Definability in mathematics

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Introduction

The Axiom of Choice implies the existence of a great variety of mathematical objects without providing explicit constructions for them.

We list some prominent examples of such objects:

- Non-Lebesgue measurable sets of real numbers.
- $\mathbb{Q}$-bases of the real numbers.
- Well-orderings of the real numbers.
- Three distinct automorphisms of the field of complex numbers.
- Stationary and co-stationary subsets of uncountable regular cardinals.

In all of the above cases, it can be shown that the existence of these objects is not a consequence of the axioms of $\mathbf{ZF}$ alone.

Three distinct automorphisms of the field of complex numbers

The field $\mathbb{C}$ of all complex numbers has two obvious automorphisms:

- The identity $\mathrm{id}_{\mathbb{C}}$.
- Complex conjugation $k_{\mathbb{C}}$.

With the help of transcendence bases, it is easy to show that $\mathbb{C}$ has many other automorphisms.

Lemma

$|\mathrm{Aut}(\mathbb{C})| = 2^{2^{\aleph_0}}$.

The proof of this lemma does not provide a concrete construction of a third automorphism of $\mathbb{C}$.

We now present results that show that the axioms of $\mathbf{ZF}$ do not imply the existence of such an automorphism.

In the following, we work in the theory $\mathbf{ZF} + \mathbf{DC}$, where $\mathbf{DC}$ denotes the *Principle of Dependent Choices*:

> *"For every nonempty set X and every binary relation R on X with $\mathrm{dom}(R) = X$, there exists a sequence $\langle x_n \in X \mid n \in \mathbb{N} \rangle$ with $x_n \; R \; x_{n+1}$ for all $n \in \mathbb{N}$".*

This choice principle suffices for the development of the topological theory of the reals and measure theory.

We now show that the identity and complex conjugation are the only field automorphisms of $\mathbb{C}$ that are topological simple.

Proposition

The maps $\mathrm{id}_{\mathbb{C}}$ *and* $k_{\mathbb{C}}$ *are the only continuous field automorphisms of* $\mathbb{C}$.

Proof.

Let $\pi \in \mathrm{Aut}(\mathbb{C})$ be continuous with $\pi(i) = i$. Since π is a field automorphism, we have $\pi \restriction \mathbb{Q} = \mathrm{id}_{\mathbb{Q}}$ therefore

$$\pi \restriction (\mathbb{Q} + i \cdot \mathbb{Q}) \ = \ \mathrm{id}_{\mathbb{Q}+i\cdot\mathbb{Q}}.$$

Since $\mathbb{Q} + i \cdot \mathbb{Q}$ is dense in $\mathbb{C}$, the continuity of π implies that $\pi = \mathrm{id}_{\mathbb{C}}$. $\square$

To extend the above result to larger classes of functions, we need to review some fundamental topological concepts.

Definition

Let X und Y be topological spaces.

- A subset N of X is *nowhere dense*, if the closure of N has empty interior.
- A subset M of X is *meager*, if it is a countable union of nowhere dense subsets.
- A subset A of X has the *Baire property*, if there is an open subset U of X such that the symmetric difference $A_\Delta U = (A \setminus U) \cup (U \setminus A)$ is meager.
- A function $f : X \longrightarrow Y$ is *Baire-measurable*, if $f^{-1}[U]$ has the Baire property in X for every open subset U of Y.

Definition

- A *Polish space* is a separable completely metrizable topological space.
- A *Polish group* is a topological group whose topology is Polish.

Example

$\langle \mathbb{C}, +, 0 \rangle$ and $\langle \mathbb{C}^*, \cdot, 1 \rangle$ are Polish groups.

Lemma

Every Baire-measurable group homomorphism between Polish groups is continuous.

Corollary

The maps $\mathrm{id}_{\mathbb{C}}$ and $k_{\mathbb{C}}$ are the only Baire-measurable field automorphisms of $\mathbb{C}$.

All of the above results were proven in the theory $\mathbf{ZF} + \mathbf{DC}$.

The following result allows to show that the existence of a third automorphism of $\mathbb{C}$ is not provable in $\mathbf{ZF} + \mathbf{DC}$.

Theorem (Shelah)

If the theory $\mathbf{ZF C}$ is consistent, then the theory

$$\mathbf{ZF} + \mathbf{DC} + \text{“\textit{Every set of reals has the Baire property}”}$$

is consistent.

Corollary

If the theory $\mathbf{ZFC}$ is consistent, then the statement

$$\text{“}\mathrm{Aut}(\mathbb{C}) = \{\mathrm{id}_{\mathbb{C}}, \mathrm{k}_{\mathbb{C}}\}\text{”}$$

is consistent with the axioms of $\mathbf{ZF} + \mathbf{DC}$.

Definability in second-order arithmetic

It is natural to ask if there is some way to distinguish between objects constructed with the help of the Axiom of Choice and the ones with explicit constructions.

Classical results from descriptive set theory provide an answer to this question by showing that certain pathological objects cannot be defined by simple formulas in second-order arithmetic.

In second-order arithmetic, we work in the two-sorted structure

$$\mathcal{A}^2 \;=\; \langle \mathbb{N}, \mathcal{P}(\mathbb{N}), \in, +, \cdot, exp, 0, 1\rangle.$$

We identify $\mathbb{Q}$ with a subset of $\mathbb{N}$ by coding triples of natural numbers into natural numbers.

We view $\mathbb{R}$ as a subset of $\mathcal{P}(\mathbb{N})$ by considering Dedekind cuts in $\mathbb{Q}$.

These identifications allow us to view products of $\mathbb{R}$ as subsets of products of $\mathcal{P}(\mathbb{N})$ and we can therefore talk about the definability of subsets of $\mathbb{R}^k$ in the structure $\mathcal{A}^2$.

We measure the complexity of such definable subsets by the number of second-order quantifiers in the defining formulas.

A formula in the language $\mathcal{L}_2$ of second-order arithmetic is a Σ^1_0-formula, if it is contained in the smallest class of formulas that contains all atomic formulas and is closed under $\neg$, $\wedge$, $\vee$, $\forall n \in \mathbb{N}$ and $\exists n \in \mathbb{N}$.

Given $n \in \mathbb{N}$, an $\mathcal{L}_2$-formula is a Σ^1_{n+1}-formula, if it is contained in the smallest class of formulas that contains all negations of Σ^1_n-formulas and is closed under $\wedge$, $\vee$, $\forall n \in \mathbb{N}$, $\exists n \in \mathbb{N}$ and $\exists x \in \mathbb{R}$.

Given $n \in \mathbb{N}$, a subset X of $\mathbb{R}^k$ is a $\boldsymbol{\Sigma}^1_n$-*definable* if there is a Σ^1_n-formula with parameters that defines X in $\mathcal{A}^2$.

Given $n \in \mathbb{N}$, a subset X of $\mathbb{R}^k$ is a $\boldsymbol{\Delta}^1_n$-*definable* if both X and $\mathbb{R}^k \setminus X$ are $\boldsymbol{\Sigma}^1_n$-definable.

The above hierarchy of complexities can also be defined topologically.

Theorem (Lusin)

A subset of $\mathbb{R}^k$ is Borel if and only if it is $\mathbf{\Delta}^1_1$-definable.

Proposition

- *A subset of $\mathbb{R}^k$ is $\mathbf{\Sigma}^1_1$-definable if and only if it is the projection of a Borel subset of $\mathbb{R}^{k+1}$.*
- *A subset of $\mathbb{R}^k$ is $\mathbf{\Sigma}^1_{n+1}$-definable if and only if it is the projection of the complement of a $\mathbf{\Sigma}^1_n$-definable subset of $\mathbb{R}^{k+1}$.*

Proposition

If a function $f : \mathbb{R}^k \longrightarrow \mathbb{R}^l$ is $\boldsymbol{\Sigma}^1_n$-definable as a subset of $\mathbb{R}^{k+l}$, then f is already $\boldsymbol{\Delta}^1_n$-definable.

Corollary

The maps $\mathrm{id}_{\mathbb{C}}$ and $k_{\mathbb{C}}$ are the only $\boldsymbol{\Sigma}^1_1$-definable field automorphisms of $\mathbb{C}$.

Proof.

Assume that $\pi \in \mathrm{Aut}(\mathbb{C})$ is $\boldsymbol{\Sigma}^1_1$-definable. By the above remarks, π is a Borel subset of $\mathbb{C} \times \mathbb{C}$. But then π is a *Borel-measurable function* (i.e. the π-preimages of open sets are Borel) and hence π is Baire-measurable. $\square$

We now consider the next level in the above complexity hierarchy.

Theorem (Gödel)

In Gödel's constructible universe L, the canonical well-ordering of $\mathbb{R}$ has the order-type ω_1 and the collection of codes for initial segments of this well-ordering is $\mathbf{\Sigma}^1_2$-definable.

This well-ordering of the $\mathbb{R}$ can be used in the construction of automorphisms through transcendence bases to obtain a simply definable discontinuous automorphism.

Corollary

In L, there is a discontinuous field automorphism of $\mathbb{C}$ that is $\mathbf{\Sigma}^1_2$-definable.

In contrast, it is also possible that such simply definable automorphisms do not exist.

Theorem

After forcing with the partial order that adds $\aleph_1$-many Cohen reals to the ground model, every $\boldsymbol{\Delta}^1_2$-definable subset of $\mathbb{R}^k$ has the Baire property.

Corollary

After forcing with the partial order that adds $\aleph_1$-many Cohen reals to the ground model, the maps $\mathrm{id}_{\mathbb{C}}$ and $k_{\mathbb{C}}$ are the only $\boldsymbol{\Sigma}^1_2$-definable field automorphisms of $\mathbb{C}$.

Large cardinal axioms

The above results show that the existence of a $\mathbf{\Sigma}^1_2$-definable discontinuous field automorphism of $\mathbb{C}$ is independent of the axioms of $\mathbf{ZFC}$.

The techniques introduced by Gödel and Cohen that were applied in the above results can be used to show that various mathematical statements are not decided by $\mathbf{ZFC}$.

The discovery of various independences initiated the programme to search for intrinsically justified extensions of these axioms that settle important mathematical questions left open by the axioms of $\mathbf{ZFC}$.

Among the axiom systems studied in the course of this programme, *strong axioms of infinities*, or *large cardinal assumptions*, play an outstanding role.

Originating from the work of Hausdorff on *cardinal arithmetics* and the work of Ulam on the *measure problem*, these axioms postulate the existence of cardinal numbers having certain properties that make them very large, and whose existence cannot be proved in $\mathbf{ZFC}$, because it implies the consistency of $\mathbf{ZFC}$ itself.

Example

A cardinal is (strongly) *inaccessible*, if it is an uncountable regular strong limit cardinal.

Example

An uncountable cardinal κ is *measurable* if there exists a κ-complete ultrafilter over κ.

Woodin cardinals are a strengthening of the property of being an inaccessible limit of measurable cardinals.

The special role of these axioms arises from two empirical facts:

- First, there is strong evidence that for every extension of $\mathbf{ZFC}$, the consistency of the given theory is either equivalent to the consistency of $\mathbf{ZFC}$, or to the consistency of some extension of $\mathbf{ZFC}$ by strong axioms of infinity.
- Second, all large cardinal axioms studied so far are linearly ordered by their consistency strength, i.e. given two such axioms, either the consistency of one of the axioms was derived from the other axiom or the consistency of both axioms was shown to be equivalent.

In combination, these two phenomena allow for an ordering of all extensions of $\mathbf{ZFC}$ (and therefore of all mathematical statements!) in a linear hierarchy based on their consistency strength.

In addition to this fundamental role in the study of mathematical theories, deep results have shown that strong axioms of infinity themselves answer many important questions left open by $\mathbf{ZFC}$ in a desirable way.

This leads many set theorists to think that these axioms should be included in the correct axiomatization of set theory, and therefore of mathematics.

In particular, seminal work of Martin, Shelah, Steel and Woodin who showed that large cardinals determine second-order arithmetics (and therefore of large parts of mathematics), in the sense that they provide a strong structure theory for definable sets of real numbers that is immune to independence phenomena.

Theorem (Solovay)

If there is a measurable cardinal, then all $\boldsymbol{\Sigma}^1_2$*-definable subsets of* $\mathbb{R}^k$ *have the Baire property.*

Theorem (Shelah–Woodin, Martin–Steel)

If there are n *Woodin cardinals below a measurable cardinal, then all* $\boldsymbol{\Sigma}^1_{n+2}$*-definable subsets of* $\mathbb{R}^k$ *have the Baire property.*

Corollary

If there are infinitely many Woodin cardinals, then the maps $\mathrm{id}_{\mathbb{C}}$ *and* $k_{\mathbb{C}}$ *are the only field automorphisms of* $\mathbb{C}$ *that are definable in* $\mathcal{A}^2$*.*

Set-theoretic definability

I now want to present results dealing with the *set-theoretic definability* of objects obtained from the Axiom of Choice, i.e. with the question whether such objects can be defined in the structure $\langle \mathrm{V}, \in \rangle$ using simple formulas.

We start by making the notion of *simple formulas* more precise.

First, we will restrict ourselves to formulas that only use cardinals and sets of small hereditary cardinality as parameters.

Next, we measure the complexity of formulas using the *Levy hierarchy*.

A formula in the language $\mathcal{L}_\in = \{\in\}$ of set theory is a Σ_0-*formula* if it is contained in the smallest collection of $\mathcal{L}_\in$-formulas that contains all atomic formulas and is closed under $\neg$, $\wedge$, $\vee$, $\forall x \in y$ and $\exists x \in y$.

Example

The sets $\mathbb{N}$ and $\{\mathbb{N}\}$ can be defined by a Σ_0-formulas without parameters.

Given $n \in \mathbb{N}$, an $\mathcal{L}_\in$-formula is a Σ_{n+1}-*formula* if it is contained in the smallest collection of $\mathcal{L}_\in$-formulas that contains all negations of Σ_n-formulas and is closed under $\wedge$, $\vee$, $\forall x \in y$ and $\exists x$.

Example

- The set ω_1 of all countable ordinals can be defined by a Σ_1-formula without parameters.
- The set $\{\omega_1\}$ cannot be defined by a Σ_1-formula with hereditary countable parameters.

Standard arguments show that complicated objects (like discontinuous automorphisms of $\mathbb{C}$) cannot be defined by Σ_0-formulas with parameters of small hereditary cardinalities.

In contrast, the question whether such objects can be defined by Σ_2-formulas without parameters is often independent of the axioms of $\mathbf{ZFC}$ together with large cardinal assumptions.

Therefore, we will focus on Σ_1-definitions of such objects.

Lemma

The following statements are equivalent for every subset X of $\mathbb{R}$:

- X *is* $\boldsymbol{\Sigma}^1_2$*-definable.*
- X *is definable by a* Σ_1*-formula with parameters in* $\mathrm{H}(\omega_1)$*.*

Corollary

If there is a measurable cardinal, then the maps $\mathrm{id}_{\mathbb{C}}$ *and* $k_{\mathbb{C}}$ *are the only field automorphisms of* $\mathbb{C}$ *that are definable by a* Σ_1*-formula with parameters in* $\mathrm{H}(\omega_1)$*.*

The next result extends the above characterization of $\mathbf{\Sigma}^1_2$-sets to the next complexity class.

Theorem (L.–Schindler–Schlicht)

If there is a measurable cardinal above a Woodin cardinal, then the following statements are equivalent for every subset X of $\mathbb{R}$:

- *X is $\mathbf{\Sigma}^1_3$-definable.*
- *X is definable by a Σ_1-formula with parameters in $\mathrm{H}(\omega_1) \cup \{\omega_1\}$.*

Corollary

If there is a measurable cardinal above a Woodin cardinal, then the maps $\mathrm{id}_{\mathbb{C}}$ and $k_{\mathbb{C}}$ are the only field automorphisms of $\mathbb{C}$ that are definable by a Σ_1-formula with parameters in $\mathrm{H}(\omega_1) \cup \{\omega_1\}$.

In contrast, the nature of Σ_1-definability completely changes when we allow the second uncountable cardinal ω_2 as a parameter.

Theorem (L.)

The existence of a well-order $\lhd$ of the reals with the property that the set of all initial segments of $\lhd$ is definable by a Σ_1-formula with parameter ω_2 is consistent with all large cardinal assumptions.

Corollary

The existence of a discontinuous field automorphism of $\mathbb{C}$ that is definable by a Σ_1-formula with parameter ω_2 is consistent with all large cardinal assumptions.

Thank you for listening!