Continuous images of closed sets in generalized Baire spaces

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Generalized Baire spaces

Let κ be an infinite cardinal with $\kappa = \kappa^{<\kappa}$.

Given a cardinal μ, we equip the set ${}^{\kappa}\mu$ consisting of all functions $x : \kappa \longrightarrow \mu$ with the topology whose basic open sets are of the form

$$N_s \ = \ \{x \in {}^{\kappa}\mu \mid s \subseteq x\},$$

where s is an element of the set ${}^{<\kappa}\mu$ of all functions $t : \alpha \longrightarrow \mu$ with $\alpha < \kappa$.

We call the space ${}^{\kappa}\kappa$ the *generalized Baire space of* κ.

$\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$

A subset A of ${}^\kappa\kappa$ is a $\mathbf{\Sigma}^1_1$-*subset* if it is equal to the projection of a closed subset of ${}^\kappa\kappa \times {}^\kappa\kappa$. We let $\mathbf{\Sigma}^1_1(\kappa)$ denote the class of all such subsets.

It is easy to see that a subset of ${}^\kappa\kappa$ is an element of $\mathbf{\Sigma}^1_1(\kappa)$ if and only if it is equal to a continuous image of a closed subset of ${}^\kappa\kappa$.

If κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$, then a subset of ${}^\kappa\kappa$ is contained in $\mathbf{\Sigma}^1_1(\kappa)$ if and only if it is definable over the structure $(\mathrm{H}_{\kappa^+}, \in)$ by a Σ_1-formula with parameters.

This shows that in this case many interesting and important sets are equal to continuous images of closed subsets.

We present an example of such a subset.

Example

The club filter

$$\mathrm{Club}_\kappa \;=\; \{x \in {}^\kappa\kappa \mid \exists C \subseteq \kappa \; \mathit{club} \; \forall \alpha \in C \; x(\alpha) = 1\}$$

is a continuous image of the space ${}^\kappa\kappa$.

Let T denote the tree consisting of all pairs (s, t) in ${}^\gamma 2 \times {}^\gamma 2$ such that $\gamma \in \mathrm{Lim} \cap \kappa$, $t(\alpha) \leq s(\alpha)$ for all $\alpha < \gamma$ and t is the characteristic function of a club subset of γ.

Then T is isomorphic to the tree ${}^{<\kappa}\kappa$, because it is closed under increasing sequences of length $<\kappa$ and every node has κ-many direct successors.

If we equip the set $[T]$ of all κ-branches through T with the topology whose basic open sets consists of all extensions of elements of T, then we obtain a topological space homeomorphic to ${}^\kappa\kappa$.

Since the projection $p : [T] \longrightarrow {}^\kappa\kappa$ onto the union of the first coordinate is continuous, we can conclude that the set Club_κ is equal to a continuous image of ${}^\kappa\kappa$.

Classes of continuous images

Given an uncountable cardinal κ with $\kappa = \kappa^{<\kappa}$, we study the following subclasses of $\mathbf{\Sigma}^1_1(\kappa)$ that arise by restricting the classes of used continuous functions and closed subsets.

- The class $\mathbf{\Sigma}^1_1(\kappa)$ of continuous images of closed subsets of ${}^\kappa\kappa$.
- The class $\mathbf{C}(\kappa)$ of continuous images of ${}^\kappa\kappa$.
- The class $\mathbf{I}_{cl}(\kappa)$ of continuous injective images of closed subsets of ${}^\kappa\kappa$.
- The class $\mathbf{I}(\kappa)$ of continuous injective images of ${}^\kappa\kappa$.

We will compare these classes with the following collections.

- The class $\mathbf{\Sigma}^0_1(\kappa)$ of open subsets of ${}^\kappa\kappa$.
- The class $\mathbf{B}(\kappa)$ of κ-Borel subsets of ${}^\kappa\kappa$, i.e. the subsets contained in the smallest algebra of sets on ${}^\kappa\kappa$ that contains all open subsets and is closed under κ-unions.

In the case "$\kappa = \omega$", the relationship of the above classes is described by the following complete diagram.

$$\begin{tikzcd}
\mathbf{\Sigma}^0_1(\omega) \arrow[r] & \mathbf{I}(\omega) \arrow[r] & \mathbf{I}_{cl}(\omega) \arrow[r, equal] & \mathbf{B}(\omega) \arrow[r] & \mathbf{\Sigma}^1_1(\omega) \arrow[r, equal] & \mathbf{C}(\omega)
\end{tikzcd}$$

Our results will show that these classes behave in a very different way if κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$. These results are summarized by the following complete diagram.

$$\begin{tikzcd}
\mathbf{B}(\kappa) \arrow[r] & \mathbf{I}_{cl}(\kappa) \arrow[r, shift right] & \mathbf{\Sigma}^1_1(\kappa) \arrow[l, dashed, shift right, "\mathrm{V=L}"'] \\
\mathbf{\Sigma}^0_1(\kappa) \arrow[r] \arrow[u] & \mathbf{I}(\kappa) \arrow[r] \arrow[u] & \mathbf{C}(\kappa) \arrow[u]
\end{tikzcd}$$

In particular, we will show that the following statements fail to generalize to higher cardinalities.

- Every closed subset of the space ${}^{\omega}\omega$ is equal to a continuous image of ${}^{\omega}\omega$.
- Every continuous injective image of the space ${}^{\omega}\omega$ is a Borel subset.

In the following, we construct the corresponding counterexamples.

Continuous images of ${}^{\kappa}\kappa$

Retracts

An important topological property of spaces of the form ${}^{\omega}\mu$ is the fact that non-empty closed subsets are retracts of the whole space, i.e. given an non-empty closed subset A of ${}^{\omega}\mu$ there is a continuous surjection $f : {}^{\omega}\mu \longrightarrow A$ such that $f \restriction A = \mathrm{id}_A$.

An easy argument shows that this property fails if κ is uncountable.

Proposition

Suppose that κ is an uncountable regular cardinal and $\mu > 1$ is a cardinal. Let A denote the set of all x in ${}^{\kappa}\mu$ such that $x(\alpha) = 1$ for only finitely many $\alpha < \kappa$. Then A is a closed subset of ${}^{\kappa}\mu$ that is not a retract of ${}^{\kappa}\mu$.

Proof.

Assume, towards a contradiction, that there is a continuous function $f : {}^{\kappa}\mu \longrightarrow A$ with $f \restriction A = \mathrm{id}_A$. We construct a strictly increasing sequence $\langle \gamma_n < \kappa \mid n < \omega \rangle$ of ordinals such that $\gamma_0 = 1$ and

$$N_{x_n \restriction \gamma_{n+1}} \subseteq f^{-1} \text{ "} N_{x_n \restriction (\gamma_n + 1)}$$

holds for all $n < \omega$ and the unique $x_n \in {}^{\kappa}2$ with

$$x_n^{-1} \text{ "} \{1\} = \{\gamma_0, \dots, \gamma_n\}.$$

Let $x \in {}^{\kappa}2$ be the unique function with

$$x^{-1} \text{ "} \{1\} = \{\gamma_n \mid n < \omega\}.$$

Then our construction yields

$$f(x) \restriction \sup_{n<\omega} \gamma_n = x \restriction \sup_{n<\omega} \gamma_n$$

and this implies that $f(x) \notin A$, a contradiction. $\square$

Continuous Images

Every closed subset of ${}^{\omega}\omega$ is a continuous image of ${}^{\omega}\omega$ and hence every $\mathbf{\Sigma}^1_1$-subset is equal to a continuous image of ${}^{\omega}\omega$.

The following result shows that this statement also does not generalize to uncountable regular cardinals.

Theorem (L.-Schlicht)

Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$. Then there is a closed non-empty subset of ${}^{\kappa}\kappa$ that is not equal to a continuous image of ${}^{\kappa}\kappa$.

Proof.

It suffices to construct a closed subset A of ${}^{\kappa}\kappa$ with the property that A is not equal to the projection $p[T]$ of a $<\kappa$-closed subtree T of ${}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$ without terminal nodes.

Given $\lambda \leq \kappa$ closed under Gödel pairing and $x \in {}^{\lambda}2$, define a binary relation $\in_x$ on λ by setting

$$\alpha \in_x \beta \iff x(\prec\alpha, \beta\succ) = 1.$$

Define

$$W = \{x \in {}^{\kappa}2 \mid (\kappa, \in_x) \text{ is a well-order}\}.$$

Then W is a closed subset of ${}^{\kappa}\kappa$.

Assume, towards a contradiction, that there is a $<\kappa$-closed subtree T of ${}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$ without terminal nodes such that $W = p[T]$.

Proof (cont.).

Given $(s,t) \in T$ *and* $\alpha < \kappa$*, define*

$$r(s,t,\alpha) \;=\; \sup\{\mathrm{rnk}_{\in_x}(\alpha) \mid x \in p([T] \cap N_{(s,t)})\} \;\leq\; \kappa^+$$

Then $r(\emptyset,\emptyset,\alpha) = \kappa^+$ *for every* $\alpha < \kappa$*.*

Claim.

Let $(s,t) \in T$ *and* $\alpha < \kappa$ *with* $r(s,t,\alpha) = \kappa^+$*. If* $\gamma < \kappa^+$*, then there is* $(u,v) \in T$ *extending* (s,t) *and* $\alpha < \beta < \kappa$ *such that* $\mathrm{dom}(u)$ *is closed under Gödel pairing,* $\beta < \mathrm{lh}(u)$*,* $\beta \in_u \alpha$*, and* $r(u,v,\beta) \geq \gamma$*.*

Proof of the Claim.

There is a $(x,y) \in [T] \cap N_{(s,t)}$ *with* $\mathrm{rnk}_{\in_x}(\alpha) \geq \gamma + \kappa$*. Hence we can find* $\alpha < \beta < \kappa$ *with* $\gamma \leq \mathrm{rnk}_{\in_x}(\beta) < \gamma + \kappa$*. Pick* $\delta > \max\{\alpha, \beta, \mathrm{lh}(s)\}$ *closed under Gödel pairing and define* (u,v) *to be the node* $(x \restriction \delta, y \restriction \delta)$ *extending* (s,t)*. Since* $\in_u$ *is a well-ordering of* $\mathrm{lh}(u)$*, we have* $\beta \in_u \alpha$*. Finally,* (x,y) *witnesses that* $r(u,v,\beta) \geq \gamma$*.* □

Proof (cont.).

Claim.

If $(s,t) \in T$ and $\alpha < \kappa$ with $r(s,t,\alpha) = \kappa^+$, then there is a node (u,v) in T extending (s,t) and $\alpha < \beta < \mathrm{lh}(u)$ such that $\mathrm{lh}(u)$ is closed under Gödel pairing, $\beta \in_u \alpha$ and $r(u,v,\beta) = \kappa^+$. □

This claim shows that there are strictly increasing sequences $\langle (s_n, t_n) \mid n < \omega \rangle$ of nodes in T and $\langle \beta_n \mid n < \omega \rangle$ of elements of κ with

- $\mathrm{lh}(s_n)$ *is closed under Gödel pairing,*
- $\beta_{n+1} \in_{s_{n+1}} \beta_n$.

Let $s = \bigcup_{n<\omega} s_n$ and $t = \bigcup_{n<\omega} t_n$. Then $(s,t) \in T$, since T is ω-closed. By our assumptions on T, there is a cofinal branch (x,y) in $[T]$ through (s,t). Then $\beta_{n+1} \in_x \beta_n$ for every $n < \omega$ and this shows that $x \notin W = p[T]$, a contradiction. □

Continuous injective images of ${}^{\kappa}\kappa$

Next we construct a continuous injective image of ${}^\kappa\kappa$ that is not a κ-Borel subset of ${}^\kappa\kappa$. In order to prove that certain sets are not κ-Borel, we need to introduce an important regularity property of subsets of ${}^\kappa\kappa$.

We say that a subset A of ${}^\kappa\kappa$ is *κ-Baire measurable* if there is an open subset U of ${}^\kappa\kappa$ and a sequence $\langle N_\alpha \mid \alpha < \kappa\rangle$ of nowhere dense subsets of ${}^\kappa\kappa$ such that the symmetric difference $A_\Delta U$ is a subset of $\bigcup_{\alpha<\kappa} N_\alpha$.

A standard prove shows that every κ-Borel subset of ${}^\kappa\kappa$ is κ-Baire measurable. Moreover it is consistent that all $\mathbf{\Delta}^1_1$-subsets of ${}^\kappa\kappa$ are κ-Baire measurable.

Theorem (L.-Schlicht)

Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$. Then there is a continuous injective image of ${}^\kappa\kappa$ that is not κ-Baire measurable.

To motivate this result, we first consider the case $\kappa = \aleph_1 = 2^{\aleph_0}$ and show that a well-known collection of combinatorial objects provides an example of a set with the above properties.

Definition

- Given $\gamma \in \mathrm{On}$, a sequence $\langle C_\alpha \mid \alpha < \gamma \rangle$ is a *coherent C-sequence* if the following statements hold for all $\alpha < \gamma$.
  - If α is a limit ordinal, then C_α is a closed unbounded subset of α.
  - If $\alpha = \bar{\alpha} + 1$, then $C_\alpha = \{\bar{\alpha}\}$.
  - If $\bar{\alpha} \in \mathrm{Lim}(C_\alpha)$, then $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$.
- A coherent C-sequence $\langle C_\alpha \mid \alpha < \gamma \rangle$ with $\gamma \in \mathrm{Lim}$ is *trivial* if there is a closed unbounded subset C_γ of γ that *threads* $\vec{C}$, i.e. the sequence $\langle C_\alpha \mid \alpha \leq \gamma \rangle$ is also a coherent C-sequence.

Let $\mathcal{Coh}(\omega_1)$ be the set of all coherent C-sequences of length ω_1 equipped with the topology whose basic open sets consist of all extensions of coherent C-sequences of limit length less than ω_1.

Claim.

The space $\mathcal{Coh}(\omega_1)$ is homeomorphic to ${}^{\omega_1}\omega_1$.

Proof of the Claim.

Let $\mathcal{T}$ denote the tree of all coherent C-sequences of limit length less than ω_1. Then $\mathcal{T}$ is isomorphic to ${}^{<\omega_1}\omega_1$, because $\mathcal{T}$ is σ-closed and every node in $\mathcal{T}$ has $\aleph_1$-many direct successors. This isomorphism gives us a homeomorphism of the above spaces. $\square$

Define $\mathcal{T}hr(\omega_1)$ to be set of all pairs $(\vec{C}, C)$ such that $\vec{C}$ is an element of $\mathcal{C}oh(\omega_1)$ and C is a thread through $\vec{C}$. We equip $\mathcal{T}hr(\omega_1)$ with the topology whose basic open sets consist of all component-wise extensions of pairs $(\vec{D}, D)$ such that $\vec{D}$ is a coherent C-sequence of length $\gamma \in \mathrm{Lim} \cap \omega_1$ and D is a thread through $\vec{D}$.

Claim.

The space $\mathcal{T}hr(\omega_1)$ is homeomorphic to ${}^{\kappa}\kappa$. □

Let $\mathcal{T}riv(\omega_1) = p[\mathcal{T}hr(\omega_1)]$ denote the set of all trivial coherent C-sequences of length ω_1.

Claim.

The set $\mathcal{T}riv(\omega_1)$ is a continuous injective image of ${}^{\kappa}\kappa$.

Proof of the Claim.

Since every coherent C-sequence of length ω_1 is threaded by at most one club subset of ω_1, the projection $p : \mathcal{T}hr(\kappa, \nu) \longrightarrow \mathcal{C}oh(\kappa, \nu)$ is injective. By the definition of the topologies, it is also continuous. □

We call a subset A of ${}^{\kappa}\kappa$ *super-dense* if $A \cap \bigcap_{\alpha<\kappa} U_\alpha \neq \emptyset$ whenever $\langle U_\alpha \mid \alpha < \kappa \rangle$ is a sequence of dense open subsets of some non-empty open subset of ${}^{\kappa}\kappa$.

Proposition

Assume that A and B are disjoint super-dense subsets of ${}^{\kappa}\kappa$. If $A \subseteq X \subseteq {}^{\kappa}\kappa \setminus B$, then X is not κ-Baire measurable. $\square$

The club filter Club_κ is always a super-dense subset of ${}^{\kappa}\kappa$.

We will show that both $\mathcal{T}\!\mathit{riv}(\omega_1)$ and its complement are super-dense. By the above claims, this shows that there is a continuous injective image of ${}^{\omega_1}\omega_1$ that is not $\aleph_1$-Baire measurable.

Let $\vec{C}_0$ be a coherent C-sequence of length $\gamma_0 < \omega_1$ and $\langle U_\alpha \mid \alpha < \kappa \rangle$ be a sequence of dense open subsets of $N_{\vec{C}_0}$.

We construct a sequence $\vec{C} = \langle C_\alpha \mid \alpha < \omega_1 \rangle$ and a strictly increasing continuous sequence $\langle \gamma_\alpha \mid \alpha < \omega_1 \rangle$ of ordinals less than ω_1 such that the following statements hold for every $\alpha < \omega_1$.

- $\langle C_\beta \mid \beta < \gamma_\alpha \rangle$ is a coherent C-sequence extending $\vec{C}_0$.
- $N_{\langle C_\beta \mid \beta < \gamma_{\alpha+1} \rangle}$ is a subset of U_α.
- If $\alpha \in \mathrm{Lim}$, then $C_{\gamma_\alpha} = \{\gamma_{\bar{\alpha}} \mid \bar{\alpha} < \alpha\}$.

Then $\vec{C}$ is a coherent C-sequence that is contained in $\bigcap_{\alpha<\kappa} U_\alpha$ and the club $C = \{\gamma_\alpha \mid \alpha < \omega_1\}$ witnesses that $\vec{C}$ is trivial.

If we replace the third statement by

- $\mathrm{otp}\,(C_{\gamma_\alpha}) \leq \omega$,

then $\vec{C}$ is a non-trivial coherent C-sequence in $\bigcap_{\alpha<\kappa} U_\alpha$.

The above constructions also allow us to prove the following result.

Theorem (L.-Schlicht)

Assume that CH *holds and every Aronszajn tree that does not contain a Souslin subtree is special. Then there is a* $\mathbf{\Delta}^1_1$*-subset of* ${}^{\omega_1}\omega_1$ *that is not* $\aleph_1$*-Baire measurable.*

Sketch of the Proof.

By considering the canonical Aronszajn tree $T(\rho_0)$ constructed from a C-sequence using Todorčević's technique of walks through such sequences, it is possible to show that the above assumption implies that the set $\mathcal{T}\!\mathit{riv}(\omega_1)$ is $\mathbf{\Delta}^1_1$-definable in $\mathcal{C}\!\mathit{oh}(\omega_1)$. $\square$

To prove the general theorem stated above, we pick an uncountable cardinal κ with $\kappa = \kappa^{<\kappa}$, fix a bijection $f : {}^{<\kappa}\kappa \times {}^{<\kappa}\kappa \longrightarrow \kappa$ and define A to be the set of all $x \in {}^{\kappa}\kappa$ such that the following statements hold for some $y \in {}^{\kappa}\kappa$ and a club subset C of κ.

- $C \ = \ \{\alpha < \kappa \mid x(\alpha) = y(\alpha)\}$.
- If $\alpha \in C$, then $x(\alpha) \ = \ f(x \restriction \alpha, \ y \restriction \alpha)$.

Given $x \in A$, it is easy to see that y and C with the above properties are uniquely determined.

A small modification of the above arguments shows that A is a continuous injective image of ${}^{\kappa}\kappa$ and a super-dense subset of ${}^{\kappa}\kappa$. Since A is disjoint from the club filter, it follows that A is not κ-Baire measurable.

The remaining implications

As mentioned above, the following diagram completely describes the provable and consistent relations between the considered classes in the case where κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$.

$$
\begin{tikzcd}
\mathbf{B}(\kappa) \arrow[r] & \mathbf{I}_{cl}(\kappa) \arrow[r, shift right] & \mathbf{\Sigma}^1_1(\kappa) \arrow[l, dashed, shift right, "\mathrm{V=L}"'] \\
\mathbf{\Sigma}^0_1(\kappa) \arrow[u] \arrow[r] & \mathbf{I}(\kappa) \arrow[u] \arrow[r] & \mathbf{C}(\kappa) \arrow[u]
\end{tikzcd}
$$

As mentioned above, the following diagram completely describes the provable and consistent relations between the considered classes in the case where κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$.

The following implications are trivial.

$$
\begin{tikzcd}
\mathbf{B}(\kappa) \arrow[d, dashed, shift right, "\times"'] & \mathbf{I}_{cl}(\kappa) \arrow[r] \arrow[d, dashed, shift right, "\times"'] & \mathbf{\Sigma}_1^1(\kappa) \\
\mathbf{\Sigma}_1^0(\kappa) \arrow[u, shift right] \arrow[r, shift right] & \mathbf{I}(\kappa) \arrow[u, shift right] \arrow[l, dashed, shift right, "\times"'] \arrow[r, shift right] & \mathbf{C}(\kappa) \arrow[u] \arrow[l, dashed, shift right, "\times"']
\end{tikzcd}
$$

As mentioned above, the following diagram completely describes the provable and consistent relations between the considered classes in the case where κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$.

The above constructions yields the following implications.

$$
\begin{tikzcd}
\mathbf{B}(\kappa) \arrow[drr, dashed, "\times" description] & \mathbf{I}_{cl}(\kappa) & \mathbf{\Sigma}^1_1(\kappa) \\
\mathbf{\Sigma}^0_1(\kappa) & \mathbf{I}(\kappa) \arrow[ul, dashed, "\times" description] & \mathbf{C}(\kappa)
\end{tikzcd}
$$

As mentioned above, the following diagram completely describes the provable and consistent relations between the considered classes in the case where κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$.

We present results that yield the following implications.

$$\begin{tikzcd}
\mathbf{B}(\kappa) \arrow[r, shift right] & \mathbf{I}_{cl}(\kappa) \arrow[l, dashed, shift right, "\times" description] & \mathbf{\Sigma}^1_1(\kappa) \arrow[l, dashed] \\
\mathbf{\Sigma}^0_1(\kappa) & \mathbf{I}(\kappa) & \mathbf{C}(\kappa) \arrow[ul, "\times" description]
\end{tikzcd}$$

Theorem

Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$ and A be a subset of ${}^{\kappa}\kappa$ such that

$$A \;=\; \{y \in {}^{\kappa}\kappa \mid \mathrm{L}[x,y] \models \varphi(x,y)\}$$

for some $x \in {}^{\kappa}\kappa$ and a Σ_1-formula $\varphi(u,v)$. Then A is a continuous injective image of a closed subset of ${}^{\kappa}\kappa$.

Corollary

Every κ-Borel subset of ${}^{\kappa}\kappa$ is a continuous injective image of a closed subset of ${}^{\kappa}\kappa$.

Corollary

There is a continuous injective image of a closed subset of ${}^{\kappa}\kappa$ that is not a κ-Borel subset of ${}^{\kappa}\kappa$.

Corollary

Assume that $\mathrm{V} = \mathrm{L}[x]$ *for some subset* x *of* κ*. Then every* $\mathbf{\Sigma}^1_1$*-subset of* ${}^\kappa\kappa$ *is a continuous injective image of a closed subset of* ${}^\kappa\kappa$*.*

Theorem

Let κ *be an uncountable regular cardinal,* $\delta > \kappa$ *be an inaccessible cardinal and* G *be* $\mathrm{Col}(\kappa, {<}\delta)$*-generic over* V*. In* $\mathrm{V}[G]$*, the club filter* Club_κ *is not equal to a continuous injective image of* ${}^\kappa\kappa$*.*

Corollary

It is consistent that there is a continuous image of ${}^\kappa\kappa$ *that is not equal to a continuous injective image of a closed subset of* ${}^\kappa\kappa$*.*

Trees of higher cardinalities

Let W be the closed set of all x in ${}^{\kappa}\kappa$ coding a well-order of κ. By the above results, W is not a continuous image of ${}^{\kappa}\kappa$. But it is easy to show that W equal to a continuous image of ${}^{\kappa}(\kappa^{+})$.

Therefore it is also interesting to investigate continuous images of closed subsets of ${}^{\mu}\kappa$ for some cardinal $\mu \geq \kappa$.

Since every subset of ${}^{\kappa}\kappa$ is a continuous image of ${}^{\kappa}(2^{\kappa})$, the above results show that

$$c(\kappa) \;=\; \min\{\mu \in \mathrm{On} \mid \textit{Closed subsets of } {}^{\kappa}\kappa \textit{ are cont. images of } {}^{\kappa}\mu\}$$

is a well-defined cardinal characteristic with

$$\kappa \;<\; c(\kappa) \;\leq\; 2^{\kappa}.$$

The following result shows that we can manipulate the value of $c(\kappa)$ by forcing.

Theorem (L.-Schlicht)

Assume that

- κ *is an uncountable cardinal with* $\kappa = \kappa^{<\kappa}$,
- $\mu \geq 2^\kappa$ *is a cardinal with* $\mu = \mu^\kappa$, *and*
- $\theta \geq \mu$ *is a cardinal with* $\theta = \theta^\kappa$.

Then the following statements hold in a cofinality preserving forcing extension $\mathrm{V}[G]$ *of the ground model* V.

- $2^\kappa = \theta$.
- *Every closed subset of* ${}^\kappa\mu$ *is an continuous image of* ${}^\kappa\mu$.
- *There is a closed subset* A *of* ${}^\kappa\kappa$ *that is not equal to an continuous image of* ${}^\kappa\bar{\mu}$ *for some* $\bar{\mu} < \mu$ *with* $\bar{\mu}^{<\kappa} < \mu$.

This statement is a consequence of the following results.

Lemma

Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$, $\mu = 2^\kappa$ and G be $\mathrm{Add}(\kappa, \theta)$-generic over V for some cardinal θ. In $\mathrm{V}[G]$, every closed subset of ${}^\kappa\mu$ is equal to a continuous image of ${}^\kappa\mu$.

Theorem

Assume that there is an inner model M such that M does not contain the reals and every countable set of ordinal in V is covered by a set that is an element of M and countable in M.

If κ is an uncountable regular cardinal, then there is a closed subset A of ${}^\kappa\kappa$ such that A is not a continuous image of ${}^\kappa\mu$ for every cardinal μ with $\mu^{<\kappa} < |(2^\kappa)^M|^{\mathrm{V}}$.

Lemma

Let κ be an uncountable regular and T be a subtree of ${}^{<\kappa}\kappa$. If μ is a cardinal with $\mu^{<\kappa} < |[T]|$ and $c : {}^{\kappa}\mu \longrightarrow [T]$ is a continuous surjection, then there is a Lipschitz embedding $i : {}^{\leq\omega}\omega \longrightarrow T$.

Sketch of the Proof of the Theorem.

Let κ be an uncountable regular cardinal. Let $T = ({}^{<\kappa}2)^M$ and $A = [T]$. Assume that there is a continuous surjection $f : {}^{\kappa}\mu \longrightarrow A$ for some $\mu < |(2^{\kappa})^M|^{\mathrm{V}}$.

- By the Lemma, there is a Lipschitz embedding $i : {}^{\leq\omega}\omega \longrightarrow T$.
- By the σ-cover property, there is a subtree T_* of T in M such that $\mathrm{ran}(i) \subseteq T_*$ and a Lipschitz embedding $j : T_* \longrightarrow {}^{\leq\omega}\omega$ in M.
- The image of ${}^{\omega}\omega$ under $(j \circ i)$ is a superperfect subset of $({}^{\omega}\omega)^M$.
- By a theorem of Velickovic and Woodin, this implies that all reals are contained in M, a contradiction. $\square$

Kurepa trees as continuous images

The techniques developed in the proofs of the above results also allows us to discuss the question whether the set of all cofinal branches through a κ-Kurepa tree can be a continuous image of ${}^{\kappa}\kappa$.

Theorem (L.-Schlicht)

- *Let ν be an infinite cardinal and $\kappa = \nu^{+} = \nu^{\aleph_0}$. If T is a κ-Kurepa subtree of ${}^{<\kappa}\kappa$, then $[T]$ is not a continuous image of ${}^{\kappa}\kappa$.*
- *Let ν be an uncountable regular cardinal, $\kappa > \nu$ be an inaccessible cardinal and $(G * H)$ be $(\mathrm{Add}(\omega, 1) * \mathrm{Col}(\nu, {<}\kappa))$-generic over V. In $\mathrm{V}[G, H]$, there is a κ-Kurepa subtree T of ${}^{<\kappa}\kappa$ such that $[T]$ is a retract of ${}^{\kappa}\kappa$.*
- *Let κ be an inaccessible cardinal and T be a slim κ-Kurepa subtree of ${}^{<\kappa}\kappa$. Then $[T]$ is not a continuous image of ${}^{\kappa}\kappa$.*

Thank you for listening!