Ascending paths and forcings that specialize higher Aronszajn trees

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Introduction

Set theoretic trees

The purpose of the work presented in this talk is to study combinatorial properties of trees of uncountable regular heights that cause these trees to be non-special in a very absolute way.

- A partial order $\mathbb{T}$ is a *tree* if it has a unique minimal element $\mathrm{root}(\mathbb{T})$ and sets of the form $\mathrm{pred}_{\mathbb{T}}(t) = \{s \in \mathbb{T} \mid s <_{\mathbb{T}} t\}$ are well-ordered by $<_{\mathbb{T}}$ for every $t \in \mathbb{T}$.
- Given a tree $\mathbb{T}$ and $t \in \mathbb{T}$, we define $\mathrm{lh}_{\mathbb{T}}(t)$ to be the order-type of $\langle \mathrm{pred}_{\mathbb{T}}(t), <_{\mathbb{T}} \rangle$ and we define $\mathrm{ht}(\mathbb{T}) = \sup_{t \in \mathbb{T}} \mathrm{lh}_{\mathbb{T}}(t)$ to be the *height* of $\mathbb{T}$.
- Given a tree $\mathbb{T}$ and $\gamma < \mathrm{ht}(\mathbb{T})$, we define $\mathbb{T}(\gamma) = \{t \in \mathbb{T} \mid \mathrm{lh}_{\mathbb{T}}(t) = \gamma\}$ and $\mathbb{T}_{<\gamma} = \{t \in \mathbb{T} \mid \mathrm{lh}_{\mathbb{T}}(t) < \gamma\}$.

Cofinal branches

One of the most basic questions about a tree $\mathbb{T}$ of infinite height is the question whether $\mathbb{T}$ has a *cofinal branch*, i.e. the question whether there is a subset B of $\mathbb{T}$ with the property that B is linearly ordered by $<_{\mathbb{T}}$ and the set $\{\mathrm{lh}_{\mathbb{T}}(t) \mid t \in B\}$ is unbounded in $\mathrm{ht}(\mathbb{T})$.

It turns out that a large class of trees of uncountable regular heights have no cofinal branches for *very special reasons*.

Special trees

Definition (Todorčević)

Let θ be an uncountable regular cardinal, let S be a subset of θ and let $\mathbb{T}$ be a tree of height θ.

- A map $r : \mathbb{T} \restriction S \longrightarrow \mathbb{T}$ is *regressive* if $r(t) <_{\mathbb{T}} t$ holds for $t \in \mathbb{T} \restriction S$ that is not minimal in $\mathbb{T}$.
- We say that S *is nonstationary with respect to* $\mathbb{T}$ if there is a regressive map $r : \mathbb{T} \restriction S \longrightarrow \mathbb{T}$ with the property that for every $t \in \mathbb{T}$ there is a function $c_t : r^{-1}\{t\} \longrightarrow \theta_t$ such that θ_t is a cardinal smaller than κ and c_t is injective on $\leq_{\mathbb{T}}$-chains.
- The tree $\mathbb{T}$ is *special* if the set θ is nonstationary with respect to $\mathbb{T}$.

Proposition

If θ is an uncountable regular cardinal, $\mathbb{T}$ is a tree of height θ and S is a stationary subset of θ that is nonstationary with respect to $\mathbb{T}$, then there are no cofinal branches through $\mathbb{T}$.

Todorčević showed that the above definition generalizes the classical definition of special trees of successor height.

Theorem (Todorčević)

If ν is an infinite cardinal, then the following statements are equivalent for every tree $\mathbb{T}$ of height ν^+:

- *$\mathbb{T}$ is special.*
- *The set $S^{\nu^+}_{\mathrm{cof}(\nu)}$ of all limit ordinals less than ν^+ of cofinality $\mathrm{cof}(\nu)$ is nonstationary with respect to $\mathbb{T}$.*
- *$\mathbb{T}$ is the union of ν-many antichains.*

The non-existence of cofinal branches through special trees is absolute in a strong sense.

If θ is an uncountable regular cardinal, $\mathbb{T}$ is a special tree of height θ and W is an outer model of the ground model V (i.e. W is a transitive model of $\mathbf{ZFC}$ with $\mathrm{V} \subseteq \mathrm{W}$ and $\mathrm{On} \cap \mathrm{V} = \mathrm{On} \cap \mathrm{W}$) with the property that θ is an uncountable regular cardinal in W, then there are no cofinal branches through $\mathbb{T}$ in W.

We want to study special reasons that cause trees without cofinal branches to be non-special in a very absolute way. Examples of such properties were already consider by Baumgartner, Cummings, Devlin, Laver, Shelah, Stanley, Todorčević and others.

For reasons described later, we will focus on the following property that directly generalizes the concept of cofinal branches and is weaker than the properties considered by the above authors.

Ascending paths

In the following, we let θ denote an uncountable regular cardinal and let $\mathbb{T}$ denote a tree of height θ.

Definition

Given a cardinal $\lambda > 0$, a sequence $\langle b_\gamma : \lambda \longrightarrow \mathbb{T}(\gamma) \mid \gamma < \theta \rangle$ of functions is an *ascending path of width* λ *through* $\mathbb{T}$ if for all $\bar{\gamma} < \gamma < \theta$, there are $\alpha, \bar{\alpha} < \lambda$ such that $b_{\bar{\gamma}}(\bar{\alpha}) <_{\mathbb{T}} b_\gamma(\alpha)$.

Then the existence of a cofinal branch through $\mathbb{T}$ is equivalent to the existence of an ascending path of width 1 through $\mathbb{T}$.

The following lemma shows that the same is true for ascending paths of finite width. We will later present a proof of this statement.

Lemma

If there is an ascending paths of finite width through $\mathbb{T}$, then $\mathbb{T}$ has a cofinal branch.

We list two more basic observations.

Proposition

- *There is an ascending path of width θ through $\mathbb{T}$.*
- *Assume that $\theta = \nu^+$. Then there is an ascending path of width ν through $\mathbb{T}$ if and only if for every $\gamma < \theta$ there is $t \in \mathbb{T}(\gamma)$ such that for every $\gamma < \delta < \theta$ there is $u \in \mathbb{T}(\delta)$ with $t <_{\mathbb{T}} u$.*

The following lemma shows how ascending paths cause certain trees to be nonspecial.

Lemma

Let $\lambda < \theta$ be a cardinal with the property that θ is not a successor of a cardinal of cofinality less than or equal to λ and let $S \subseteq S^{\theta}_{>\lambda}$ be stationary in θ. If S is nonstationary with respect to $\mathbb{T}$, then there is no ascending path of width λ through $\mathbb{T}$.

Corollary

Let $\lambda < \theta$ be a cardinal with the property that θ is not a successor of a cardinal of cofinality less than or equal to λ. If $\mathbb{T}$ contains an ascending path of width λ, then $\mathbb{T}$ is not special.

Note that the above result does not answer the following question.

Question

Given a singular cardinal ν and a cardinal $\mathrm{cof}(\nu) \leq \lambda < \nu$, is it possible that a special tree of height ν^+ contains an ascending path of width λ?

The above corollary shows that ascending paths cause trees to be nonspecial in a very absolute way: in the situation of the corollary, the tree $\mathbb{T}$ remains nonspecial in every outer model in which θ satisfies the assumptions of the corollary.

We will later show that, if θ and λ satisfy certain cardinal arithmetic assumptions, then the converse of this implication also holds true, i.e. if there is no ascending path of width λ through $\mathbb{T}$, then $\mathbb{T}$ is special in a forcing extension of the ground model in which the above assumptions on θ is hold.

Forcings that specialize trees

We present results showing that the existence of ascending paths is closely connected to chain conditions of the following canonical partial order that specializes the tree $\mathbb{T}$.

Definition

Let κ be an infinite regular cardinal. We define $\mathbb{P}_\kappa(\mathbb{T})$ to be the partial order whose conditions are partial functions from $\mathbb{T}$ to κ of cardinality less than κ that are injective on chains in $\mathbb{T}$ and whose ordering is given by reversed inclusion.

It is easy to see that partial orders of the form $\mathbb{P}_\kappa(\mathbb{T})$ are $<\kappa$-closed, forcing with $\mathbb{P}_\kappa(\mathbb{T})$ collapses every cardinal in the interval $(\kappa, \mathrm{ht}(\mathbb{T}))$ and, if θ is a regular in a $\mathbb{P}_\kappa(\mathbb{T})$-generic extension, then $\mathbb{T}$ is special in this extension.

Therefore it is natural to ask under which conditions the regularity of θ is preserved by forcing with $\mathbb{P}_\kappa(\mathbb{T})$.

The following lemma gives a necessary conditions for this preservation.

Lemma

Given $\kappa \leq \mu < \theta$, there is a forcing projection

$$\pi : \mathbb{P}_\kappa(\mathbb{T}) \longrightarrow \mathrm{Add}(\kappa, \theta) \times \mathrm{Col}(\kappa, \mu).$$

In particular, if forcing with $\mathbb{P}_\kappa(\mathbb{T})$ preserves the regularity of θ, then $\mu^{<\kappa} < \theta$ for all $\mu < \theta$.

The following result shows that, under the above cardinal arithmetic assumptions, the desired preservation is equivalent to the nonexistence of ascending paths of width less than κ.

Theorem

The following statements are equivalent for every infinite regular cardinal $\kappa < \theta$ with $\mu^{<\kappa} < \theta$ for all $\mu < \theta$:

- *There is no ascending path of width less than κ through $\mathbb{T}$.*
- *The partial order $\mathbb{P}_\kappa(\mathbb{T})$ satisfies the θ-chain condition.*
- *Forcing with the partial order $\mathbb{P}_\kappa(\mathbb{T})$ preserves the regularity of θ.*

By the above remarks, in the setting of the above theorem the three statements are also equivalent to the statement that there is some outer model W of the ground model V such that κ and θ are regular cardinals in W and $\mathbb{T}$ is a special tree in W.

Given an uncountable regular cardinal κ with $\kappa = \kappa^{<\kappa}$, the above theorem allows us to shows that the collection of *potentially special* trees of height κ^+ (i.e. the collection of all trees that special in a cofinality preserving outer model of the ground model V) is equal to the class of trees of height κ^+ that do not contain an ascending paths of width less than κ and therefore this class is definable in V.

Since we will later show that it is consistent that CH fails and every tree of height ω_2 that contains an ascending path of width ω has a cofinal branch, it is interesting to consider the following questions.

Question

If κ is an infinite regular cardinal and $\mathbb{T}$ is a tree of height κ^+ that does not contain an ascending path of width less than κ, is $\mathbb{T}$ potentially special?

Question

If κ is an infinite regular cardinal, is the class of potentially special trees of height κ^+ definable in V?

We present an application of the above result to questions regarding potential generalizations of Martin's Axiom to larger cardinalities.

Given an uncountable regular cardinal κ and a partial order $\mathbb{P}$, we let $\mathbf{FA}_\kappa(\mathbb{P})$ denote the statement that for every collection $\mathcal{D}$ of κ-many dense subsets of $\mathbb{P}$, there is a $\mathcal{D}$-generic filter on $\mathbb{P}$.

Shelah showed that for every uncountable regular cardinal κ, there is a $<\kappa$-closed partial order satisfying the κ^+-chain condition with the property that $\mathbf{FA}_{\kappa^+}(\mathbb{P})$ fails.

The above partial order is not *well-met*, i.e. there are compatible conditions without a greatest lower bound. Since generalizations of Martin's Axiom for classes of $<\kappa$-closed, well-met partial orders satisfying certain strengthenings of the κ^+-chain condition are known to be consistent, it is natural to ask the following question.

Question

Is it consistent that there is an uncountable regular cardinal κ such that $\mathbf{FA}_{\kappa^+}(\mathbb{P})$ holds for all $<\kappa$-closed, well-met partial orders satisfying the κ^+-chain condition?

A combination of the above theorem and a result of Todorčević on the existence of *nonspecial square sequences* shows that the consistency strength of a positive answer to the above question is at least a weakly compact cardinal.

Theorem

Let κ be an uncountable regular cardinal. If κ^+ is not weakly compact in L, then there is a $<\kappa$-closed, well-met partial order $\mathbb{P}$ satisfying the κ^+-chain condition with the property that $\mathbf{FA}_{\kappa^+}(\mathbb{P})$ fails.

Productivity of the Knaster property

We discuss another application of the notion of ascent paths. This application deals with questions on the *productivity* of certain chain conditions and characterizations of weak compactness.

Recall that a partial order is θ-*Knaster* if every θ-sized collection of conditions can be refined to a θ-sized set of pairwise compatible conditions.

Note that, if θ is weakly compact, then the class of θ-Knaster partial orders is closed under ν-support products for all $\nu < \theta$.

The starting point of this connection is the following result.

Theorem (Cox-L.)

If θ is weakly compact cardinal, then there is a partial order $\mathbb{P}$ such that the following statements hold in $\mathrm{V}[G]$ whenever G is $\mathbb{P}$-generic over V.

- *θ is inaccessible and not weakly compact.*
- *For every $\nu < \theta$, the class of θ-Knaster partial orders is closed under ν-support products.*

In contrast, we can use ascending paths to prove the following result.

Theorem

Let $\mathrm{L}[E]$ *be a Jensen-style extender model. In* $\mathrm{L}[E]$*, the following statements are equivalent for every uncountable regular cardinal* θ*:*

- θ *is weakly compact.*
- *The class of* θ*-Knaster partial orders is closed under* ν*-support products for all* $\nu < \theta$*.*

Moreover, if θ *is not the successor of a subcompact cardinal in* $\mathrm{L}[E]$*, then the above statements are also equivalent to the following statement:*

- *The class of* θ*-Knaster partial orders is closed under countable support products.*

In particular, the statement that the countable productivity of the θ-Knaster property implies that θ is weakly compact is independent from the axioms of $\mathbf{ZFC}$.

The proof of the above theorem relies on the following theorem and results that show that certain square principles allow us to construct trees $\mathbb{T}$ containing ascent paths of width ω with the property that the partial order $\mathbb{P}_\omega(\mathbb{T})$ is θ-Knaster.

Theorem

The following statements are equivalent for every infinite regular cardinal $\kappa < \theta$ with $\mu^{<\kappa} < \theta$ for all $\mu < \theta$.

- *There is no ascending path of width less than κ through $\mathbb{T}$.*
- *Products of the partial order $\mathbb{P}_\omega(\mathbb{T})$ with $<\kappa$-support satisfy the θ-chain condition.*

Some nonexistence results

Theorem

- *Every tree of height θ that contains an ascending path of finite width has a cofinal branch.*
- *If θ is weakly compact, then every tree of height θ that contains an ascending path of width less than θ has a cofinal branch.*
- *If $\kappa \leq \theta$ is a θ-compact cardinal, then every tree of height θ that contains an ascending path of width less than κ has a cofinal branch.*
- *If θ is weakly compact, $\kappa < \theta$ is an uncountable regular cardinal and G is $\mathrm{Col}(\kappa, {<}\theta)$-generic over V, then in $\mathrm{V}[G]$ every tree of height θ that contains an ascending path of width less than κ has a cofinal branch.*
- *If $\kappa \leq \theta$ is supercompact, $\nu < \kappa$ is an uncountable regular cardinal and G is $\mathrm{Col}(\nu, {<}\kappa)$-generic over V, then in $\mathrm{V}[G]$ every tree of height θ that contains an ascending path of width less than κ has a cofinal branch.*

The first three statements of the above theorem are a consequence of the following lemma.

Lemma

Let λ be a cardinal with the property that for every collection $\mathcal{S}$ of θ-many subsets of θ there is a $<\lambda^+$-closed $\mathcal{S}$-ultrafilter on θ consisting of unbounded subsets. Then every tree of height θ that contains an ascending path of width λ has a cofinal branch.

Using the notion of *narrow systems* introduced by Magidor and Shelah and a recent result by Lambie-Hanson, it is possible to derive a stronger result that implies the last two statements of the above theorem.

Lemma

Let λ be a cardinal with the property that for every collection $\mathcal{S}$ of θ-many subsets of θ there is a $<\lambda^+$-closed partial order $\mathbb{P}$ such that forcing with $\mathbb{P}$ adds a $<\lambda^+$-closed $\mathcal{S}$-ultrafilter on θ consisting of unbounded subsets. Then every tree of height θ that contains an ascending path of width λ has a cofinal branch.

Finally, it is possible to use results of Lambie-Hanson to establish a global result.

Theorem (Lambie-Hanson)

Assume that there is a proper class of supercompact cardinals. Then the following statement holds in a class forcing extension of the ground model: if θ is an uncountable regular cardinal and λ is a cardinal with $\lambda^+ < \theta$, then every tree of height θ that contains an ascending path of width λ has a cofinal branch.

In the following, we will present results that show that the above assumption implies that $\square_\kappa$ fails for every uncountable cardinal κ.

Some existence results

We present results that show how trees without cofinal branches containing ascending paths of small width can be constructed from *square principles*.

Definition (Todorčević)

- A sequence $\vec{C} = \langle C_\gamma \subseteq \gamma \mid \gamma < \theta \rangle$ is a *C-sequence of length θ* if the following statements hold for all $\gamma < \theta$:
  - If γ is a limit ordinal, then C_γ is a closed unbounded subset of γ.
  - If $\gamma = \bar{\gamma} + 1$, then $C_\gamma = \{\bar{\gamma}\}$.
- A C-sequence $\vec{C} = \langle C_\gamma \subseteq \gamma \mid \gamma < \theta \rangle$ is a $\square(\theta)$-sequence if the following statements hold:
  - If $\gamma \in \theta \cap \mathrm{Lim}$ and $\bar{\gamma} \in \mathrm{Lim}(C_\gamma)$, then $C_{\bar{\gamma}} = C_\gamma \cap \bar{\gamma}$.
  - There is no club C in θ with $C_\gamma = C \cap \gamma$ for all $\gamma \in \mathrm{Lim}(C)$.

Theorem (Todorčević)

If κ is a regular cardinal such that θ is not the successor of a cardinal of cofinality κ and there is a $\square(\theta)$-sequence $\vec{C} = \langle C_\gamma \subseteq \gamma \mid \gamma < \theta \rangle$ with the property that the set $\{\gamma < \theta \mid \mathrm{otp}(C_\gamma) = \kappa\}$ is stationary, then there is a θ-Aronszajn tree that contains an ascending paths of width κ.

Corollary

Assume that $\theta > \omega_1$ is not a Mahlo cardinal in L.

- *If θ is not a successor of a cardinal of cofinality ω, then there is a θ-Aronszajn tree that contains an ascending paths of width ω.*
- *If θ is a successor of a cardinal of cofinality ω, then there is a θ-Aronszajn tree that contains an ascending paths of width ω_1.*

In combination with the above results, this raises the following question.

Question

Let κ be an uncountable regular cardinal with the property that every tree of height κ^+ that contains an ascending path of width less than κ has a cofinal branch. Is κ^+ weakly compact in L?

Note that Jensen's construction of a $\square(\theta)$-sequences in L shows that the above question has a positive answer if every constructible subset of κ^+ that is stationary in L is also stationary in V.

We present another example of a construction of such trees from partial square sequences.

Definition (Baumgartner)

Let κ be an infinite regular cardinal and let $\vec{C} = \langle C_\gamma \mid \gamma < \kappa^+ \rangle$ be a C-sequence. We say that $S \subseteq \kappa^+$ *witnesses that* $\vec{C}$ *is a* $\square^B_\kappa$*-sequence* if the following statements hold:

- $S^{\kappa^+}_\kappa \subseteq S$.
- $\mathrm{otp}(C_\gamma) \leq \kappa$ for all $\gamma < \kappa^+$.
- If $\gamma \in S \cap \mathrm{Lim}$ and $\bar{\gamma} \in \mathrm{Lim}(C_\gamma)$, then $\bar{\gamma} \in S$ and $C_{\bar{\gamma}} = C_\gamma \cap \bar{\gamma}$.

Baumgartner showed that the existence of such sequences can be forced by $<\kappa$-directed closed, $(\kappa + 1)$-strategically closed partial orders.

Theorem

Let κ be an uncountable regular cardinal that satisfies $2^\kappa = \kappa^+$ and $2^{\kappa^+} = \kappa^{++}$. If there is a $\square^B_{\kappa^+}$-sequence, then there is a κ^{++}-Souslin tree that contains a ascending path of width κ.

PFA and ascending paths

We close by presenting some results that concern the influence of the proper forcing axiom on the existence and nonexistence of trees containing ascending paths.

Theorem

Assume that PFA holds. If $\theta > \omega_1$, then every tree of height θ that contains an ascending path of width ω has a cofinal branch.

This leads to the following question.

Question

Assume that PFA holds. Is every tree of height ω_2 without a cofinal branch potentially special?

We can use results presented above to show that PFA does not decide the statement that tree of height greater than ω_1 that contain an ascending path of width ω_1 have a cofinal branch.

Theorem

Assume that PFA holds.

- *There is a partial order $\mathbb{P}$ with the property that, whenever G is $\mathbb{P}$-generic over V, then PFA holds in $\mathrm{V}[G]$ and in $\mathrm{V}[G]$ there is an ω_3-Souslin tree that contains an ascending path of width ω_1.*
- *If κ is a supercompact compact cardinal and G is $\mathrm{Col}(\omega_2, {<}\kappa)$-generic over V, then PFA holds in $\mathrm{V}[G]$ and in $\mathrm{V}[G]$ every tree whose height is a regular cardinal greater than ω_2 that contains an ascending path of width ω_1 has a cofinal branch.*

Thank you for listening!