Infinite fields with free automorphism groups

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The results to be presented in this talk are joint work with Saharon Shelah. They appeared in the following paper.

PHILIPP LÜCKE and SAHARON SHELAH.
Free groups and automorphism groups of infinite structures.
Forum of Mathematics, Sigma, 2.

Introduction

In this talk, we want to consider questions about the existence of certain algebraic objects with prescribed automorphism groups whose answers depend on the underlying set theory.

Definition

Let κ be an infinite cardinal.

- Given a group G, we say that G is *the automorphism group of a* κ*-structure* if there is a first-order language $\mathcal{L}$ whose signature has cardinality κ and an $\mathcal{L}$-structure $\mathcal{M}$ of cardinality κ such that the group $\mathrm{Aut}(\mathcal{M})$ of all automorphisms of $\mathcal{M}$ is isomorphic to G.
- We let $\mathcal{AG}(\kappa)$ denote the class of all automorphism groups of κ-structures.

Related results

We start by presenting some known related results.

- If $G \in \mathcal{AG}(\kappa)$, then there is a connected graph Γ of cardinality κ such that G is isomorphic to $\mathrm{Aut}(\Gamma)$.

- **[Fried & Kollár, 1982]** If $G \in \mathcal{AG}(\kappa)$ and p is either 0 or a prime number, then there is a field K of characteristic p and cardinality κ such that the groups G and $\mathrm{Aut}(K)$ are isomorphic.

- The class $\mathcal{AG}(\kappa)$ contains all groups of cardinality at most κ.

- **[Just, Shelah & Thomas, 1999]** If $\aleph_0 < \kappa = \kappa^{<\kappa}$ and G is a subgroup of $\mathrm{Sym}(\kappa)$, then $G \in \mathcal{AG}(\kappa)$ holds in a cofinality-preserving forcing extension of the ground model.

- If $G \in \mathcal{AG}(\kappa)$, then G embeds into the group $\mathrm{Sym}(\kappa)$ of all permutations of κ and therefore has cardinality at most 2^κ.

  In particular, there is a group of cardinality 2^κ that is not contained in the class $\mathcal{AG}(\kappa)$.

- **[De Bruijn, 1957]** The group $\mathrm{Fin}(\kappa^+)$ of all finite permutations of κ^+ does not embed into $\mathrm{Sym}(\kappa)$ and therefore is not contained in the class $\mathcal{AG}(\kappa)$.

- **[Sanerib, 1974]** The group $\mathrm{Sym}(\kappa)$ has 2^{2^κ}-many pairwise non-isomorphic subgroups of the form

  $$S_{\mathcal{U}} = \{\pi \in \mathrm{Sym}(\kappa) \mid \forall X \subseteq \kappa\, [X \in \mathcal{U} \longleftrightarrow \pi[X] \in \mathcal{U}]\}$$

  for some non-principal ultrafilter $\mathcal{U}$ on κ.

  In particular, there is a subgroup of $\mathrm{Sym}(\kappa)$ that is not contained in the class $\mathcal{AG}(\kappa)$.

Free groups

Remember that, given a set A, the *free group with basis* A is the group $\mathrm{F}(A)$ consisting of the set of all reduced words in the alphabet

$$\{\mathrm{x}_a^i \mid a \in A,\ i = \pm 1\}$$

equipped with the operation that sends two words to the unique reduced word that is equivalent to their concatenation.

In the remainder of this talk, we focus on free groups and the following question.

Question

Given infinite cardinals $\kappa < \lambda$, is the free group $F(\lambda)$ contained in the class $\mathcal{AG}\,(\kappa)$?

In the case $\kappa = \aleph_0$, Shelah answered the above question in the negative. This answered a question of David Evans.

Theorem (Shelah, 2003)

If $\lambda > \aleph_0$, then $F(\lambda)$ is not contained in $\mathcal{AG}(\aleph_0)$.

The method developed in the proof of this result can also be used to answers the question in the negative for singular strong limit cardinals of countable cofinality.

Theorem (Shelah, 2003)

Let $\langle \kappa_n \mid n < \omega \rangle$ be a sequence of infinite cardinals with $2^{\kappa_n} < 2^{\kappa_{n+1}}$ for all $n < \omega$. Define $\kappa = \sum_{n<\omega} \kappa_n$. If $\lambda \geq \sum_{n<\omega} 2^{\kappa_n}$, then $F(\lambda)$ is not contained in $\mathcal{AG}(\kappa)$.

In contrast, the forcing construction mentioned above can be used to produce a positive answers to the above question.

Lemma

If κ is an infinite cardinal, then $\mathrm{F}(2^\kappa)$ embeds into $\mathrm{Sym}(\kappa)$.

Theorem (Just, Shelah & Thomas, 1999)

If $\aleph_0 < \kappa = \kappa^{<\kappa}$, then $F(2^\kappa) \in \mathcal{AG}(\kappa)$ holds in a cofinality-preserving forcing extension of the ground model.

We will show that the axioms of set theory already imply positive answers to the above question by proving the following theorem.

Theorem (L. & Shelah, 2014)

Let κ be a cardinal with $\kappa = \kappa^{\aleph_0}$. Then the free group $F(2^\kappa)$ is contained in the class $\mathcal{AG}(\kappa)$.

This theorem shows that the above question has a positive answer for some uncountable cardinal.

Corollary

The free group $F\left(2^{2^{\aleph_0}}\right)$ is contained in the class $\mathcal{AG}\left(2^{\aleph_0}\right)$.

A combination of the above results allows us to completely answer the above question under certain cardinal arithmetic assumptions.

Corollary

Assume that the Continuum Hypothesis and the Singular Cardinal Hypothesis hold. Then the following statements are equivalent for every infinite cardinal κ*.*

- *There is a cardinal* $\lambda > \kappa$ *with* $F(\lambda) \in \mathcal{AG}(\kappa)$*.*
- *Either* $\mathrm{cof}(\kappa) > \omega$ *or there is a cardinal* $\nu < \kappa$ *with* $2^{\nu} > \kappa$*.*

The methods developed in the proof of the above theorem also allow us to show that for a cardinal κ of uncountable cofinality, the assumption $F(\kappa^+) \notin \mathcal{AG}(\kappa)$ implies the existence of large cardinals in inner models.

In particular, the assumption $\kappa = \kappa^{\aleph_0}$ is consistently not necessary for the above conclusion.

Theorem (L. & Shelah, 2014)

Let κ be a cardinal of uncountable cofinality such that $F(\kappa^+)$ is not contained in the class $\mathcal{AG}(\kappa)$.

- *If κ is regular, then κ^+ is inaccessible in L.*
- *If κ is singular, then there is an inner model with a Woodin cardinal.*

We sketch the main construction behind the proofs of the above results.

Definition

Given an infinite cardinal $\kappa < \nu$, we say that an inverse system

$$\mathbb{I} \;=\; \langle\langle A_p \mid p \in \mathbb{D}\rangle,\ \langle f_{p,q} \mid p \leq_{\mathbb{D}} q\rangle\rangle$$

of sets over a directed set $\mathbb{D}$ is κ-*good* if the following statements hold.

- $\mathbb{D}$ is σ-directed and has cardinality at most κ.
- If $p \in \mathbb{D}$, then A_p has cardinality at most κ.
- The inverse limit

$$A_{\mathbb{I}} \;=\; \{(a_p)_{p\in\mathbb{D}} \in \prod_{p\in\mathbb{D}} A_p \mid f_{p,q}(a_q) = a_p \text{ for all } p \leq_{\mathbb{D}} q\}$$

  of $\mathbb{I}$ has cardinality ν.

Proposition

If $\kappa = \kappa^{\aleph_0}$, then there is a $(\kappa, 2^\kappa)$-good inverse system of sets.

Proof.

Define $\mathbb{D}$ to be the σ-directed set $\langle [\kappa]^{\aleph_0}, \subseteq \rangle$.

Given $u, v \in [\kappa]^{\aleph_0}$ with $u \subseteq v$, define

$$f_{u,v} : {}^{v}2 \longrightarrow {}^{u}2; \; s \longmapsto s \restriction u.$$

Let

$$\mathbb{I}(\kappa) \; = \; \langle \langle {}^{u}2 \mid u \in [\kappa]^{\aleph_0} \rangle, \langle f_{u,v} \mid u, v \in [\kappa]^{\aleph_0}, \; u \subseteq v \rangle \rangle$$

denote the resulting inverse system of sets over $\mathbb{D}$. Then the map $\left[x \mapsto (x \restriction u)_{u \in [\kappa]^{\aleph_0}} \right]$ is a bijection between the sets ${}^{\kappa}2$ and $A_{\mathbb{I}(\kappa)}$. □

If certain inner models compute κ^+ correctly, then we can use trees of height κ from these models to construct (κ, κ^+)-good inverse systems.

- Let κ be an infinite cardinal, $\mathbb{I}$ be a (κ,ν)-good inverse system of sets over a directed set $\mathbb{D}$ and $A_{\mathbb{I}}$ be the inverse limit of $\mathbb{I}$.

- Let F : Set $\longrightarrow$ Grp be the canonical functor that sends a set A to the corresponding free group $F(A)$. t Then

$$\mathbb{I}_G \;=\; \langle\langle F(A_p) \mid p \in \mathbb{D}\rangle, \langle F(f_{p,q}) \mid p \leq_{\mathbb{D}} q\rangle\rangle$$

  is a inverse system of groups.

- Show that the inverse limit

$$G_{\mathbb{I}} \;=\; \{(g_p)_{p\in\mathbb{D}} \in \prod_{p\in\mathbb{D}} F(A_p) \mid F(f_{p,q})(g_q) = g_p \textit{ for all } p \leq_{\mathbb{D}} q\}$$

  of $\mathbb{I}_G$ is contained in the class $\mathcal{AG}(\kappa)$.

- Show that the induced homomorphism $f_{\mathbb{I}} : F(A_{\mathbb{I}}) \longrightarrow G_{\mathbb{I}}$ with $f_{\mathbb{I}}(\mathrm{x}_{(a_p)_{p\in\mathbb{D}}}) = (\mathrm{x}_{a_p})_{p\in\mathbb{D}}$ is an isomorphism.

The above conclusion are consequences of the following results.

Theorem

Let κ be an infinite cardinal and

$$\mathbb{I} \;=\; \langle\langle G_q \mid q \in \mathbb{D}\rangle, \langle h_{q,r} \mid q \leq_{\mathbb{D}} r\rangle\rangle$$

be an inverse system of groups over a directed set $\mathbb{D}$ such that $|\mathbb{D}| \leq \kappa$ and $|G_p| \leq \kappa$ for all $p \in \mathbb{D}$.

Then the inverse limit of $\mathbb{I}$ is contained in the class $\mathcal{AG}(\kappa)$.

Theorem

Let $\mathbb{I}$ be an inverse system of sets over a directed set $\mathbb{D}$ with inverse limit $A_{\mathbb{I}}$, $\mathbb{I}_G$ be the induced inverse system of free groups with inverse limit $G_{\mathbb{I}}$ and $f_{\mathbb{I}} : \mathrm{F}(A_{\mathbb{I}}) \longrightarrow G_{\mathbb{I}}$ be the induced homomorphism.

If $\mathbb{D}$ is σ-directed, then $f_{\mathbb{I}}$ is an isomorphism.

We close this talk with questions raised by the above results.

Question

Is it consistent that there is a cardinal κ of uncountable cofinality with $F(2^\kappa) \notin \mathcal{AG}(\kappa)$?

Question

Is it consistent that there is a cardinal κ of uncountable cofinality with $F(\lambda) \notin \mathcal{AG}(\kappa)$ for every cardinal $\lambda > \kappa$?

Question

Is it consistent that there is a singular cardinal κ of uncountable cofinality with the property that there is no tree of cardinality and height κ with more than κ-many branches of order-type κ?

Thank you for listening!