Generalized Baire spaces and closed Maximality Principles

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Σ^1_1-subsets of generalized Baire spaces

Throughout this talk, we let κ denote an uncountable cardinal satisfying $\kappa = \kappa^{<\kappa}$.

The *generalized Baire space of* κ is the set ${}^\kappa\kappa$ of all functions from κ to κ equipped with the topology whose basic open sets are of the form

$$N_s \ = \ \{x \in {}^\kappa\kappa \mid s \subseteq x\}$$

for some s contained in the set ${}^{<\kappa}\kappa$ of all functions of the form $t : \alpha \longrightarrow \kappa$ with $\alpha < \kappa$.

We are interested in the definable subsets of this space and their structural properties.

A subset A of ${}^{\kappa}\kappa$ is closed in the above topology if and only if there is a subtree T of ${}^{<\kappa}\kappa$ such that A is equal to the set $[T]$ of all cofinal branches through T.

We generalize notions of complexity from the classical Baire space ${}^{\omega}\omega$ to our uncountable setting.

A subset of ${}^{\kappa}\kappa$ is a κ-*Borel set* if it is contained in the smallest algebra of sets that contains all open subsets of ${}^{\kappa}\kappa$ and is closed under κ-unions.

A subset of ${}^{\kappa}\kappa$ is a $\mathbf{\Sigma}^1_1$-*set* if it is equal to the projection of a closed subset of ${}^{\kappa}\kappa \times {}^{\kappa}\kappa$.

The following folklore result shows that the class of $\mathbf{\Sigma}^1_1$-sets contains many interesting objects.

Proposition

As subset of ${}^\kappa\kappa$ is a $\mathbf{\Sigma}^1_1$-set if and only if it is definable over the structure $\langle \mathrm{H}(\kappa^+), \in\rangle$ by a Σ_1-formula with parameters.

This result can be used to show that the structural properties of the above classes differ from the properties of their counterparts in the countable setting.

Corollary

There is a $\mathbf{\Delta}^1_1$-subset of ${}^\kappa\kappa$ that is not κ-Borel.

We present more examples of results that emphasize differences to the classical setting.

Theorem (Halko-Shelah)

There are disjoint $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa$ that cannot be separated by a κ-Borel set.

Theorem (L.-Schlicht)

There is a closed subset of ${}^\kappa\kappa$ that is not a continuous image of ${}^\kappa\kappa$.

Theorem (L.-Schlicht)

There is a sequence $\langle i_\gamma : {}^\kappa\kappa \longrightarrow {}^\kappa\kappa \mid \gamma < 2^\kappa\rangle$ of continuous injections such that for all $\gamma < \delta < 2^\kappa$, the sets $\mathrm{ran}(i_\gamma)$ and $\mathrm{ran}(i_\delta)$ are disjoint and cannot be separated by a κ-Borel subset of ${}^\kappa\kappa$.

In particular, there is a continuous injection $i : {}^\kappa\kappa \longrightarrow {}^\kappa\kappa$ such that $\mathrm{ran}(i)$ is not a κ-Borel subset of ${}^\kappa\kappa$.

Moreover, it is well-known that many basic structural questions about the class of $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$ are not settled by the axioms of $\mathbf{ZFC}$ together with large cardinal axioms.

In the following, we discuss four examples of such questions that motivated the work presented in this talk.

Separating the club filter from the nonstationary ideal

Given $S \subseteq \kappa$, we define

$$\mathrm{Club}(S) = \{x \in {}^{\kappa}\kappa \mid \exists C \subseteq \kappa \textit{ club } \forall \alpha \in C \cap S \ \ x(\alpha) > 0\}$$

and

$$\mathrm{NStat}(S) = \{x \in {}^{\kappa}\kappa \mid \exists C \subseteq \kappa \textit{ club } \forall \alpha \in C \cap S \ \ x(\alpha) = 0\}.$$

Then the *club filter* $\mathrm{Club}(\kappa)$ and the *non-stationary ideal* $\mathrm{NStat}(\kappa)$ are disjoint $\boldsymbol{\Sigma}^1_1$-subsets of ${}^{\kappa}\kappa$.

The proof of the Halko-Shelah result mentioned above shows that these sets cannot be separated by a κ-Borel subset of ${}^{\kappa}\kappa$.

This motivates the following question.

Question

Is there a $\mathbf{\Delta}^1_1$-subset of ${}^{\kappa}\kappa$ that separates $\mathrm{Club}(\kappa)$ and $\mathrm{NStat}(\kappa)$?

If S is a stationary subset of κ, then $\mathsf{Club}(S)$ is a $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa$ that separates $\mathsf{Club}(\kappa)$ from $\mathsf{NStat}(\kappa)$.

The following result builds upon results of Mekler-Shelah and Hyttinen-Rautila on the existence of κ-*Canary trees*. It shows that, for many regular cardinals, it is possible to force the existence of $\mathbf{\Delta}^1_1$-definable sets of the above form.

Theorem (Friedman-Hyttinen-Kulikov)

Assume that GCH *holds and* κ *is not the successor of a singular cardinal. Then* $\mathsf{Club}(S^\kappa_\omega)$ *is a* $\mathbf{\Delta}^1_1$*-subset of* ${}^\kappa\kappa$ *in a cofinality preserving forcing extension of the ground model.*

This shows that a positive answer to the above question is consistent for such cardinals.

In contrast, it is possible to combine results of Halko-Shelah and Friedman-Hyttinen-Kulikov (or L.-Schlicht) to show that a negative answer to the above question is also consistent.

Theorem

If G is $\mathrm{Add}(\kappa, \kappa^+)$-generic over V, then there is no $\mathbf{\Delta}^1_1$-subset A of ${}^{\kappa}\kappa$ that separates Club(κ) *from* NStat(κ) *in $\mathrm{V}[G]$.*

Lengths of $\mathbf{\Sigma}^1_1$-definable well-orders

We call well-order $\langle A, \lessdot \rangle$ a $\mathbf{\Sigma}^1_1$-*well-ordering of a subset of* ${}^\kappa\kappa$ if $\lessdot$ is a $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa \times {}^\kappa\kappa$.

It is easy to see that for every $\alpha < \kappa^+$, there is a $\mathbf{\Sigma}^1_1$-well-ordering of a subset of ${}^\kappa\kappa$ of order-type α.

Moreover, if there is an $x \subseteq \kappa$ such that κ^+ is not inaccessible in $\mathrm{L}[x]$, then there is a $\mathbf{\Sigma}^1_1$-well-ordering of a subset of ${}^\kappa\kappa$ of order-type κ^+.

The following question is motivated by the classical *Kunen-Martin Theorem*.

Question

What is the least upper bound for the order-types of $\mathbf{\Sigma}^1_1$-well-orderings of subsets of ${}^\kappa\kappa$?

With the help of generic coding techniques, it is possible to force the existence of long $\mathbf{\Sigma}^1_1$-well-orderings.

Theorem (Holy-L.)

There is a cofinality preserving forcing extension of the ground model that contains a $\mathbf{\Sigma}^1_1$-definable well-ordering of ${}^\kappa\kappa$.

Moreover, these techniques allow us to make arbitrary subsets of ${}^\kappa\kappa$ contained in the ground model $\mathbf{\Sigma}^1_1$-definable in a cofinality preserving forcing extension. This yields the following result.

Theorem

Given $\alpha < (2^\kappa)^+$, there is a cofinality preserving forcing extension $\mathrm{V}[G]$ of the ground model V such that $(2^\kappa)^{\mathrm{V}} = (2^\kappa)^{\mathrm{V}[G]}$ and $\mathrm{V}[G]$ contains a $\mathbf{\Sigma}^1_1$-well-orderings of subsets of ${}^\kappa\kappa$ of order-type α.

In the other direction, both κ^+ and $2^\kappa > \kappa^+$ can consistently be upper bounds for the lengths of such well-orders.

Theorem

Let $\nu > \kappa$ be a cardinal, G be $\mathrm{Add}(\kappa, \nu)$-generic over V and $\langle A, \lessdot\rangle$ be a $\mathbf{\Sigma}^1_1$-well-ordering of a subset of ${}^\kappa\kappa$ in $\mathrm{V}[G]$. Then $A \neq ({}^\kappa\kappa)^{\mathrm{V}[G]}$ and the order-type of $\langle A, \lessdot\rangle$ has cardinality at most $(2^\kappa)^{\mathrm{V}}$ in $\mathrm{V}[G]$.

Theorem

If $\nu > \kappa$ is inaccessible, $G \times H$ is $(\mathrm{Col}(\kappa, {<}\nu) \times \mathrm{Add}(\kappa, \nu))$-generic over V and $\langle A, \lessdot\rangle$ is a $\mathbf{\Sigma}^1_1$-well-ordering of a subset of ${}^\kappa\kappa$ in $\mathrm{V}[G, H]$, then A has cardinality κ in $\mathrm{V}[G, H]$.

Note that the conclusion of the last theorem implies that κ^+ is inaccessible in $\mathrm{L}[x]$ for every $x \subseteq \kappa$.

The Hurewicz dichotomy

Classical results of Hurewicz, Kechris and Saint-Raymond show that a $\mathbf{\Sigma}^1_1$-subset of a Polish space is either contained in a countable union of compact subsets or contains a closed subset homeomorphic to ${}^\omega\omega$.

We consider generalizations of this dichotomy to subsets of ${}^\kappa\kappa$ that replace the use of countable unions compact subsets by κ-unions of κ-compact subsets (i.e. subsets A with the property that every open cover of A contains a subcover of cardinality less than κ).

Definition

We say that a subset A of ${}^\kappa\kappa$ satisfies the *Hurewicz dichotomy* if either A is contained in the union of κ-many κ-compact subsets of ${}^\kappa\kappa$, or A contains a closed subset of ${}^\kappa\kappa$ homeomorphic to ${}^\kappa\kappa$.

Note that the two alternatives in the above definition are mutually exclusive.

Question

Does every $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa$ satisfy the *Hurewicz dichotomy*?

Theorem (L.-Motto Ros-Schlicht)

Every model of $\mathbf{ZFC} + \mathrm{GCH}$ *has a cofinality, large cardinals and* GCH *preserving class forcing extension with the property that for every uncountable regular cardinal* ν*, every* $\mathbf{\Sigma}^1_1$*-subset of* ${}^\nu\nu$ *satisfies the Hurewicz dichotomy.*

Theorem (L.-Motto Ros-Schlicht)

Assume that either $\mathrm{V{=}L}$ *or* V *is an* $\mathrm{Add}(\omega, 1)$*-generic extension of a ground model. Then there is a closed subset of* ${}^\kappa\kappa$ *that does not satisfy the Hurewicz dichotomy.*

The bounding and dominating number of $\langle \mathcal{TO}_\kappa, \leq \rangle$

Let $\mathcal{T}_\kappa$ denote the class of all trees of cardinality and height κ and $\mathcal{TO}_\kappa$ denote the class of all trees in $\mathcal{T}_\kappa$ without a branch of length κ.

Given $\mathbb{T}_0, \mathbb{T}_1 \in \mathcal{T}_\kappa$, we write $\mathbb{T}_0 \leq \mathbb{T}_1$ if there is a function $f : \mathbb{T}_0 \longrightarrow \mathbb{T}_1$ such that $f(s) <_{\mathbb{T}_1} f(t)$ holds for all $s, t \in \mathbb{T}_0$ with $s <_{\mathbb{T}_0} t$.

The elements of the resulting partial order $\langle \mathcal{TO}_\kappa, \leq \rangle$ can be viewed as generalizations of countable ordinals.

We can identify $\mathcal{TO}_\kappa$ with a $\mathbf{\Pi}^1_1$-subset of ${}^\kappa\kappa$ and the ordering $\leq$ with a $\mathbf{\Sigma}^1_1$-definable relation on this set.

We are interested in the order-theoretic properties of the resulting partial order $\langle \mathcal{TO}_\kappa, \leq \rangle$.

More specifically, we interested in the value of the following cardinal characteristics.

- The *bounding number of* $\langle \mathcal{TO}_\kappa, \leq \rangle$ is the smallest cardinal $\mathfrak{b}_{\mathcal{TO}_\kappa}$ with the property that there is a $U \subseteq \mathcal{TO}_\kappa$ of this cardinality such that there is no tree $\mathbb{T} \in \mathcal{TO}_\kappa$ with $\mathbb{S} \leq \mathbb{T}$ for all $\mathbb{S} \in U$.
- The *dominating number of* $\langle \mathcal{TO}_\kappa, \leq \rangle$ is the smallest cardinal $\mathfrak{d}_{\mathcal{TO}_\kappa}$ with the property that there is a subset $D \subseteq \mathcal{TO}_\kappa$ of this cardinality such that for every $\mathbb{S} \in \mathcal{TO}_\kappa$ there is a $\mathbb{T} \in D$ with $\mathbb{S} \leq \mathbb{T}$.

It is easy to see that

$$\kappa^+ \leq \mathfrak{b}_{\mathcal{TO}_\kappa} \leq \mathfrak{d}_{\mathcal{TO}_\kappa} \leq 2^\kappa$$

holds. In particular, $2^\kappa = \kappa^+$ implies that these cardinal characteristics are equal. We may therefore ask if this is always the case.

Question

Is $\mathfrak{b}_{\mathcal{TO}_\kappa}$ equal to $\mathfrak{d}_{\mathcal{TO}_\kappa}$?

With the help of κ-Cohen forcing, it is possible to show that a negative answer to this question is also consistent.

Theorem

If G is $\mathrm{Add}(\kappa, (2^\kappa)^+)$-generic over V, then

$$\mathfrak{b}_{\mathcal{TO}_\kappa}^{\mathrm{V}[G]} \leq (2^\kappa)^{\mathrm{V}} < (2^\kappa)^{\mathrm{V}[G]} = \mathfrak{d}_{\mathcal{TO}_\kappa}^{\mathrm{V}[G]}.$$

The results presented above show that there are many interesting questions about $\mathbf{\Sigma}^1_1$-subsets that are not settled by the axioms of $\mathbf{ZFC}$ together with large cardinal axioms. In particular, these axioms do not provide a rich structure theory for the class of $\mathbf{\Sigma}^1_1$-sets.

This observation leads us to the following question.

Question

Are there natural extensions of $\mathbf{ZFC}$ that settle these questions by providing a strong structure theory for the class of $\mathbf{\Sigma}^1_1$-sets?

In the following, we will show that forcing axioms called *closed maximality principle* are examples of such extensions of $\mathbf{ZFC}$.

Closed maximality principles

We will present forcing axioms that are variations of the *maximality principles* introduced by Stavi-Väänänen and Hamkins.

We say that a sentence φ in the language of set theory is *forceably necessary* if there is a partial order $\mathbb{P}$ such that $\mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}} \Vdash \varphi$ holds for every $\mathbb{P}$-name $\dot{\mathbb{Q}}$ for a partial order.

Example

The sentence "$\omega_1 > \omega_1^{\mathrm{L}}$" is forceably necessary.

The *maximality principle for forcing* is the scheme of axioms stating that every forceably necessary sentence in the language of set theory is true.

This formulation is motivated by the *maximality principle*

$$\Diamond \Box \varphi \longrightarrow \varphi$$

of modal logic by interpreting the statement $\Diamond\varphi$ ("φ *is possible*") as "φ *holds in some forcing extension of the ground model*" and the statement $\Box\varphi$ ("φ *is necessary*") as "φ *holds in every forcing extension of the ground model*".

Following Fuchs, Leibman and Stavi-Väänänen, we will modify this principle in the following ways.

- By restricting the complexity of the considered formulas.
- By restricting the class of forcings that can be used to witness that a given statement is possible.
- By restricting the class of forcings that need to be considered in order to check that a given statement is necessary.
- By allowing statements containing parameters.

Definition

Let $\Phi(v_0, v_1)$ be a formula and z be a set.

- We say that a statement $\psi(x_0, \ldots, x_{n-1})$ is $\Phi(\cdot\,, z)$-*forceably necessary* if there is a partial order $\mathbb{P}$ with $\Phi(\mathbb{P}, z)$ and $\mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}} \Vdash \psi(\check{x}_0, \ldots, \check{x}_{n-1})$ for every $\mathbb{P}$-name $\dot{\mathbb{Q}}$ for a partial order with $\mathbb{1}_{\mathbb{P}} \Vdash \Phi(\dot{\mathbb{Q}}, \check{z})$.
- Given an infinite cardinal ν and $0 < n < \omega$, we let $\mathrm{MP}^{\Phi}_{n}(z, \nu)$ denote the statement that every $\Phi(\cdot\,, z)$-forceably necessary Σ_n-statement with parameters in $\mathrm{H}(\nu)$ is true.

Note that, with the help of a universal Σ_n-formula, the principle $\mathrm{MP}^{\Phi}_{n}(z, \nu)$ can be expressed by a single statement using the parameters ν and z.

Let $\Phi_{cl}(v_0, v_1)$ be the canonical formula defining the class of $<\kappa$-closed partial orders using the parameter κ.

We write $\mathrm{CMP}_n(\kappa)$ instead of $\mathrm{MP}_n^{\Phi_{cl}}(\kappa, \kappa^+)$.

The principles $\mathrm{CMP}_n(\kappa)$ were studied in depth by Gunter Fuchs.

The following remark shows that they may be viewed as strengthenings of the fact that Σ_1-statements with parameters in $\mathrm{H}(\kappa^+)$ are absolute with respect to $<\kappa$-closed forcings.

Proposition

The principle $\mathrm{CMP}_1(\kappa)$ *is true.*

The following result of Fuchs gives bounds for the consistency strength of these principles.

Remember that a cardinal δ is Σ_n-*reflecting* if it is inaccessible and $\langle \mathrm{V}_\delta, \in \rangle$ is a Σ_n-elementary submodel of $\langle \mathrm{V}, \in \rangle$.

Theorem (Fuchs)

Let $0 < n < \omega$.

- *If* $\delta > \kappa$ *is a* Σ_{n+2}*-reflecting cardinal and* G *is* $\mathrm{Col}(\kappa, \delta)$*-generic over* V*, then* $\mathrm{CMP}_n(\kappa)$ *holds in* $\mathrm{V}[G]$*.*
- *If* $\mathrm{CMP}_{n+1}(\kappa)$ *holds and* $\delta = \kappa^+$*, then* δ *is* Σ_{n+1}*-reflecting in* L*.*

The axiom $\mathrm{CMP}_2(\kappa)$ induces a strong structure theory for $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$. In particular, it settles the first three questions posed above.

Theorem

If $\mathrm{CMP}_2(\kappa)$ *holds, then there is no* $\mathbf{\Delta}^1_1$*-subset of* ${}^\kappa\kappa$ *that separates* $\mathsf{Club}(\kappa)$ *and* $\mathsf{NStat}(\kappa)$*.*

Sketch of the proof.

- *A subset of ${}^{\kappa}\kappa$ is κ-meager if it is contained in the union of κ-many nowhere dense subsets of ${}^{\kappa}\kappa$.*
- *A subset A of ${}^{\kappa}\kappa$ has the κ-Baire property if there is an open subset U of ${}^{\kappa}\kappa$ such that $A_{\Delta}U$ is κ-meager.*
- $\mathrm{CMP}_2(\kappa)$ *implies* $\mathbf{\Sigma}^1_2$*-absoluteness for* $<\kappa$*-closed forcings.*
- $\mathbf{\Sigma}^1_2$*-absoluteness for* $\mathrm{Add}(\kappa,1)$ *implies that all* $\mathbf{\Delta}^1_1$*-sets have the* κ*-Baire property.*
- *A subset A of ${}^{\kappa}\kappa$ is super-dense if $A \cap \bigcap_{\alpha<\kappa} U_\alpha \neq \emptyset$ whenever $\langle U_\alpha \mid \alpha < \kappa\rangle$ is a sequence of dense open subsets of some non-empty open subset U of ${}^{\kappa}\kappa$.*
- *Two disjoint super-dense subsets of ${}^{\kappa}\kappa$ cannot be separated by a subset of ${}^{\kappa}\kappa$ with the κ-Baire property.*
- *The sets* Club(κ) *and* NStat(κ) *are super-dense.*

Theorem

If $\mathrm{CMP}_2(\kappa)$ holds, then the least upper bound for the order-types of $\mathbf{\Sigma}^1_1$-well-orderings of subsets of ${}^\kappa\kappa$ is equal to κ^+.

Sketch of the proof.

- *If a $<\kappa$-closed forcing adds an element to a $\mathbf{\Sigma}^1_1$-set, then this set contains a closed subset homeomorphic to ${}^\kappa 2$.*
- *This shows that $\mathrm{CMP}_2(\kappa)$ implies that all $\mathbf{\Sigma}^1_1$-sets have the perfect set property, i.e. every such set either has cardinality at most κ or contains a closed subset homeomorphic to ${}^\kappa 2$.*
- *$\mathbf{\Sigma}^1_2$-absoluteness for $\mathrm{Add}(\kappa, 1)$ implies that the domains of $\mathbf{\Sigma}^1_1$-well-orderings of subsets of ${}^\kappa\kappa$ do not contain closed subset homeomorphic to ${}^\kappa 2$.*

Theorem

If $\mathrm{CMP}_2(\kappa)$ holds, then all $\boldsymbol{\Sigma}^1_1$-subsets of ${}^{\kappa}\kappa$ satisfy the Hurewicz dichotomy.

In contrast, such principles do not answer the fourth question.

To show this, we consider closed maximality principles for statements of arbitrary complexities.

Let $\mathcal{L}_{\in,\dot{\nu}}$ denote the language of set theory extended by an additional constant symbol $\dot{\nu}$.

Let REFL denote the $\mathcal{L}_{\in,\dot{\nu}}$-theory consisting of $\mathbf{ZFC}$ together with the scheme of $\mathcal{L}_{\in,\dot{\nu}}$-sentences stating that $\dot{\nu}$ is Σ_n-reflecting for all $0 < n < \omega$.

Let CMP denote the $\mathcal{L}_{\in,\dot{\nu}}$-theory consisting of $\mathbf{ZFC}$ together with the scheme of $\mathcal{L}_{\in,\dot{\nu}}$-sentences stating that $\mathrm{CMP}_n(\dot{\nu})$ holds for all $0 < n < \omega$.

Corollary (Fuchs)

- *Assume that* $\langle \mathrm{V}, \in, \delta \rangle$ *is a model of* REFL *with* $\delta > \kappa$. *If* G *is* $\mathrm{Col}(\kappa, \delta)$*-generic over* V*, then* $\langle \mathrm{V}[G], \in, \kappa \rangle$ *is a model of* CMP.
- *Assume that* $\langle \mathrm{V}, \in, \kappa \rangle$ *is a model of* CMP *and* $\delta = \kappa^+$. *Then* $\langle \mathrm{L}, \in, \delta \rangle$ *is a model of* REFL.

A result of Fuchs shows that $\langle \mathrm{V}[G, H], \in, \kappa \rangle$ is a model of CMP whenever $\langle \mathrm{V}, \in, \delta \rangle$ is a model of REFL with $\delta > \kappa$ and $G \times H$ is $(\mathrm{Col}(\kappa, \delta) \times \mathrm{Add}(\kappa, \delta^+))$-generic over V.

Together with the above result on the values of the cardinal characteristics in $\mathrm{Add}(\kappa, (2^\kappa)^+)$-generic extensions, this yields the following statement.

Theorem

If the theory CMP *is consistent, then it does not decide the statement* $\mathfrak{b}_{\mathcal{TO}_{\dot{\nu}}} = \mathfrak{d}_{\mathcal{TO}_{\dot{\nu}}}$.

Closed maximality principles with more parameters

The proof of the above negative result suggests that we consider maximality principles for statements containing parameters of higher cardinalities. To do so we have to restrict ourselves to forcings that preserve more cardinals.

Natural candidates are classes of all $<\kappa$-closed partial orders satisfying the κ^+-chain condition.

It turns out that such principles are connected to generalizations of classical forcing axioms to κ.

Given a partial order $\mathbb{P}$ and an infinite cardinal ν, we let $\mathsf{FA}_\nu(\mathbb{P})$ denote the statement that for every collection $\mathcal{D}$ of ν-many dense subsets of $\mathbb{P}$, there is a filter G on $\mathbb{P}$ that meets all elements of $\mathcal{D}$.

Proposition

Let $\Phi(v_0, v_1)$ be a formula, z be set and $\nu \geq \kappa$ be a cardinal.

- *If $\mathrm{MP}_1^\Phi(z, \nu^+)$ holds and $\mathbb{P}$ is a partial order of cardinality at most ν with $\Phi(\mathbb{P}, z)$, then $\mathsf{FA}_\nu(\mathbb{P})$ holds.*
- *Assume that every partial order $\mathbb{P}$ with $\Phi(\mathbb{P}, z)$ satisfies the κ^+-chain condition and $\mathsf{FA}_\nu(\mathbb{P})$ holds for all such $\mathbb{P}$. Then $\mathrm{MP}_1^\Phi(z, \nu^+)$ holds*

A result of Shelah shows that there is a $<\kappa$-closed partial order $\mathbb{P}$ of cardinality κ^+ satisfying the κ^+-chain condition such that $\mathsf{FA}_{\kappa^+}(\mathbb{P})$ fails.

Together with the above observation, this shows that we have to restrict our class of forcings even further to obtain a consistent maximality principle. In particular, $\mathsf{FA}_{\kappa^+}(\mathbb{P})$ should consistently hold for every partial orders $\mathbb{P}$ in this class.

An example of such a class can be found in Shelah's work on generalization of Martin's Axiom to higher cardinalities and work of Stavi-Väänänen on modified maximality principles.

We say that a partial order $\mathbb{P}$ has the property $\mathcal{S}(\kappa)$ if for every sequence $\langle p_\gamma \mid \gamma < \kappa^+ \rangle$ of conditions in $\mathbb{P}$, there is a club C in κ^+ and a regressive function $r : \kappa^+ \longrightarrow \kappa^+$ such that $\inf_{\mathbb{P}}\{p_\gamma, p_\delta\}$ exists whenever $\gamma, \delta \in C \cap S^{\kappa^+}_\kappa$ with $r(\gamma) = r(\delta)$.

$\mathrm{GMA}(\kappa)$ is the assumption that $\mathsf{FA}_\nu(\mathbb{P})$ holds for all $\nu < 2^\kappa$ and every $<\kappa$-closed partial order $\mathbb{P}$ with property $\mathcal{S}(\kappa)$.

Let $\Phi_{\mathcal{S}}(v_0, v_1)$ be the canonical formula that defines the class of all $<\kappa$-closed partial orders with the property $\mathcal{S}(\kappa)$ using the parameter κ.

We study the maximality principles $\mathrm{MP}^{\Phi_{\mathcal{S}}}_n(\kappa, 2^\kappa)$ associated to this class. We abbreviate these principles by $\mathrm{SMP}_n(\kappa)$.

These principles may be viewed as strengthenings of $\mathrm{GMA}(\kappa)$.

Proposition

If $\nu^{<\kappa} < 2^\kappa$ for all $\nu < 2^\kappa$, then the principles $\mathrm{GMA}(\kappa)$ and $\mathrm{SMP}_1(\kappa)$ are equivalent.

Theorem

If $\mathrm{SMP}_2(\kappa)$ holds, then 2^κ is a weakly inaccessible cardinal and $\nu^{<\kappa} < 2^\kappa$ holds for all $\nu < 2^\kappa$.

Corollary

$\mathrm{SMP}_2(\kappa)$ *implies* $\mathrm{GMA}(\kappa)$.

The consistency strength of this principle can be bounded in similar way as for the principles discussed in the last section.

Theorem

Let $0 < n < \omega$.

- *Given an inaccessible cardinal* $\delta > \kappa$, *there is a partial order* $\mathbb{S}(\kappa, \delta)$ *that is uniformly definable in parameters* κ *and* δ *with the property that, if* δ *is* Σ_{n+2}*-reflecting, then* $\mathrm{SMP}_n(\kappa)$ *holds in every* $\mathbb{S}(\kappa, \delta)$*-generic extension of the ground model* V.
- *If* $\mathrm{SMP}_{n+1}(\kappa)$ *holds and* $\delta = 2^\kappa$, *then* δ *is* Σ_{n+1}*-reflecting in* L.

The axiom $\mathrm{SMP}_2(\kappa)$ provides a strong structure theory for $\mathbf{\Sigma}^1_1$-sets that settles all of the above questions.

Theorem

If $\mathrm{SMP}_2(\kappa)$ holds, then there is no $\mathbf{\Delta}^1_1$-subset of ${}^\kappa 2$ that separates Club(κ) *and* NStat(κ)*.*

Sketch of the proof.

- $\mathrm{SMP}_2(\kappa)$ *implies $\mathbf{\Sigma}^1_2$-absoluteness for* $\mathrm{Add}(\kappa, 1)$*.*
- $\mathbf{\Sigma}^1_2$*-absoluteness for* $\mathrm{Add}(\kappa, 1)$ *implies that all $\mathbf{\Delta}^1_1$-sets have the κ-Baire property.*
- *No set with the κ-Baire property separates* Club(κ) *and* NStat(κ)*.*

Theorem

If $\mathrm{SMP}_2(\kappa)$ holds, then the least upper bound for the order-types of $\mathbf{\Sigma}^1_1$-well-orderings of subsets of ${}^\kappa\kappa$ is equal to 2^κ and every $\gamma < 2^\kappa$ is equal to the order-type of such a well-ordering.

Sketch of the proof.

- *If a $<\kappa$-closed forcing adds an element to a $\mathbf{\Sigma}^1_1$-set, then this set contains a closed subset homeomorphic to ${}^\kappa 2$.*
- *This shows that $\mathrm{SMP}_2(\kappa)$ implies that every $\mathbf{\Sigma}^1_1$-set of cardinality 2^κ contains a closed subset homeomorphic to ${}^\kappa 2$.*
- *$\mathbf{\Sigma}^1_2$-absoluteness for $\mathrm{Add}(\kappa, 1)$ implies that the domains of $\mathbf{\Sigma}^1_1$-well-orderings of subsets of ${}^\kappa\kappa$ do not contain closed subsets homeomorphic to ${}^\kappa 2$.*
- *Using almost disjoint coding forcing at κ, it can be seen that $\mathrm{SMP}_2(\kappa)$ implies that every subset of ${}^\kappa\kappa$ of cardinality less than 2^κ is equal to the union of κ-many closed subsets of ${}^\kappa\kappa$.*

Theorem

If $\mathrm{SMP}_2(\kappa)$ holds, then every $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa$ satisfies the Hurewicz dichotomy.

Theorem

If $\mathrm{SMP}_2(\kappa)$ holds, then $\mathfrak{b}_{\mathcal{TO}_\kappa} = \mathfrak{d}_{\mathcal{TO}_\kappa} = 2^\kappa$.

Sketch of the proof.

- $\mathrm{SMP}_2(\kappa)$ *implies that every subset of ${}^\kappa\kappa$ of cardinality less than 2^κ is a $\mathbf{\Sigma}^1_1$-set.*
- *A result of Mekler-Väänänen (Boundedness Lemma for $\mathcal{TO}_\kappa$) shows that for every $\mathbf{\Sigma}^1_1$-subset A of $\mathcal{TO}_\kappa$ there is a $\mathbb{T} \in \mathcal{TO}_\kappa$ with $\mathbb{S} \leq \mathbb{T}$ for all $\mathbb{S} \in A$.*
- *Together, this shows that $\mathrm{SMP}_2(\kappa)$ implies that $\mathfrak{b}_{\mathcal{TO}_\kappa} = 2^\kappa$.*

Further results and open questions

The above results show that the axioms $\mathrm{CMP}_2(\kappa)$ and $\mathrm{SMP}_2(\kappa)$ decide the least upper bounds for the lengths of $\mathbf{\Sigma}^1_1$-definable well-orders.

Motivated by the results of classical descriptive set theory, it is natural to ask the same question for prewell-orders.

Question

Is the least upper bound of the lengths of $\mathbf{\Delta}^1_1$-prewell-orders on subsets of ${}^\kappa\kappa$ determined by axioms of the form $\mathrm{CMP}_n(\kappa)$ or $\mathrm{SMP}_n(\kappa)$?

There are bigger classes of $<\kappa$-closed partial orders satisfying the κ^+-chain condition such that the corresponding maximality principle is consistent.

In the light of classical forcing axioms, it is natural to ask the following question.

Question

Is it possible to classify the classes of $<\kappa$-closed partial orders satisfying the κ^+-chain condition with the property that consistently this class consists of all such partial orders such that $\mathsf{FA}_{\kappa^+}(\mathbb{P})$ holds?

It there a unique maximal class with this property?

We proposed the above maximality principles as candidates for extensions of $\mathbf{ZFC}$ that provide a strong structure theory for $\mathbf{\Sigma}^1_1$-sets.

Therefore it is natural to ask whether these axioms can hold globally, i.e. is it consistent that $\mathrm{SMP}_n(\nu)$ (or $\mathrm{CMP}_n(\nu)$) holds for every uncountable cardinal ν with $\nu = \nu^{<\nu}$?

Theorem (Fuchs)

The class of all uncountable cardinals ν with $\nu = \nu^{<\nu}$ and $\mathrm{CMP}_3(\nu)$ is bounded in On.

Theorem

The class of all uncountable cardinals ν with $\nu = \nu^{<\nu}$ and $\mathrm{SMP}_2(\nu)$ is bounded in On.

Question

Is the class of all uncountable cardinals ν with $\nu = \nu^{<\nu}$ and $\mathrm{CMP}_2(\nu)$ always bounded in On?

Following Fuchs, we may consider weakenings of the above principles called *localized maximality principles*.

These principles can consistently hold at every uncountable cardinal ν with $\nu = \nu^{<\nu}$ and all of the above consequences of maximality principles also follow from these restricted versions.

Thank you for listening!