with parameters.*

The forcing that witnesses the above theorem is constructed inductively using techniques developed by David Asperó and Sy Friedman in their paper "*Definable well-orders of* $\mathrm{H}(\omega_2)$ *and* GCH".

In the following, we sketch the construction of this forcing.

Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$ and 2^κ regular.

We will later show that there is a family $\langle \mathbb{C}(A) \mid A \subseteq {}^\kappa\kappa \rangle$ of partial orders such that the following statements hold for all $A \subseteq {}^\kappa\kappa$.

- $\mathbb{C}(A)$ is $<\kappa$-closed and satisfies the κ^+-chain condition.
- If G is $\mathbb{C}(A)$-generic over V, then A is definable in $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in \rangle$ by a Σ_1-formula with parameters.
- If $B \subseteq A$, then $\mathbb{C}(B)$ is a complete subforcing of $\mathbb{C}(A)$.

We construct a sequence $\langle \mathbb{P}_\gamma \mid \gamma \leq 2^\kappa \rangle$ of partial orders such that

- Forcing with $\mathbb{P}_\gamma$ preserves cofinalities $\leq 2^\kappa$.
- If $\bar{\gamma} < \gamma$, then $\mathbb{P}_{\bar{\gamma}}$ is a complete subforcing of $\mathbb{P}_\gamma$ and there is a projection map $\pi_{\bar{\gamma},\gamma} : \mathbb{P}_\gamma \longrightarrow \mathbb{P}_{\bar{\gamma}}$.
- A condition in $\mathbb{P}_\gamma$ is a pair $p = \langle \vec{z}_p, \bar{p} \rangle$ such that
  - $\vec{z}_p = \langle \dot{z}_{p,\delta} \mid \delta < \gamma_p \rangle$ with $\gamma_p < \min\{\gamma + 1,\ 2^\kappa\}$ such that each $\dot{z}_{p,\delta}$ is a $\mathbb{P}_\delta$-name for an element of ${}^\kappa\kappa$ and every condition in $\mathbb{P}_\delta$ forces the sequence $\vec{z}_p \upharpoonright (\delta + 1)$ to be injective.
  - $\bar{p}$ is a condition in $\mathbb{C}(A_p)$, where $A_p \subseteq {}^\kappa 2$ consists of functions that encode the information about the $\dot{z}_{p,\delta}$ decided by $\pi_{\delta,\gamma}(p)$.
- We have $p \leq_{\mathbb{P}_\gamma} q$ iff $\vec{z}_q \subseteq \vec{z}_p$, $A_q \subseteq A_p$ and $\bar{p} \leq_{\mathbb{C}(A_p)} \bar{q}$.

Let G be $\mathbb{P}_{2^\kappa}$-generic over V. Given $x, y \in ({}^\kappa\kappa)^{\mathrm{V}[G]}$, the relation

$$x \prec y \iff \exists p \in G\ \exists \delta_0 < \delta_1 < \gamma_p\ [x = \dot{z}^G_{p,\delta_0}\ \wedge\ y = \dot{z}^G_{p,\delta_1}]$$

is a well-ordering of $({}^\kappa\kappa)^{\mathrm{V}[G]}$. With the help of a preparatory forcing, we can ensure that this relation is a locally Π_1-definable.

We now show how to construct the family $\langle \mathbb{C}(A) \mid A \subseteq {}^{\kappa}\kappa \rangle$ of coding forcings with the desired properties.

The starting point is the generalization of Solovay's *almost disjoint coding forcing* to uncountable cardinals.

Equip ${}^{\kappa}\kappa$ with the topology whose basic open subsets are of the form

$$N_s \ = \ \{x \in {}^{\kappa}\kappa \mid s \subseteq x\}$$

for some function $s : \alpha \longrightarrow \kappa$ with $\alpha < \kappa$.

Note that a subset of ${}^{\kappa}\kappa$ is closed in this topology if and only if it is equal to the set $[T]$ of all cofinal branches through a subtree T of ${}^{<\kappa}\kappa$.

Given $A \subseteq {}^{\kappa}\kappa$, define a partial order $\mathbb{C}(A)$ by the following clauses.

- A condition in $\mathbb{C}(A)$ is a tuple $p = \langle \alpha_p, a_p, s_p, t_p \rangle$ with
  - $\alpha_p < \kappa$.
  - $a_p \in [A]^{<\kappa}$.
  - $s_p : \alpha_p \longrightarrow {}^{<\kappa}\kappa$.
  - $t_p : \alpha_p \longrightarrow 2$.
- We have $q \leq_{\mathbb{C}(A)} p$ if and only if

$$\alpha_p \leq \alpha_q, \quad a_p \subseteq a_q, \quad s_p \subseteq s_q, \quad t_p \subseteq t_q$$

  and

$$s_q(\beta) \subseteq x \ \longrightarrow \ t_q(\beta) = 1$$

  for every $x \in a_p$ and $\alpha_p \leq \beta < \alpha_q$.

Proposition

$\mathbb{C}(A)$ is $<\kappa$-closed and satisfies the κ^+-chain condition.

Let G be $\mathbb{C}(A)$-generic over V. Define

$$s_G = \bigcup_{p \in G} s_p : \kappa \longrightarrow {}^{<\kappa}\kappa \quad \textit{and} \quad t_G = \bigcup_{p \in G} t_p : \kappa \longrightarrow 2.$$

Given $\alpha < \kappa$, define

$$T_\alpha^G \;=\; \{u \in {}^{<\kappa}\kappa \mid \forall \alpha < \beta < \kappa \, [s_G(\beta) \subseteq u \longrightarrow t_G(\beta) = 1]\}.$$

Then T_α^G is a subtree of ${}^{<\kappa}\kappa$.

Proposition

If $x \in A$ and $p \in G$ with $x \in a_p$, then $x \in [T_{\alpha_p}]$.

Proof.

Pick $\alpha_p < \beta < \kappa$ with $s_G(\beta) \subseteq x$. Then we can find $q \in G$ with $\beta < \alpha_q$ and $q \leq_{\mathbb{C}(A)} p$. Since $s_q(\beta) = s_G(\beta) \subseteq x \in a_p$ holds, we can conclude that $t_G(\beta) = t_q(\beta) = 1$. This shows that $x \in [T_{\alpha_p}]$. $\square$

Lemma

If $\alpha < \kappa$*, then* $[T_\alpha]^{\mathrm{V}[G]} \subseteq A$*.*

Proof.

Let $\dot{x}$ be a $\mathbb{C}(A)$-name for a function from κ to κ not contained in A.

Given $p_0 \in \mathbb{C}(A)$ with $\alpha_{p_0} > \alpha$, we inductively construct sequences $\langle p_n \in \mathbb{C}(A) \mid n < \omega \rangle$ and $\langle u_n \in {}^{<\kappa}\kappa \mid n < \omega \rangle$ such that the following statements hold for all $n < \omega$.

- $p_{n+1} \leq_{\mathbb{C}(A)} p_n$ and $p_{n+1} \Vdash$ "$\check{u}_n \subseteq \dot{x}$".
- $\alpha_{p_n} < \mathrm{lh}(u_n) < \alpha_{p_{n+1}}$.
- $u_n \nsubseteq y$ for every $y \in a_{p_n}$.

Set $\alpha_\omega = \sup_{n<\omega} \alpha_{p_n}$. There is a unique condition p_ω in $\mathbb{C}(A)$ such that

- $p_\omega \leq_{\mathbb{C}(A)} p_n$ for all $n < \omega$.
- $\alpha_{p_\omega} = \alpha_\omega + 1 > \alpha$ and $a_{p_\omega} = \bigcup_{n<\omega} a_{p_n}$.
- $s_{p_\omega}(\alpha_\omega) = \bigcup_{n<\omega} u_n$ and $t_p(\alpha_\omega) = 0$.

If $\dot{T}_\alpha$ is the canonical name for T_α^G, then $p_\omega \Vdash$ "$\dot{x} \notin [\dot{T}_\alpha]$". □

The following theorem summarizes the above computations.

Theorem

Let G be $\mathbb{C}(A)$-generic over V. Then A is equal to a union of κ-many closed subsets of ${}^{\kappa}\kappa$ in $\mathrm{V}[G]$ (i.e. A is a $\mathbf{\Sigma}^0_2$-subset of ${}^{\kappa}\kappa$ in $\mathrm{V}[G]$).

This shows that the family $\langle \mathbb{C}(A) \mid A \subseteq {}^{\kappa}\kappa \rangle$ of partial orders satisfies the first and the second property listed above.

Unfortunately, $\mathbb{C}(A)$ does not satisfy the "*complete subforcing*" property.

But it is possible to modify the above definition to obtain a family of coding forcings that satisfies all three properties.

The following construction and results are contained in joint work with David Asperó and Peter Holy.

Given $A \subseteq {}^\kappa\kappa$, define a partial order $\mathbb{C}^*(A)$ by the following clauses.

- A condition in $\mathbb{C}^*(A)$ is a tuple

$$p \;=\; \langle \alpha_p,\; s_p,\; t_p,\; \langle c_{p,x} \mid x \in a_p \rangle \rangle$$

  with

  - $\alpha_p < \kappa$ is a successor ordinal.
  - $s_p : \alpha_p \longrightarrow {}^{<\kappa}\kappa$.
  - $t_p : \alpha_p \longrightarrow 2$.
  - $a_p \in [A]^{<\kappa}$ and $c_{p,x}$ is a closed subset of α_p with

$$s_p(\beta) \subseteq x \quad\longrightarrow\quad t_p(\beta) = 1$$

    for every $x \in a_p$ and $\beta \in c_{p,x}$.

- We have $q \leq_{\mathbb{C}^*(A)} p$ if and only if $a_p \subseteq a_q$, $s_p \subseteq s_q$, $t_p \subseteq t_q$ and $c_{p,x} = c_{q,x} \cap \alpha_p$ for every $x \in a_p$.

Proposition

The partial order $\mathbb{C}^*(A)$ is $<\kappa$-closed and satisfies the κ^+-chain condition.

Proposition

If $B \subsetneq A \subset {}^\kappa\kappa$, then $\mathbb{C}^*(B)$ is a complete subforcing of $\mathbb{C}^*(A)$.

Theorem (Asperó–Holy–L.)

Let G be $\mathbb{C}^*(A)$-generic over V. Define s_G and t_G as above.

In $\mathrm{V}[G]$, the set A is equal to the set of all $x \in {}^\kappa\kappa$ such that

$$\exists C \subseteq \kappa \text{ club} \quad \forall \beta \in C \quad [s_G(\beta) \subseteq \beta \longrightarrow t_G(\beta) = 1]\,.$$

In particular, A is definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in \rangle$ by a Σ_1-formula.

Lightface Π_1-definitions

The parameter used in the above Π_1-definitions is a subset of κ added by the forcing and therefore is a very complicated object.

We want to construct models of set theory with local Π_1-definitions of well-orders that only use *simple parameters*.

The first step towards this goal is the next result that produces Π_1-definitions in the generic extension whose parameters are elements of the ground model.

The construction of the corresponding forcing combines the techniques of the above proof with iterated club shooting and variations of Shelah's notion of S*-complete forcings*.

Theorem (Holy–L.)

Assume that CH *holds and* 2^{ω_1} *is regular. If* $\vec{S}$ *is an* ω_1*-sequence of pairwise disjoint stationary subsets of* ω_1*, then there is a partial order* $\mathbb{P}_{\vec{S}}$ *with the following properties.*

- $\mathbb{P}_{\vec{S}}$ *is* σ*-distributive and forcing with* $\mathbb{P}$ *preserves cofinalities* $\leq 2^{\omega_1}$ *and the value of* 2^{ω_1}*.*
- *If* G *is* $\mathbb{P}_{\vec{S}}$*-generic over* V*, then there is a well-order of* $({}^{\omega_1}\omega_1)^{\mathrm{V}[G]}$ *that is definable over* $\langle \mathrm{H}(\omega_2)^{\mathrm{V}[G]}, \in \rangle$ *by a* Π_1*-formula with parameter* $\vec{S}$*.*

By forcing over ground models that are close to Gödel's constructible universe L, we can use the above result to produce Π_1-definitions of well-orders that use no parameters.

Theorem (Holy–L.)

Assume that CH *holds,* 2^{ω_1} *is regular and every stationary subset of* ω_1 *in* L *is stationary in* V*. Then there is an* ω_1*-sequence* $\vec{S}$ *of pairwise disjoint stationary subsets of* ω_1 *with the property that whenever* G *is* $\mathbb{P}_{\vec{S}}$*-generic over* V*, then there is a well-ordering of* $({}^{\omega_1}\omega_1)^{\mathrm{V}[G]}$ *that is definable over* $\langle \mathrm{H}(\omega_2)^{\mathrm{V}[G]}, \in\rangle$ *by a* Π_1*-formula without parameters.*

Corollary

The existence of a well-ordering of ${}^{\omega_1}\omega_1$ *that is definable over* $\langle \mathrm{H}(\omega_2), \in\rangle$ *by a* Π_1*-formula without parameters is consistent with a failure of the* GCH *at* ω_1*.*

A similar argument can be made if we take a filter U witnessing the measurability of some uncountable cardinal δ and replace the inner model L by $\mathrm{L}[U]$ in the above statement.

Theorem (Holy–L.)

Let U be a normal ultrafilter over some uncountable cardinal δ. Assume that CH holds, 2^{ω_1} is regular and every stationary subset of ω_1 in $\mathrm{L}[U]$ is stationary in V. Then there is an ω_1-sequence $\vec{S}$ of pairwise disjoint stationary subsets of ω_1 with $|\mathbb{P}_{\vec{S}}| < \delta$ and the property that whenever G is $\mathbb{P}_{\vec{S}}$-generic over V, then there is a well-ordering of $({}^{\omega_1}\omega_1)^{\mathrm{V}[G]}$ that is definable over $\langle \mathrm{H}(\omega_2)^{\mathrm{V}[G]}, \in\rangle$ by a Π_1-formula without parameters.

Corollary

If the existence of a measurable cardinal is consistent, then the existence of a measurable cardinal is consistent with the existence of a well-ordering of ${}^{\omega_1}\omega_1$ that is definable over $\langle \mathrm{H}(\omega_2), \in\rangle$ by a Π_1-formula without parameters.

In contrast, it is possible to use results of Woodin on the Π_2-maximality of the $\mathbb{P}_{max}$-extension of $\mathrm{L}(\mathbb{R})$ to show that the existence of larger large cardinals is incompatible with the existence of such definable well-orders.

Proposition

Assume that there are infinitely many Woodin cardinals with a measurable cardinal above them all. If there is a well-order of ${}^{\omega_1}\omega_1$ *that is definable over* $\langle \mathrm{H}(\omega_2), \in \rangle$ *by a* Π_1*-formula with parameter* $z \subseteq \omega_1$*, then* $z \notin \mathrm{L}(\mathbb{R})$*.*

Proof.

Assume that there is a Π_1-formula φ and $z \in \mathcal{P}(\omega_1)^{\mathrm{L}(\mathbb{R})}$ that define a well-order of ${}^{\omega_1}\omega_1$ over $\langle \mathrm{H}(\omega_2), \in \rangle$. Let G be $\mathbb{P}_{\mathsf{max}}$-generic over $\mathrm{L}(\mathbb{R})$.

By the Π_2-maximality of the $\mathbb{P}_{max}$-extension, we know that φ and z define a well-order of $({}^{\omega_1}\omega_1)^{\mathrm{L}(\mathbb{R})[G]}$ over $\langle \mathrm{H}(\omega_2)^{\mathrm{L}(\mathbb{R})[G]}, \in \rangle$. This shows that there is well-ordering of $\mathbb{R}$ that is definable in $\mathrm{L}(\mathbb{R})[G]$ by a formula with parameter $z \in \mathrm{L}(\mathbb{R})$.

Since the partial order $\mathbb{P}_{\mathsf{max}}$ is homogeneous in $\mathrm{L}(\mathbb{R})$, this implies that there is a well-ordering of $\mathbb{R}$ in $\mathrm{L}(\mathbb{R})$. But results of Woodin show that our assumptions imply that AD holds in $\mathrm{L}(\mathbb{R})$, a contradiction. $\square$

Thank you for listening!