## Continuous images of closed sets in generalized Baire spaces

ESI Workshop: Forcing and Large Cardinals

#### Philipp Moritz Lücke (joint work with Philipp Schlicht)

Mathematisches Institut, Rheinische Friedrich-Wilhelms-Universität Bonn http://www.math.uni-bonn.de/people/pluecke/

Vienna, 09/26/2013

#### Generalized Baire spaces

Let  $\kappa$  be an infinite cardinal with  $\kappa = \kappa^{<\kappa}$ .

Given a cardinal  $\mu$ , we equip the set  ${}^{\kappa}\mu$  consisting of all functions  $x: \kappa \longrightarrow \mu$  with the topology whose basic open sets are of the form

$$N_s = \{ x \in {}^{\kappa} \mu \mid s \subseteq x \},$$

where s is an element of the set  ${}^{<\kappa}\mu$  of all functions  $t: \alpha \longrightarrow \mu$  with  $\alpha < \kappa$ .

We call the space  ${}^{\kappa}\kappa$  the generalized Baire space of  $\kappa$ .

#### $\Sigma_1^1$ -subsets of $\kappa \kappa$

A subset A of  $\kappa \kappa$  is a  $\Sigma_1^1$ -subset if it is equal to the projection of a closed subset of  $\kappa \kappa \times \kappa \kappa$ . We let  $\Sigma_1^1(\kappa)$  denote the class of all such subsets.

It is easy to see that a subset of  ${}^{\kappa}\kappa$  is an element of  $\Sigma_1^1(\kappa)$  if and only if it is equal to a continuous image of a closed subset of  ${}^{\kappa}\kappa$ .

If  $\kappa$  is an uncountable cardinal with  $\kappa = \kappa^{<\kappa}$ , then a subset of  ${}^{\kappa}\kappa$  is contained in  $\Sigma_1^1(\kappa)$  if and only if it is definable over the structure  $(H_{\kappa^+}, \in)$  by a  $\Sigma_1$ -formula with parameters.

This shows that in this case many interesting and important sets are equal to continuous images of closed subsets.

We present an example of such a subset.

#### Example

The club filter

 $\operatorname{Club}_{\kappa} = \{ x \in {}^{\kappa}\kappa \mid \exists C \subseteq \kappa \ club \ \forall \alpha \in C \ x(\alpha) = 1 \}$ 

is a continuous image of the space  ${}^\kappa\kappa.$ 

Let T denote the tree consisting of all pairs (s,t) in  $\gamma 2 \times \gamma 2$  such that  $\gamma \in \text{Lim} \cap \kappa$ ,  $t(\alpha) \leq s(\alpha)$  for all  $\alpha < \gamma$  and t is the characteristic function of a club subset of  $\gamma$ .

Then T is isomorphic to the tree  ${}^{<\kappa}\kappa$ , because it is closed under increasing sequences of length  $<\kappa$  and every node has  $\kappa$ -many direct successors.

If we equip the set [T] of all  $\kappa$ -branches through T with the topology whose basic open sets consists of all extensions of elements of T, then we obtain a topological space homeomorphic to  $\kappa \kappa$ .

Since the projection  $p:[T] \longrightarrow {}^{\kappa}\kappa$  onto the union of the first coordinate is continuous, we can conclude that the set  $\operatorname{Club}_{\kappa}$  is equal to a continuous image of  ${}^{\kappa}\kappa$ .

#### Classes of continuous images

Given an uncountable cardinal  $\kappa$  with  $\kappa = \kappa^{<\kappa}$ , we study the following subclasses of  $\Sigma_1^1(\kappa)$  that arise by restricting the classes of used continuous functions and closed subsets.

- The class  $\Sigma_1^1(\kappa)$  of continuous images of closed subsets of  ${}^{\kappa}\kappa$ .
- The class  $\mathbf{C}(\kappa)$  of continuous images of  $\kappa \kappa$ .
- The class  $\mathbf{I}_{cl}(\kappa)$  of continuous injective images of closed subsets of  ${}^{\kappa}\kappa$ .
- The class  $\mathbf{I}(\kappa)$  of continuous injective images of  $\kappa \kappa$ .

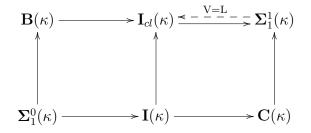
We will compare these classes with the following collections.

- The class  $\Sigma_1^0(\kappa)$  of open subsets of  ${}^{\kappa}\kappa$ .
- The class  $\mathbf{B}(\kappa)$  of  $\kappa$ -Borel subsets of  $\kappa \kappa$ , i.e. the subsets contained in the smallest algebra of sets on  $\kappa \kappa$  that contains all open subsets and is closed under  $\kappa$ -unions.

In the case " $\kappa = \omega$ ", the relationship of the above classes is described by the following complete diagram.

$$\Sigma_1^0(\omega) \longrightarrow \mathbf{I}(\omega) \longrightarrow \mathbf{I}_{cl}(\omega) = \mathbf{B}(\omega) \longrightarrow \Sigma_1^1(\omega) = \mathbf{C}(\omega)$$

Our results will show that these classes behave in a very different way if  $\kappa$  is an uncountable cardinal with  $\kappa = \kappa^{<\kappa}$ . These results are summarized by the following complete diagram.



In particular, we will show that the following statements fail to generalize to higher cardinalities.

- Every closed subset of the space  ${}^{\omega}\omega$  is equal to a continuous image of  ${}^{\omega}\omega$ .
- Every continuous injective image of the space  ${}^{\omega}\omega$  is a Borel subset.

In the following, we construct the corresponding counterexamples.

Continuous images of  ${}^\kappa\kappa$ 

### Continuous images of $\kappa \kappa$

#### Retracts

An important topological property of spaces of the form  ${}^{\omega}\mu$  is the fact that non-empty closed subsets are retracts of the whole space, i.e. given an non-empty closed subset A of  ${}^{\omega}\mu$  there is a continuous surjection  $f:{}^{\omega}\mu \longrightarrow A$  such that  $f \upharpoonright A = \mathrm{id}_A$ .

An easy argument shows that this property fails if  $\kappa$  is uncountable.

#### Proposition

Suppose that  $\kappa$  is an uncountable regular cardinal and  $\mu > 1$  is a cardinal. Let A denote the set of all x in  ${}^{\kappa}\mu$  such that  $x(\alpha) = 1$  for only finitely many  $\alpha < \kappa$ . Then A is a closed subset of  ${}^{\kappa}\mu$  that is not a retract of  ${}^{\kappa}\mu$ .

#### Proof.

Assume, towards a contradiction, that there is a continuous function  $f: {}^{\kappa}\mu \longrightarrow A$  with  $f \upharpoonright A = \mathrm{id}_A$ . We construct a strictly increasing sequence  $\langle \gamma_n < \kappa \mid n < \omega \rangle$  of ordinals such that  $\gamma_0 = 1$  and

$$N_{x_n \upharpoonright \gamma_{n+1}} \subseteq f^{-1} " N_{x_n \upharpoonright (\gamma_n+1)}$$

holds for all  $n < \omega$  and the unique  $x_n \in {}^{\kappa}2$  with

$$x_n^{-1}$$
 " {1} = { $\gamma_0, \ldots, \gamma_n$  }.

Let  $x \in {}^{\kappa}2$  be the unique function with

$$x^{-1}$$
 " {1} = { $\gamma_n \mid n < \omega$  }.

Then our construction yields

$$f(x) \upharpoonright \sup_{n < \omega} \gamma_n = x \upharpoonright \sup_{n < \omega} \gamma_n$$

and this implies that  $f(x) \notin A$ , a contradiction.

#### Continuous Images

Every closed subset of  ${}^{\omega}\omega$  is a continuous image of  ${}^{\omega}\omega$  and hence every  $\Sigma_1^1$ -subset is equal to a continuous image of  ${}^{\omega}\omega$ .

The following result shows that this statement also does not generalize to uncountable regular cardinals.

#### Theorem (L.-Schlicht)

Let  $\kappa$  be an uncountable cardinal with  $\kappa = \kappa^{<\kappa}$ . Then there is a closed non-empty subset of  ${}^{\kappa}\kappa$  that is not equal to a continuous image of  ${}^{\kappa}\kappa$ .

#### Proof.

It suffices to construct a closed subset A of  $\kappa \kappa$  with the property that A is not equal to the projection p[T] of a  $<\kappa$ -closed subtree T of  $<\kappa \kappa < <\kappa \kappa$  without terminal nodes.

Given  $\lambda \leq \kappa$  closed under Gödel pairing and  $x \in {}^{\lambda}2$ , define a binary relation  $\in_x$  on  $\lambda$  by setting

$$\alpha \in_x \beta \iff x(\prec \alpha, \beta \succ) = 1.$$

Define

$$W = \{x \in {}^{\kappa}2 \mid (\kappa, \in_x) \text{ is a well-order}\}.$$

Then W is a closed subset of  $\kappa \kappa$ .

Assume, towards a contradiction, that there is a  $<\kappa$ -closed subtree T of  ${}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$  without terminal nodes such that W = p[T].

#### Proof (cont.).

Given  $(s,t) \in T$  and  $\alpha < \kappa$ , define

$$r(s,t,\alpha) = \sup\{\operatorname{rnk}_{\in_x}(\alpha) \mid x \in p([T] \cap N_{(s,t)})\} \leq \kappa^+$$

Then  $r(\emptyset, \emptyset, \alpha) = \kappa^+$  for every  $\alpha < \kappa$ .

#### Claim.

Let  $(s,t) \in T$  and  $\alpha < \kappa$  with  $r(s,t,\alpha) = \kappa^+$ . If  $\gamma < \kappa^+$ , then there is  $(u,v) \in T$  extending (s,t) and  $\alpha < \beta < \kappa$  such that dom(u) is closed under Gödel pairing,  $\beta < \ln(u)$ ,  $\beta \in_u \alpha$ , and  $r(u,v,\beta) \ge \gamma$ .

#### Proof of the Claim.

There is a  $(x, y) \in [T] \cap N_{(s,t)}$  with  $\operatorname{rnk}_{\in x}(\alpha) \geq \gamma + \kappa$ . Hence we can find  $\alpha < \beta < \kappa$  with  $\gamma \leq \operatorname{rnk}_{\in x}(\beta) < \gamma + \kappa$ . Pick  $\delta > \max\{\alpha, \beta, \ln(s)\}$ closed under Gödel pairing and define (u, v) to be the node  $(x \upharpoonright \delta, y \upharpoonright \delta)$ extending (s, t). Since  $\in_u$  is a well-ordering of  $\ln(u)$ , we have  $\beta \in_u \alpha$ . Finally, (x, y) witnesses that  $r(u, v, \beta) \geq \gamma$ .

#### Proof (cont.).

#### Claim.

If  $(s,t) \in T$  and  $\alpha < \kappa$  with  $r(s,t,\alpha) = \kappa^+$ , then there is a node (u,v)in T extending (s,t) and  $\alpha < \beta < \ln(u)$  such that  $\ln(u)$  is closed under Gödel pairing,  $\beta \in_u \alpha$  and  $r(u,v,\beta) = \kappa^+$ .

This claim shows that there are strictly increasing sequences  $\langle (s_n, t_n) \mid n < \omega \rangle$  of nodes in T and  $\langle \beta_n \mid n < \omega \rangle$  of elements of  $\kappa$  with  $\blacksquare \ln(s_n)$  is closed under Gödel pairing,

Let  $s = \bigcup_{n < \omega} s_n$  and  $t = \bigcup_{n < \omega} t_n$ . Then  $(s, t) \in T$ , since T is  $\omega$ -closed. By our assumptions on T, there is a cofinal branch (x, y) in [T] through (s, t). Then  $\beta_{n+1} \in_x \beta_n$  for every  $n < \omega$  and this shows that  $x \notin W = p[T]$ , a contradiction.

# Continuous injective images of $\kappa \kappa$

Next we construct a continuous injective image of  $\kappa \kappa$  that is not a  $\kappa$ -Borel subset of  $\kappa \kappa$ . In order to prove that certain sets are not  $\kappa$ -Borel, we need to introduce an important regularity property of subsets of  $\kappa \kappa$ .

We say that a subset A of  ${}^{\kappa}\kappa$  is  $\kappa$ -Baire measurable if there is an open subset U of  ${}^{\kappa}\kappa$  and a sequence  $\langle N_{\alpha} \mid \alpha < \kappa \rangle$  of nowhere dense subsets of  ${}^{\kappa}\kappa$  such that the symmetric difference  $A_{\Delta}U$  is a subset of  $\bigcup_{\alpha < \kappa} N_{\alpha}$ .

A standard prove shows that every  $\kappa$ -Borel subset of  ${}^{\kappa}\kappa$  is  $\kappa$ -Baire measurable. Moreover it is consistent that all  $\Delta_1^1$ -subsets of  ${}^{\kappa}\kappa$  are  $\kappa$ -Baire measurable.

#### Theorem (L.-Schlicht)

Let  $\kappa$  be an uncountable cardinal with  $\kappa = \kappa^{<\kappa}$ . Then there is a continuous injective image of  $\kappa$  that is not  $\kappa$ -Baire measurable.

To motivate this result, we first consider the case  $\kappa = \aleph_1 = 2^{\aleph_0}$ and show that a well-known collection of combinatorial objects provides an example of a set with the above properties.

#### Definition

- Given  $\gamma \in On$ , a sequence  $\langle C_{\alpha} \mid \alpha < \gamma \rangle$  is a *coherent C*-sequence if the following statements hold for all  $\alpha < \gamma$ .
  - If  $\alpha$  is a limit ordinal, then  $C_{\alpha}$  is a closed unbounded subset of  $\alpha$ .
  - If  $\alpha = \bar{\alpha} + 1$ , then  $C_{\alpha} = \{\bar{\alpha}\}.$
  - If  $\bar{\alpha} \in \text{Lim}(C_{\alpha})$ , then  $C_{\bar{\alpha}} = C_{\alpha} \cap \bar{\alpha}$ .
- A coherent C-sequence  $\langle C_{\alpha} \mid \alpha < \gamma \rangle$  with  $\gamma \in \text{Lim is trivial if}$ there is a closed unbounded subset  $C_{\gamma}$  of  $\gamma$  that threads  $\vec{C}$ , i.e. the sequence  $\langle C_{\alpha} \mid \alpha \leq \gamma \rangle$  is also a coherent C-sequence.

Let  $Coh(\omega_1)$  be the set of all coherent *C*-sequences of length  $\omega_1$  equipped with the topology whose basic open sets consist of all extensions of coherent *C*-sequences of limit length less than  $\omega_1$ .

#### Claim.

The space  $Coh(\omega_1)$  is homeomorphic to  $\omega_1 \omega_1$ .

#### Proof of the Claim.

Let  $\mathcal{T}$  denote the tree of all coherent *C*-sequences of limit length less than  $\omega_1$ . Then  $\mathcal{T}$  is isomorphic to  ${}^{<\omega_1}\omega_1$ , because  $\mathcal{T}$  is  $\sigma$ -closed and every node in  $\mathcal{T}$  has  $\aleph_1$ -many direct successors. This isomorphism gives us a homeomorphism of the above spaces.  $\Box$  Define  $\mathcal{Thr}(\omega_1)$  to be set of all pairs  $(\vec{C}, C)$  such that  $\vec{C}$  is an element of  $\mathcal{Coh}(\omega_1)$  and C is a thread through  $\vec{C}$ . We equip  $\mathcal{Thr}(\omega_1)$  with the topology whose basic open sets consist of all component-wise extensions of pairs  $(\vec{D}, D)$  such that  $\vec{D}$  is a coherent C-sequence of length  $\gamma \in \text{Lim} \cap \omega_1$  and D is a thread through  $\vec{D}$ .

#### Claim.

The space  $Thr(\omega_1)$  is homeomorphic to  $\kappa \kappa$ .

Let  $Triv(\omega_1) = p[Thr(\omega_1)]$  denote the set of all trivial coherent *C*-sequences of length  $\omega_1$ .

#### Claim.

The set  $Triv(\omega_1)$  is a continuous injective image of  $\kappa \kappa$ .

#### Proof of the Claim.

Since every coherent C-sequence of length  $\omega_1$  is threaded by at most one club subset of  $\omega_1$ , the projection  $p: Thr(\kappa, \nu) \longrightarrow Coh(\kappa, \nu)$  is injective. By the definition of the topologies, it is also continuous. We call a subset A of  ${}^{\kappa}\kappa$  super-dense if  $A \cap \bigcap_{\alpha < \kappa} U_{\alpha} \neq \emptyset$  whenever  $\langle U_{\alpha} \mid \alpha < \kappa \rangle$  is a sequence of dense open subsets of some non-empty open subset of  ${}^{\kappa}\kappa$ .

#### Proposition

Assume that A and B are disjoint super-dense subsets of  $\kappa \kappa$ . If  $A \subseteq X \subseteq \kappa \wedge B$ , then X is not  $\kappa$ -Baire measurable.

The club filter  $\text{Club}_{\kappa}$  is always a super-dense subset of  $\kappa \kappa$ .

We will show that both  $Triv(\omega_1)$  and its complement are super-dense. By the above claims, this shows that there is a continuous injective image of  $\omega_1 \omega_1$  that is not  $\aleph_1$ -Baire measurable. Let  $\vec{C}_0$  be a coherent *C*-sequence of length  $\gamma_0 < \omega_1$  and  $\langle U_\alpha \mid \alpha < \kappa \rangle$  be a sequence of dense open subsets of  $N_{\vec{C}_0}$ .

We construct a sequence  $\vec{C} = \langle C_{\alpha} \mid \alpha < \omega_1 \rangle$  and a strictly increasing continuous sequence  $\langle \gamma_{\alpha} \mid \alpha < \omega_1 \rangle$  of ordinals less than  $\omega_1$  such that the following statements hold for every  $\alpha < \omega_1$ .

•  $\langle C_{\beta} \mid \beta < \gamma_{\alpha} \rangle$  is a coherent *C*-sequence extending  $\vec{C}_0$ .

• 
$$N_{\langle C_{\beta} | \beta < \gamma_{\alpha+1} \rangle}$$
 is a subset of  $U_{\alpha}$ .

• If  $\alpha \in \text{Lim}$ , then  $C_{\gamma_{\alpha}} = \{\gamma_{\bar{\alpha}} \mid \bar{\alpha} < \alpha\}.$ 

Then  $\vec{C}$  is a coherent *C*-sequence that is contained in  $\bigcap_{\alpha < \kappa} U_{\alpha}$ and the club  $C = \{\gamma_{\alpha} \mid \alpha < \omega_1\}$  witnesses that  $\vec{C}$  is trivial.

If we replace the third statement by

• 
$$\operatorname{otp}(C_{\gamma_{\alpha}}) \leq \omega,$$

then  $\vec{C}$  is a non-trivial coherent *C*-sequence in  $\bigcap_{\alpha < \kappa} U_{\alpha}$ .

The above constructions also allow us to prove the following result.

#### Theorem (L.-Schlicht)

Assume that CH holds and every Aronszajn tree that does not contain a Souslin subtree is special. Then there is a  $\Delta_1^1$ -subset of  $\omega_1 \omega_1$  that is not  $\aleph_1$ -Baire measurable.

#### Sketch of the Proof.

By considering the canonical Aronszajn tree  $T(\rho_0)$  constructed from a *C*-sequence using Todorčević's technique of walks through such sequences, it is possible to show that the above assumption implies that the set  $Triv(\omega_1)$  is  $\Delta_1^1$ -definable in  $Coh(\omega_1)$ . To prove the general theorem stated above, we pick an uncountable cardinal  $\kappa$  with  $\kappa = \kappa^{<\kappa}$ , fix a bijection  $f: {}^{<\kappa}\kappa \times {}^{<\kappa}\kappa \longrightarrow \kappa$  and define A to be the set of all  $x \in {}^{\kappa}\kappa$  such that the following statements hold for some  $y \in {}^{\kappa}\kappa$  and a club subset C of  $\kappa$ .

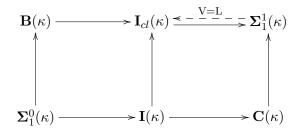
$$\label{eq:constraint} \blacksquare \ C \ = \ \{\alpha < \kappa \ | \ x(\alpha) = y(\alpha)\}.$$

• If  $\alpha \in C$ , then  $x(\alpha) = f(x \upharpoonright \alpha, y \upharpoonright \alpha)$ .

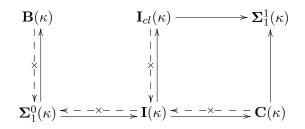
Given  $x \in A$ , it is easy to see that y and C with the above properties are uniquely determined.

A small modification of the above arguments shows that A is a continuous injective image of  $\kappa \kappa$  and a super-dense subset of  $\kappa \kappa$ . Since A is disjoint from the club filter, it follows that A is not  $\kappa$ -Baire measurable. The remaining results

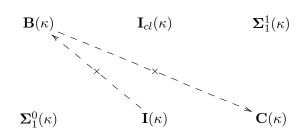
## The remaining implications



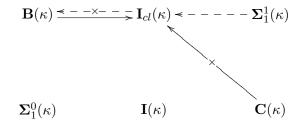
The following implications are trivial.



The above constructions yields the following implications.



We present results that yield the following implications.



#### Theorem

Let  $\kappa$  be an uncountable cardinal with  $\kappa = \kappa^{<\kappa}$  and A be a subset of  ${}^{\kappa}\kappa$  such that

$$A = \{ y \in {}^{\kappa}\kappa \mid \mathbf{L}[x, y] \models \varphi(x, y) \}$$

for some  $x \in {}^{\kappa}\kappa$  and a  $\Sigma_1$ -formula  $\varphi(u, v)$ . Then A is a continuous injective image of a closed subset of  ${}^{\kappa}\kappa$ .

#### Corollary

Every  $\kappa$ -Borel subset of  $\kappa \kappa$  is a continuous injective image of a closed subset of  $\kappa \kappa$ .

#### Corollary

There is a continuous injective image of a closed subset of  $\kappa \kappa$  that is not a  $\kappa$ -Borel subset of  $\kappa \kappa$ .

#### Corollary

Assume that V = L[x] for some subset x of  $\kappa$ . Then every  $\Sigma_1^1$ -subset of  $\kappa \kappa$  is a continuous injective image of a closed subset of  $\kappa \kappa$ .

#### Theorem

Let  $\kappa$  be an uncountable regular cardinal,  $\delta > \kappa$  be an inaccessible cardinal and G be  $\operatorname{Col}(\kappa, <\delta)$ -generic over V. In V[G], the club filter  $\operatorname{Club}_{\kappa}$  is not equal to a continuous injective image of  $\kappa \kappa$ .

#### Corollary

It is consistent that there is a continuous image of  $\kappa \kappa$  that is not equal to a continuous injective image of a closed subset of  $\kappa \kappa$ .

Trees of higher cardinalities

## Trees of higher cardinalities

Let W be the closed set of all x in  $\kappa \kappa$  coding a well-order of  $\kappa$ . By the above results, W is not a continuous image of  $\kappa \kappa$ . But it is easy to show that W equal to a continuous image of  $\kappa (\kappa^+)$ .

Therefore it is also interesting to investigate continuous images of closed subsets of  ${}^{\mu}\kappa$  for some cardinal  $\mu \geq \kappa$ .

Since every subset of  $\kappa \kappa$  is a continuous image of  $\kappa(2^{\kappa})$ , the above results show that

 $c(\kappa) = \min\{\mu \in \operatorname{On} | Closed \ subsets \ of \ \kappa \ are \ cont. \ images \ of \ \kappa \mu\}$ 

is a well-defined cardinal characteristic with

$$\kappa < c(\kappa) \leq 2^{\kappa}.$$

The following result shows that we can manipulate the value of  $c(\kappa)$  by forcing.

#### Theorem (L.-Schlicht)

Assume that

- $\kappa$  is an uncountable cardinal with  $\kappa = \kappa^{<\kappa}$ ,
- $\mu \geq 2^{\kappa}$  is a cardinal with  $\mu = \mu^{\kappa}$ , and
- $\theta \ge \mu$  is a cardinal with  $\theta = \theta^{\kappa}$ .

Then the following statements hold in a cofinality preserving forcing extension V[G] of the ground model V.

- $\bullet 2^{\kappa} = \theta.$
- Every closed subset of  ${}^{\kappa}\mu$  is an continuous image of  ${}^{\kappa}\mu$ .
- There is a closed subset A of <sup>κ</sup>κ that is not equal to an continuous image of <sup>κ</sup>μ for some μ < μ with μ<sup><κ</sup> < μ.</li>

This statement is a consequence of the following results.

#### Lemma

Let  $\kappa$  be an uncountable cardinal with  $\kappa = \kappa^{<\kappa}$ ,  $\mu = 2^{\kappa}$  and G be Add $(\kappa, \theta)$ -generic over V for some cardinal  $\theta$ . In V[G], every closed subset of  $^{\kappa}\mu$  is equal to a continuous image of  $^{\kappa}\mu$ .

#### Theorem

Assume that there is an inner model M such that M does not contain the reals and every countable set of ordinal in V is covered by a set that is an element of M and countable in M. If  $\kappa$  is an uncountable regular cardinal, then there is a closed subset A of  $\kappa \kappa$  such that A is not a continuous image of  $\kappa \mu$  for every cardinal  $\mu$  with  $\mu^{<\kappa} < |(2^{\kappa})^M|^V$ .

#### Lemma

Let  $\kappa$  be an uncountable regular and T be a subtree of  ${}^{<\kappa}\kappa$ . If  $\mu$  is a cardinal with  $\mu^{<\kappa} < |[T]|$  and  $c : {}^{\kappa}\mu \longrightarrow [T]$  is a continuous surjection, then there is a Lipschitz embedding  $i : {}^{\leq \omega}\omega \longrightarrow T$ .

#### Sketch of the Proof of the Theorem.

Let  $\kappa$  be an uncountable regular cardinal. Let  $T = ({}^{<\kappa}2)^M$  and A = [T]. Assume that there is a continuous surjection  $f : {}^{\kappa}\mu \longrightarrow A$  for some  $\mu < |(2^{\kappa})^M|^V$ .

- By the Lemma, there is a Lipschitz embedding  $i : {}^{\leq \omega}\omega \longrightarrow T$ .
- By the  $\sigma$ -cover property, there is a subtree  $T_*$  of T in M such that  $\operatorname{ran}(i) \subseteq T_*$  and a Lipschitz embedding  $j: T_* \longrightarrow {}^{\leq \omega} \omega$  in M.
- The image of  ${}^{\omega}\omega$  under  $(j \circ i)$  is a superperfect subset of  $({}^{\omega}\omega)^M$ .
- By a theorem of Velickovic and Woodin, this implies that all reals are contained in *M*, a contradiction.

# Kurepa trees as continuous images

The techniques developed in the proofs of the above results also allows us to discuss the question whether the set of all cofinal branches through a  $\kappa$ -Kurepa tree can be a continuous image of  $\kappa \kappa$ .

#### Theorem (L.-Schlicht)

- Let  $\nu$  be an infinite cardinal and  $\kappa = \nu^+ = \nu^{\aleph_0}$ . If T is a  $\kappa$ -Kurepa subtree of  ${}^{<\kappa}\kappa$ , then [T] is not a continuous image of  ${}^{\kappa}\kappa$ .
- Let  $\nu$  be an uncountable regular cardinal,  $\kappa > \nu$  be an inaccessible cardinal and (G \* H) be  $(Add(\omega, 1) * Col(\nu, <\kappa))$ -generic over V. In V[G, H], there is a  $\kappa$ -Kurepa subtree T of  ${}^{<\kappa}\kappa$  such that [T] is a retract of  ${}^{\kappa}\kappa$ .
- Let  $\kappa$  be an inaccessible cardinal and T be a slim  $\kappa$ -Kurepa subtree of  ${}^{<\kappa}\kappa$ . Then [T] is not a continuous image of  ${}^{\kappa}\kappa$ .

## Thank you for listening!