Squares, ascent paths, and chain conditions

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Introduction

In this introduction, we discuss three questions that motivated the work presented in this talk.

These questions ask about the weakest combinatorial principles needed for the construction of certain objects.

More specifically, we will use *Todorčević's square principle* $\square(\kappa)$ for regular cardinals $\kappa > \omega_1$ to construct:

- a non-specializable tree of height κ.
- a failure of the countable productivity of the κ-Knaster property.
- a κ-Knaster partial order that is not κ-stationarily layered.

Non-specializable tree

Laver and Shelah established the consistency of the $\aleph_2$-Souslin Hypothesis from a weakly compact cardinal.

Their construction can be modified to obtain a model in which all $\aleph_2$-Aronszajn trees are special, i.e. all of these trees can be represented as unions of $\aleph_1$-many antichains.

Results of Shelah and Stanley and, independently, Todorčević show that this statement causes ω_2 to be a weakly compact cardinal in L.

These arguments rely on classical results of Jensen and the concept of *ascent paths through trees* introduced by Laver.

Definition

Let $\lambda < \kappa$ be cardinals with κ uncountable and regular, let $\mathbb{T}$ be a tree of height κ, and let

$$\vec{b} \;=\; \langle b_\alpha : \lambda \longrightarrow \mathbb{T}(\alpha) \mid \alpha < \kappa \rangle$$

be a sequence of functions. Then $\vec{b}$ is a λ-*ascent path through* $\mathbb{T}$ if, for all $\alpha < \beta < \kappa$, there is an $i < \lambda$ such that $b_\alpha(j) <_{\mathbb{T}} b_\beta(j)$ holds for all $i \leq j < \lambda$.

Lemma (Shelah)

Let $\lambda < \kappa$ be cardinals with κ uncountable and regular. If κ is not the successor of a cardinal of cofinality at most λ, then special trees of height κ do not contain λ-ascent paths.

The following definition was isolated by Todorčević from Jensen's seminal work on the existence of κ-Souslin trees.

Definition

Let κ be an uncountable regular cardinal. A sequence $\langle C_\alpha \mid \alpha \in \mathrm{Lim} \cap \kappa \rangle$ is a $\Box(\kappa)$-sequence if the following statements hold for all $\alpha \in \mathrm{Lim} \cap \kappa$:

- C_α is a closed unbounded subset of α.
- If $\beta \in \mathrm{Lim}(C_\alpha)$, then $C_\beta = C_\alpha \cap \beta$.
- There is no closed unbounded subset C of κ with the property that $C_\alpha = C \cap \alpha$ holds for all $\alpha \in \mathrm{Lim}(C)$.

In addition, we say that such a sequence *avoids* a subset S of κ if $\mathrm{Lim}(C_\alpha) \cap S = \emptyset$ holds for all $\alpha \in \mathrm{Lim}(\kappa)$.

Note that the existence of a $\Box(\kappa)$-sequence does not imply the existence of a $\Box(\kappa)$-sequence that avoids a stationary subset of κ.

Theorem (Todorčević)

Let κ be an uncountable regular cardinal. If there is a $\square(\kappa)$-sequence, then there is a κ-Aronszajn tree.

Theorem (Jensen, Todorčević)

Let κ be an uncountable regular cardinal. If κ is not weakly compact in L, then there is a $\square(\kappa)$-sequence that is an element of L.

Theorem (Jensen, Schimmerling, Zeman)

In canonical inner models of set theory, an uncountable regular cardinal κ is not weakly compact if and only if for every stationary subset S of κ, there is a $\square(\kappa)$-sequence that avoids a stationary subset of S.

Theorem (Shelah–Stanley, Todorčević)

Assume that there are infinite regular cardinals $\lambda < \kappa$ with the property that there exists a $\square(\kappa)$-sequence that avoids a stationary subset of κ that consists of limit ordinals of cofinality λ. Then there exists a κ-Aronszajn tree with a λ-ascent path.

Theorem (Laver–Shelah)

Let κ be weakly compact, let $\nu < \kappa$ be a regular cardinal, let G be $\mathrm{Col}(\nu, {<}\kappa)$-generic over G and let $\mathbb{T}$ be a tree of height κ in $\mathrm{V}[G]$. If $\mathbb{T}$ contains a λ-ascent path for some $\lambda < \nu$ in $\mathrm{V}[G]$, then $\mathbb{T}$ contains a cofinal branch in $\mathrm{V}[G]$.

Question

Given $\kappa > \omega_1$ regular, does the existence of a $\square(\kappa)$-sequence imply the existence of a κ-Aronszajn tree with an ω-ascent path?

The productivity of the κ-Knaster property

Another motivation of our work comes from the investigation of the productivity of chain conditions.

An easy argument shows that the κ-chain condition is productive if κ is weakly compact.

Todorčević asked whether the converse implication holds for regular cardinals $\kappa > \omega_1$. This question is still open.

A series of deep results by Rinot, Shelah, Todorčević, and others shows that the productivity of the chain condition entails many consequences of weak compactness, like Mahloness, stationary reflection and the non-existence of $\square(\kappa)$-sequences.

In this talk, we are interested in the productivity of stronger chain conditions.

Definition

Given an uncountable regular cardinal κ, a partial order $\mathbb{P}$ is κ-*Knaster* if every set of κ-many conditions in $\mathbb{P}$ contains a subset of cardinality κ consisting of pairwise compatible conditions.

This property clearly strengthens the κ-chain condition and it is easy to see that it is finitely productive.

Moreover, if κ is weakly compact, then the full-support product of less than κ-many κ-Knaster partial orders is again κ-Knaster.

Therefore, it is natural to consider the question whether weak compactness can be characterized by the productivity of the κ-Knaster property.

The following result shows that a negative answer to this variation of Todorčević's question is consistent.

Theorem (Cox–L.)

Let κ be an inaccessible cardinal such that there is a κ-Souslin tree $\mathbb{T}$ with

$$\mathbb{1}_{\mathbb{T}} \Vdash \text{“}\check{\kappa} \text{ is weakly compact”}.$$

Then there the class of all κ-Knaster partial orders is closed under full-support products of size less than κ.

In contrast, the following result shows that this characterization of weak compactness holds in canonical inner models.

Theorem (L.)

Let $\kappa > \omega_1$ be a regular cardinal with the property that there is a $\square(\kappa)$-sequence that avoids a stationary subset of κ that consists of limit ordinals of countable cofinality. Then the class of κ-Knaster partial is not closed under countable support products.

The above results raise the question whether the existence of a $\square(\kappa)$-sequence alone implies the above conclusion.

Question

Given a regular cardinal $\kappa > \omega_1$, does the countable productivity of the κ-Knaster property imply the non-existence of $\square(\kappa)$-sequences?

Stationary layeredness

Finally, we are interested in the following strengthening of the κ-Knaster property introduced by Cox:

Definition

Given an uncountable regular cardinal κ, a partial order $\mathbb{P}$ is called *κ-stationarily layered* if the set $\mathrm{Reg}_\kappa(\mathbb{P})$ of all regular suborders of $\mathbb{P}$ of cardinality less than κ is stationary in the collection $\mathcal{P}_\kappa(\mathbb{P})$ of all subsets of $\mathbb{P}$ of cardinality less than κ.

Lemma

All κ-stationarily layered partial order are κ-Knaster.

Theorem (Cox–L.)

The following statements are equivalent for every uncountable regular cardinal κ:

- *κ is weakly compact.*
- *The κ-chain condition is equivalent to κ-stationary layeredness.*

The above result raised the question whether weak compactness can also be characterized by the statement that all κ-Knaster partial orders are κ-stationarily layered.

The following result shows that this characterization holds in certain inner models.

Theorem (Cox–L.)

Let κ be an uncountable regular cardinal with the property that every κ-Knaster partial orders is κ-stationarily layered. Then κ is Mahlo and every stationary subset of κ reflects.

In particular, the existence of a $\square(\kappa)$-sequence that avoids a stationary subset of κ implies the existence of a κ-Knaster partial order that is not κ-stationarily layered.

In contrast, it is consistent that these two chain conditions coincide at a non-weakly compact cardinal.

Theorem (Cox–L.)

Let κ be an inaccessible cardinal such that there is a κ-Souslin tree $\mathbb{T}$ with

$$\mathbb{1}_{\mathbb{T}} \Vdash \text{“}\check{\kappa} \text{ is weakly compact”}.$$

Then every κ-Knaster partial order is κ-stationarily layered.

These results motivate the following question:

Question

Does the existence of a $\square(\kappa)$-sequence imply the existence of a κ-Knaster partial order that is not κ-stationarily layered?

Square principles

The concept of a $\Box(\kappa)$-sequence can be weakened in the following way:

Definition

Given an uncountable regular cardinal κ and a cardinal $1 < \lambda \leq \kappa$, a sequence $\langle \mathcal{C}_\alpha \mid \alpha \in \mathrm{Lim} \cap \kappa \rangle$ is a $\Box(\kappa,<\lambda)$-*sequence* if the following statements hold for all $\alpha \in \mathrm{Lim} \cap \kappa$:

- $\mathcal{C}_\alpha$ is a collection of closed unbounded subsets of α with $0 < |\mathcal{C}_\alpha| < \lambda$.
- If $C \in \mathcal{C}_\alpha$ and $\beta \in \mathrm{Lim}(C)$, then $C \cap \beta \in \mathcal{C}_\beta$.
- There is no closed unbounded subset C of κ with the property that $C \cap \alpha \in \mathcal{C}_\alpha$ holds for all $\alpha \in \mathrm{Lim}(C)$.

Theorem (Todorčević)

Given an uncountable regular cardinal κ, the tree property at κ is equivalent to the non-existence of $\Box(\kappa,<\kappa)$-sequences.

The following notion again strengthens the concept of a $\square(\kappa,{<}\lambda)$-sequence. It is a modification of the indexed square notions studied by Cummings and Schimmerling.

Definition (Lambie-Hanson)

Let $\lambda < \kappa$ be infinite regular cardinals. A $\square^{\mathrm{ind}}(\kappa,\lambda)$-sequence is a matrix

$$\langle C_{\alpha,i} \mid \alpha < \kappa,\ i(\alpha) \leq i < \lambda \rangle$$

satisfying the following statements for all $\alpha \in \mathrm{Lim} \cap \kappa$:

- $i(\alpha) < \lambda$.
- If $i(\alpha) \leq i < \lambda$, then $C_{\alpha,i}$ is a closed unbounded subset of α.
- If $i(\alpha) \leq i < j < \lambda$, then $C_{\alpha,i} \subseteq C_{\alpha,j}$.
- If $i(\alpha) \leq i < \lambda$ and $\beta \in \mathrm{Lim}(C_{\alpha,i})$, then $i \geq i(\beta)$ and $C_{\beta,i} = C_{\alpha,i} \cap \beta$.
- If $\beta \in \mathrm{Lim} \cap \alpha$, then there is an $i(\alpha) \leq i < \lambda$ with $\beta \in \mathrm{Lim}(C_{\alpha,i})$.
- There is no closed unbounded subset C of κ with the property that, for all $\alpha \in \mathrm{Lim}(C)$, there is an $i(\alpha) \leq i < \lambda$ with $C_{\alpha,i} = C \cap \alpha$.

It is immediate that the existence of a $\square^{\mathrm{ind}}(\kappa, \lambda)$-sequence implies the existence of a $\square(\kappa, {<}\lambda^+)$-sequence.

The following result shows that the above definition can be slightly simplified.

Proposition (Hayut–Lambie-Hanson)

The above definition is unchanged if we replace the last statement by the following seemingly weaker condition:

- *There is no closed unbounded subset C of κ and $i < \lambda$ with the property that $i \geq i(\alpha)$ and $C_{\alpha,i} = C \cap \alpha$ holds for all $\alpha \in \mathrm{Lim}(C)$.*

The following technical result will allow us to answer all questions posed in the introduction.

Theorem (Lambie-Hanson – L.)

Let $\lambda < \kappa$ be infinite regular cardinals and let S be a stationary subset of κ. If there is a $\square(\kappa)$-sequence, then there is a $\square^{\mathrm{ind}}(\kappa, \lambda)$-sequence

$$\langle C_{\alpha,i} \mid \alpha \in \mathrm{Lim} \cap \kappa, \ i(\alpha) \leq i < \lambda \rangle$$

with the following properties:

- *If $i < \lambda$, then the set $\{\alpha \in S \mid i(\alpha) = i\}$ is stationary in κ.*
- *There is a $\square(\kappa)$-sequence $\langle D_\alpha \mid \alpha < \kappa \rangle$ with the property that $\mathrm{Lim}(D_\alpha) \subseteq \mathrm{Lim}(C_{\alpha,i(\alpha)})$ holds for all $\alpha \in \mathrm{Lim}(\kappa)$.*

The partial order $\mathbb{P}_{\vec{C}}$

In the following, we use the $\square^{\mathrm{ind}}(\kappa,\lambda)$-sequence constructed from a $\square(\kappa)$-sequence to prove the following result:

Theorem (Lambie-Hanson–L.)

Let κ be an uncountable regular. If there is a $\square(\kappa)$-sequence, then the there is a κ-Knaster partial order $\mathbb{P}$ with the property that the full support product $\mathbb{P}^{\omega}$ does not satisfy the κ-chain condition.

Definition

Let $\lambda < \kappa$ be infinite regular cardinals, let

$$\vec{\mathcal{C}} \;=\; \langle C_{\alpha,i} \mid \alpha \in \mathrm{Lim} \cap \kappa,\; i(\alpha) \leq i < \lambda \rangle$$

be a $\square^{\mathrm{ind}}(\kappa,\lambda)$-sequence and let $i < \lambda$.

- We define
$$S^{\vec{\mathcal{C}}}_{\leq i} \;=\; \{\alpha \in \mathrm{Lim} \cap \kappa \mid i(\alpha) \leq i\}.$$
- We let $\mathbb{T}^{\vec{\mathcal{C}}}_i$ denote the tree with underlying set $S^{\vec{\mathcal{C}}}_{\leq i}$ and
$$\alpha <_{\mathbb{T}^{\vec{\mathcal{C}}}_i} \beta \iff \alpha \in \mathrm{Lim}(C_{\beta,i}).$$

Definition

Given a tree $\mathbb{T}$, we let $\mathbb{P}(\mathbb{T})$ denote the partial order consisting of finite partial functions $f : \mathbb{T} \xrightarrow{part} \omega$ that are injective on $<_{\mathbb{T}}$-chains and that are ordered by reverse inclusion.

Definition

Let $\lambda < \kappa$ be infinite regular cardinals and let $\vec{C}$ be a $\square^{\mathrm{ind}}(\kappa, \lambda)$-sequence. We let $\mathbb{P}_{\vec{C}}$ denote the lottery sum of the sequence $\langle \mathbb{P}(\mathbb{T}_i^{\vec{C}}) \mid i < \lambda \rangle$.

Proposition

Given infinite regular cardinals $\lambda < \kappa$ and a $\square^{\mathrm{ind}}(\kappa, \lambda)$-sequence $\vec{C}$, the full support product $\mathbb{P}_{\vec{C}}^{\lambda}$ does not satisfy the κ-chain condition.

Proof.

Given $\alpha \in \mathrm{Lim} \cap \kappa$ and $i < \lambda$, the function $\{\langle \alpha, 0 \rangle\}$ is a condition in the partial order $\mathbb{P}(\mathbb{T}^{\vec{\mathcal{C}}}_{\max\{i,i(\alpha)\}})$. This shows that for every $\alpha \in \mathrm{Lim} \cap \kappa$, there is a unique condition $\vec{p}_\alpha$ in $\mathbb{P}^\lambda_{\vec{\mathcal{C}}}$ with

$$\vec{p}_\alpha(i) \;=\; \langle \{\langle \alpha, 0 \rangle\}, \max\{i, i(\alpha)\} \rangle$$

for all $i < \lambda$.

Fix $\alpha, \beta \in \mathrm{Lim} \cap \kappa$ with $\beta < \alpha$. Then there is an $i(\alpha) \leq i < \lambda$ with the property that $\beta \in \mathrm{Lim}(C_{\alpha,i})$.

This implies that $i \geq i(\beta)$, $C_{\beta,i} = C_{\alpha,i} \cap \beta$ and $\beta <_{\mathbb{T}^{\vec{\mathcal{C}}}_i} \alpha$.

We can conclude that the conditions $\vec{p}_\alpha(i) = \langle \{\langle \alpha, 0 \rangle\}, i \rangle$ and $\vec{p}_\beta(i) = \langle \{\langle \beta, 0 \rangle\}, i \rangle$ are incompatible in $\mathbb{P}_{\vec{\mathcal{C}}}$ and therefore the condition $\vec{p}_\alpha$ and $\vec{p}_\beta$ are incompatible in $\mathbb{P}^\lambda_{\vec{\mathcal{C}}}$.

These computations show that the sequence $\langle \vec{p}_\alpha \mid \alpha \in \mathrm{Lim} \cap \kappa \rangle$ enumerates an antichain in $\mathbb{P}^\lambda_{\vec{\mathcal{C}}}$. $\square$

Antichains in $\mathbb{P}_{\vec{C}}$

It remains to show that the partial order $\mathbb{P}_{\vec{\mathcal{C}}}$ is κ-Knaster, whenever $\vec{\mathcal{C}}$ is a $\square^{\mathrm{ind}}(\kappa, \lambda)$-sequence constructed from a $\square(\kappa)$-sequence.

The proof of this statement makes use of the following notion:

Definition (Todorčević)

Let κ be an uncountable regular cardinal, let $\mathbb{T}$ be a tree of height at most κ, and let S be a subset of κ.

- A map $r : \mathbb{T} \upharpoonright S \longrightarrow \mathbb{T}$ is *regressive* if $r(t) <_{\mathbb{T}} t$ holds for all $t \in \mathbb{T} \upharpoonright S$ with $\mathrm{lh}_{\mathbb{T}}(t) > 0$.
- The subset S is *non-stationary with respect to* $\mathbb{T}$ if there is a regressive map $r : \mathbb{T} \upharpoonright S \longrightarrow \mathbb{T}$ such that, for every $t \in \mathbb{T}$, there is a $\theta_t < \kappa$ and a function $c_t : r^{-1}``\{t\} \longrightarrow \theta_t$ that is injective on $<_{\mathbb{T}}$-chains.
- The tree $\mathbb{T}$ is *special* if κ is non-stationary with respect to $\mathbb{T}$.

Todorčević showed that for successor cardinals ν^+, this notion of special trees coincides with the classical notion of special trees, i.e. trees of height ν^+ that can be represented as the union of ν-many antichains.

Lemma

Let κ be an uncountable regular cardinal and let $\mathbb{T}$ be a tree of cardinality at most κ. If there is a stationary subset of κ that is non-stationary with respect to $\mathbb{T}$, then the partial order $\mathbb{P}(\mathbb{T})$ is κ-Knaster.

The following lemma is the last ingredient needed for the proof of our main result.

Lemma

Let

$$\vec{\mathcal{C}} = \langle C_{\alpha,i} \mid \alpha < \kappa, \ i(\alpha) \leq i < \lambda \rangle$$

be a $\square^{\mathrm{ind}}(\kappa, \lambda)$-sequence. If $i < \lambda$ has the property that the tree $\mathbb{T}_i^{\vec{\mathcal{C}}}$ has height κ, then the set $S_{>i}^{\vec{\mathcal{C}}}$ is non-stationary with respect to $\mathbb{T}_i^{\vec{\mathcal{C}}}$.

Proof of the Theorem.

Assume that there is a $\square(\kappa)$-sequence. Then our theorem yields a $\square^{\mathrm{ind}}(\kappa, \omega)$-sequence

$$\vec{C} \;=\; \langle C_{\alpha,i} \mid \alpha \in \mathrm{Lim} \cap \kappa,\; i(\alpha) \leq i < \omega\rangle$$

with $S^{\vec{\mathcal{C}}}_{>i} = \{\alpha \in \mathrm{Lim} \cap \kappa \mid i(\alpha) > i\}$ stationary in κ for all $i < \omega$.

Given $i < \lambda$, the stationary set $S^{\vec{\mathcal{C}}}_{>i}$ is non-stationary with respect to $\mathbb{T}^{\vec{\mathcal{C}}}_i$ and this implies that the partial order $\mathbb{P}(\mathbb{T}^{\vec{\mathcal{C}}}_i)$ is κ-Knaster. This shows that $\mathbb{P}_{\vec{\mathcal{C}}}$ is a lottery sum of less than κ-many κ-Knaster partial orders and hence $\mathbb{P}_{\vec{\mathcal{C}}}$ itself is κ-Knaster.

Finally, the above computations show that the full support product $\mathbb{P}^{\omega}_{\vec{\mathcal{C}}}$ does not satisfy the κ-chain condition. $\square$

Motivated by the above result, Lambie-Hanson and Rinot continued investigating the productivity of Knaster properties.

The following theorem answers one of the main questions left open by our work.

Theorem (Lambie-Hanson–Rinot)

If ν is an uncountable cardinal, then the class of ν^+-Knaster partial orders is not closed under countable support products.

Stationary layeredness

We will use the above results and the partial order $\mathbb{P}_{\vec{\mathcal{C}}}$ to prove the following result.

Theorem (Lambie-Hanson–L.)

Let κ be an uncountable regular. If there is a $\square(\kappa)$-sequence, then the there is a κ-Knaster partial order that is not κ-stationarily layered.

Proof.

By the above results, we may assume that κ is Mahlo and every stationary subset of κ reflects. Our main theorem yields a $\square^{\mathrm{ind}}(\kappa,\omega)$-sequence

$$\vec{\mathcal{C}} = \langle C_{\alpha,i} \mid \alpha < \kappa,\ i(\alpha) \leq i < \omega \rangle$$

with the property that there exists a $\square(\kappa)$-sequence $\langle D_\alpha \mid \alpha < \kappa \rangle$ such that $\mathrm{Lim}(D_\alpha) \subseteq \mathrm{Lim}(C_{\alpha,i(\alpha)})$ holds for all $\alpha \in \mathrm{Lim} \cap \kappa$.

Then stationary reflection yields a club C in κ of strong limit cardinals such that $\mathrm{otp}(D_\alpha) = \alpha$ holds for all $\alpha \in C$.

Proof (cont.).

Assume that $\mathbb{P}_{\vec{\mathcal{C}}}$ is κ-stationarily layered.

By standard arguments, we find a regular cardinal $\theta > \kappa$, an elementary substructure M of $\mathrm{H}(\theta)$ of cardinality less than κ and $\alpha \in C$ such that $\alpha = \kappa \cap M$, $\vec{\mathcal{C}} \in M$, and $\mathbb{P}_{\vec{\mathcal{C}}} \cap M$ is a regular suborder of $\mathbb{P}_{\vec{\mathcal{C}}}$.

Set $p_0 = \{\langle \alpha, 0 \rangle\}$. Then $p = \langle p_0, i(\alpha) \rangle$ is a condition $\mathbb{P}_{\vec{\mathcal{C}}}$ and there is $q_0 \in \mathbb{P}(\mathbb{T}^{\vec{\mathcal{C}}}_{i(\alpha)}) \cap M$ such that $q = \langle q_0, i(\alpha) \rangle$ is a reduct of p in $\mathbb{P}_{\vec{\mathcal{C}}} \cap M$.

Since the conditions p_0 and q_0 are compatible in $\mathbb{P}(\mathbb{T}^{\vec{\mathcal{C}}}_{i(\alpha)})$, we know that $q_0(\beta) \neq 0$ holds for all $\beta \in \mathrm{dom}(q_0)$ with $\beta <_{\mathbb{T}^{\vec{\mathcal{C}}}_{i(\alpha)}} \alpha$.

Since $\alpha = \mathrm{otp}(D_\alpha)$ is a cardinal and $\mathrm{dom}(q_0)$ is a finite subset of α, there is a $\gamma \in \mathrm{Lim}(D_\alpha)$ with $\mathrm{dom}(q_0) \subseteq \gamma$.

Then $\gamma \in \mathrm{Lim}(C_{\alpha, i(\alpha)})$, $i(\gamma) \leq i(\alpha)$ and $\gamma <_{\mathbb{T}^{\vec{\mathcal{C}}}_{i(\alpha)}} \alpha$.

Moreover, the above remarks show that $r = \langle q_0 \cup \{\langle \gamma, 0 \rangle\}, i(\alpha) \rangle \in M$ is a condition in $\mathbb{P}_{\vec{\mathcal{C}}}$ that strengthens q. But the conditions p and r are incompatible in $\mathbb{P}_{\vec{\mathcal{C}}}$, a contradiction.

Ascent paths

Using Todorčević's technique of *walks on ordinals*, it is possible to prove the following result.

Theorem (Lambie-Hanson–L.)

If $\lambda < \kappa$ are infinite regular cardinals such that there is a $\square^{\mathrm{ind}}(\kappa, \lambda)$-sequence, then there is a κ-Aronszajn tree with a λ-ascent path.

In combination with our main result, this theorem allows us to answer the remaining question posed in the introduction.

Theorem (Lambie-Hanson–L.)

Let $\lambda < \kappa$ be infinite, regular cardinals. If there is a $\square(\kappa)$-sequence, then there is a κ-Aronszajn tree with a λ-ascent path.

The above results allow us to provide a complete picture of the consistency strengths of statements relating the interactions of $\aleph_2$-Aronszajn trees with ascent paths and special $\aleph_2$-Aronszajn trees.

	$\forall \mathbb{T}.A(\mathbb{T})$	$\exists \mathbb{T}.A(\mathbb{T})$	$\forall \mathbb{T}.\neg A(\mathbb{T})$
$\forall \mathbb{T}.S(\mathbb{T})$	$0 = 1$	$0 = 1$	Weakly compact
$\exists \mathbb{T}.S(\mathbb{T})$	$0 = 1$	ZFC	Weakly compact
$\forall \mathbb{T}.\neg S(\mathbb{T})$	Weakly compact	Mahlo	Weakly compact

Besides the above $\square(\kappa)$-constructions, the following result is the other main new ingredient in the determination of the consistency strengths in the above table:

Theorem (Lambie-Hanson–L.)

Let $\lambda < \kappa$ be infinite, regular cardinals such that $\kappa = \kappa^{<\kappa}$ and $\mathbb{1}_{\mathrm{Add}(\kappa,1)} \Vdash \mathrm{TP}(\check{\kappa})$. Then the following statements hold in a cofinality-preserving forcing extension of the ground model:

- *There are κ-Aronszajn trees.*
- *Every κ-Aronszajn tree contains a λ-ascent path.*

Thank you for listening!