DEFINABLE BI-STATIONARY SETS

Philipp Moritz Lücke

Mathematisches Institut
Rheinische Friedrich-Wilhelms-Universität Bonn
http://www.math.uni-bonn.de/people/pluecke/

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Introduction

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Moreover, in many important cases, it can be shown that the existence of these objects is not a consequence of the remaining axioms of set theory.

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- Non-Lebesgue measurable sets of real numbers.
- $\mathbb{Q}$-bases of the real numbers.
- Well-orderings of power sets of infinite cardinals.
- Bi-stationary (i.e. stationary and co-stationary) subsets of uncountable regular cardinals.
- Colourings witnessing failures of weak compactness at accessible cardinals.

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For paradoxical sets consisting of real numbers, classical results from descriptive set theory show that these objects cannot be defined by simple formulas in second-order arithmetic.

Moreover, both strong large cardinal assumptions and forcing axioms imply that this implication can be extended to arbitrary formulas.

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In many important cases, these objects can be identified with subsets of *higher Baire spaces*, canonical generalizations of the space of real numbers to larger cardinalities, and can be studied using results from *generalized descriptive set theory*.

I will focus on the definability of bi-stationary subsets of uncountable regular cardinals.

Definition

A formula $\varphi(v_0, \dots, v_n)$ in the language $\mathcal{L}_\in = \{\epsilon\}$ of set theory and parameters $z_0, \dots, z_{n-1}$ *define* a set X if

$$X \;=\; \{x \mid \varphi(x, z_0, \dots, z_{n-1})\}.$$

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The above restriction of parameters is supposed to exclude trivial definitions that use bi-stationary subsets as parameters.

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Moreover, an $\mathcal{L}_\in$-formula is a Σ_{n+1}-*formula* for some $n < \omega$ if it is of the form $\exists x \, \neg\varphi$ for some Σ_n-formula φ.

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Note that the class of all formulas that are $\mathbf{ZFC}$-provably equivalent to a Σ_{n+1}-formula is closed under existential quantification, bounded quantification, conjunctions and disjunctions.

The Σ_n-club property

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Given $n < \omega$, an uncountable regular cardinal κ has the $\boldsymbol{\Sigma}_n$*-club property* if for every bi-stationary subset E of κ, the set $\{E\}$ is not definable by a Σ_n-formula with parameters in $\mathrm{H}(\kappa) \cup \{\kappa\}$.

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Lemma

Every uncountable regular cardinal has the $\mathbf{\Sigma}_0$*-club property.*

In contrast, there are many cardinals without the $\mathbf{\Sigma}_1$-club property.

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- *If ν is an uncountable cardinal, then ν^+ does not have the $\mathbf{\Sigma}_1$-club property.*
- *A cardinal has the $\mathbf{\Sigma}_2$-club property if and only if it has the $\mathbf{\Sigma}_n$-club property for all $n < \omega$.*

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- *If ν is an uncountable cardinal, then ν^+ does not have the $\mathbf{\Sigma}_1$-club property.*
- *A cardinal has the $\mathbf{\Sigma}_2$-club property if and only if it has the $\mathbf{\Sigma}_n$-club property for all $n < \omega$.*
- *Cardinals greater than ω_1 do not have the $\mathbf{\Sigma}_2$-club property.*

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Theorem (L.)

If κ is an inaccessible cardinal with the $\mathbf{\Sigma}_1$-club property and G is either $\mathrm{Add}(\omega, \kappa)$- or $\mathrm{Col}(\omega, {<}\kappa)$-generic over V, then κ has the $\mathbf{\Sigma}_1$-club property in $\mathrm{V}[G]$.

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If κ is an uncountable regular cardinal with the Σ_1-club property, then κ is an inaccessible cardinal with the Σ_1-club property in the Dodd-Jensen core model.

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Proposition

If $\mathrm{V} = \mathrm{HOD}$ *holds, then* ω_1 *does not have the* $\mathbf{\Sigma}_2$*-club property.*

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Successors of singular cardinals

Given a singular cardinal λ, the above results show that the cardinal λ^+ does not have the $\mathbf{\Sigma}_1$-club property.

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This failure is witnessed by the set $S^{\lambda^+}_\omega$ consisting of all elements of λ^+ of countable cofinality.

The set $\{S^{\lambda^+}_\omega\}$ can be defined by a Σ_1-formula that uses λ^+ and the set of all regular cardinals less than λ as parameters.

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- *There is a measurable cardinal.*

The main argument in proof of the above result relies on the *Covering Lemma* for the *Dodd–Jensen core model* and the representation of this model as the union of all *lower parts of mice*.

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- *There are infinitely many measurable cardinals.*

Open Questions

Question

Given a singular cardinal λ, is there a bi-stationary subset E of λ^+ with the property that the set $\{E\}$ is definable by a Σ_1-formula with parameters λ and λ^+?

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Let λ be a singular cardinal with the property that for every bi-stationary subset E of λ^+, the set $\{E\}$ is not definable by a Σ_1-formula with parameter λ^+.

Is there an inner model with a measurable cardinal?

Question

Given a cardinal $\kappa > \omega_1$, is there an uncountable regular cardinal $\delta < \kappa$ with the property that the set $\{\delta\}$ is definable by a Σ_1-formula with parameter κ?

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Question

Is it consistent that the set $\{\omega_1\}$ is not definable by a Σ_1-formula with parameter ω_ω?

Note that if ω_ω is Rowbottom, then the set $\{\omega_1\}$ is not definable by a Σ_1-formula with parameter ω_ω.

Thank you for listening!