The height of the automorphism tower of a centreless group

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Automorphism towers

Let G be a group with trivial centre. For each $g \in G$, we let ι_g denote the corresponding *inner automorphism*. The map

$$\iota_G : G \longrightarrow \mathrm{Aut}(G);\ g \mapsto \iota_g$$

is an embedding of groups that maps G onto the subgroup $\mathrm{Inn}(G)$ of all inner automorphisms of G. An easy computation shows that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$ and $\mathrm{Aut}(G)$ is again a group with trivial centre.

By iterating this process, we construct the *automorphism tower of* G.

Definition

A sequence $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ of groups is *an automorphism tower of a centreless group* G if the following statements hold.

- $G = G_0$.
- If $\alpha \in \mathrm{On}$, then G_α is a normal subgroup of $G_{\alpha+1}$ and the induced homomorphism

$$\varphi_\alpha : G_{\alpha+1} \longrightarrow \mathrm{Aut}(G_\alpha);\ g \mapsto \iota_g \restriction G_\alpha$$

  is an isomorphism.
- If $\alpha \in \mathrm{Lim}$, then $G_\alpha = \bigcup\{G_\beta \mid \beta < \alpha\}$.

In this definition we replaced $\mathrm{Aut}(G_\alpha)$ by an isomorphic copy $G_{\alpha+1}$ that contains G_α as a normal subgroup. This allows us to take unions at limit stages.

Given a centreless group G, we can construct an automorphism tower of G and it is unique up to isomorphisms that induce the identity on G.

We say that the automorphism tower of a centerless group terminates if there is an $\alpha \in \mathrm{On}$ with $G_\alpha = G_{\alpha+1}$ and therefore $G_\alpha = G_\beta$ for all $\beta \geq \alpha$. It is natural to ask whether the automorphism tower of every centreless group finally terminates.

Theorem (S. Thomas)

If G is an infinite centreless group of cardinality κ, then there is an $\alpha < (2^\kappa)^+$ with $G_\alpha = G_{\alpha+1}$.

This result allows us to make the following definitions.

Definition

Given a centreless group G, we let $\tau(G)$ denote the least ordinal α satisfying $G_\alpha = G_{\alpha+1}$ and call this ordinal the *height of the automorphism tower of G*. If κ is an infinite cardinal, then we define

$$\tau_\kappa = \mathrm{lub}\{\tau(G) \mid G \textit{ is a centreless group of cardinality } \kappa\}.$$

There are only 2^κ-many centreless groups of cardinality κ and therefore the above result implies "$\tau_\kappa < (2^\kappa)^+$" for all infinite cardinals κ. Simon Thomas also proved "$\tau_\kappa \geq \kappa^+$".

It is known that $(2^\kappa)^+$ is the best *cardinal* upper bound for τ_κ provable in ZFC for regular uncountable cardinal κ (Just/Shelah/Thomas). In addition, Simon Thomas showed that "$(\forall\kappa$ *regular*$)$ $\tau_\kappa < 2^\kappa$" is consistent with the axioms of ZFC.

The following problem is still open.

Problem (The automorphism tower problem)

Find a model M of ZFC and an infinite cardinal κ in M such that it is possible to "compute" the exact value of τ_κ in M.

In this talk I present a new upper bound for τ_κ.

A new upper bound

To define this bound, we need to introduce some notions from *abstract recursion theory*.

Let $\mathcal{L}_\in$ denote the language of set theory.

- An $\mathcal{L}_\in$-formula is Δ_0 if it is contained in the smallest class of $\mathcal{L}_\in$-formulae that contains every atomic formula and is closed under negation, conjunction and bounded existential quantification.
- An $\mathcal{L}_\in$-formula φ is Σ_1 if there is a Δ_0-formula φ_0 such that $\varphi \equiv (\exists x)\ \varphi_0(x, v_0, \ldots, v_{n-1})$.

Admissible sets

Definition

A set $\mathbb{A}$ is *admissible* if it has the following properties.

- $\mathbb{A}$ is nonempty, transitive and closed under pairing and union.
- $\langle \mathbb{A}, \in \rangle$ satisfies Δ_0-*Seperation*, i.e.

$$(\forall x_0, \ldots, x_n)(\exists y)(\forall z)\ [z \in y \leftrightarrow [y \in x_0 \wedge \varphi(x_0, \ldots, x_n, z)]]$$

  holds in $\langle \mathbb{A}, \in \rangle$ for every Δ_0-formula $\varphi(u_0, \ldots, u_n, v)$.
- $\langle \mathbb{A}, \in \rangle$ satisfies Δ_0-*Collection*, i.e.

$$\begin{aligned}(\forall x_0, \ldots, x_n)[(\forall y \in x_0)(\exists z)\ \varphi(x_0, \ldots, x_n, y, z) \\ \rightarrow (\exists w)(\forall y \in x_0)(\exists z \in w)\ \varphi(x_0, \ldots, x_n, y, z)]\end{aligned}$$

  holds in $\langle \mathbb{A}, \in \rangle$ for every Δ_0-formula $\varphi(u_0, \ldots, u_n, v_0, v_1)$.

The result

Definition

Given a set x, an ordinal α is x-admissible if there is an admissible set $\mathbb{A}$ with $x \in \mathbb{A}$ and $\alpha = \mathbb{A} \cap \mathrm{On}$.

We are now ready to state our result.

Theorem

Let κ be an infinite cardinal and α be $\mathcal{P}(\kappa)$-admissible. If $\tau_\kappa \neq \alpha + 1$, then $\tau_\kappa < \alpha$.

In the proof of this statement, we use results of Itay Kaplan and Saharon Shelah to establish a connection between automorphism towers and admissible set theory by representing automorphism towers as *inductive definitions* on a certain structure.

Inductive Definitions

Let $\mathcal{L}$ be a finite first order language, $\mathcal{M}$ be an $\mathcal{L}$-structure and $n < \omega$. We let $\mathcal{L}^n_{\mathcal{M}}$ denote the first order language that extends $\mathcal{L}$ by a new n-ary predicate $\dot{R}$ and a constant symbol $\dot{x}$ for every $x \in M$. If X is a subset of M^n, then we define $\mathcal{M}(X)$ to be the unique $\mathcal{L}^n_{\mathcal{M}}$-expansion of $\mathcal{M}$ with $\dot{R}^{\mathcal{M}(X)} = X$ and $\dot{x}^{\mathcal{M}(X)} = x$ for all $x \in M$.

Given an $\mathcal{L}^n_{\mathcal{M}}$-formula $\varphi \equiv \varphi(v_0, \ldots, v_{n-1})$ with n free variables, we define a sequence $\langle I^\alpha_\varphi \mid \alpha \in \mathrm{On}\rangle$ of subsets of M^n in the following way.

- $I^0_\varphi = \{\vec{x} \in M^n \mid \mathcal{M}(\emptyset) \models \varphi(\vec{x})\}$.
- $I^{\alpha+1}_\varphi = I^\alpha_\varphi \cup \{\vec{x} \in M^n \mid \mathcal{M}(I^\alpha_\varphi) \models \varphi(\vec{x})\}$ for all $\alpha \in \mathrm{On}$.
- $I^\alpha_\varphi = \bigcup_{\beta<\alpha} I^\beta_\varphi$ for all $\alpha \in \mathrm{Lim}$.

The following definition will allow us to construct automorphism towers as inductive definition.

Let G be a group and $\mathcal{L}_G$ be the language of group theory expanded by a constant symbol $\dot{g}$ for every $g \in G$. We let $\mathcal{T}_G^n$ denote the set of all $\mathcal{L}_G$-terms $t \equiv t(v_0, \ldots, t_{n-1})$ with exactly n free variables. If H is a group containing G, then we interpret H as an $\mathcal{L}_G$-structure and for each $\vec{h} \in H^n$ we define

$$\mathsf{qft}_{H,G}(\vec{h}) = \{t(\vec{v}) \in \mathcal{T}_G^n \mid H \models \text{“}t(\vec{h}) = \mathbb{1}\text{”}\}.$$

Theorem (I. Kaplan & S. Shelah)

If $\langle G_\alpha \mid \alpha \in \mathrm{On}\rangle$ is the automorphism tower of a centreless group G, then the map

$$\mathsf{qft}_{G_\alpha,G} : G_\alpha \longrightarrow \mathcal{P}(\mathcal{T}_G^1); \; g \longmapsto \mathsf{qft}_{G_\alpha,G}(\vec{g})$$

is injective for all $\alpha \in \mathrm{On}$.

If $g \in G_{\tau(G)}$ and $\alpha \in \mathrm{On}$, then $\mathsf{qft}_{G_\alpha,G}(g) = \mathsf{qft}_{G_{\tau(G)},G}(g)$ and we can call this set $\mathsf{qft}_G(g)$.

We reformulate results of Itay Kaplan and Saharon Shelah to our setting.

Theorem (I. Kaplan & S. Shelah)

There is a finite first order language $\mathcal{L}$ such that for every infinite cardinal κ there is an $\mathcal{L}$-structure $\mathcal{M}_\kappa = \langle M_\kappa; \dots \rangle$ and a $\mathcal{L}^4_{\mathcal{M}_\kappa}$-formula $\varphi \equiv \varphi(v_0, \dots, v_3)$ with the following properties.

- *$\mathcal{M}_\kappa$ is an element of every admissible set that contains $\mathcal{P}(\kappa)$.*
- *If G is a centreless group with domain κ and $\langle G_\alpha \mid \alpha \in \mathrm{On} \rangle$ is an automorphism tower of G, then $\mathcal{P}(\mathcal{T}^2_G) \subseteq M_\kappa$ and*

$$\langle x, y, z, \mathsf{qft}_G(\mathbb{1}_G) \rangle \in I^\alpha_\varphi \iff$$
$$(\exists g_0, g_1 \in G_\alpha)[x = \mathsf{qft}_G(g_0) \wedge y = \mathsf{qft}_G(g_1) \wedge z = \mathsf{qft}_G(g_0 \circ g_1)]$$

holds for all $\alpha \in \mathrm{On}$ and $x, y, z \in M_\kappa \times M_\kappa \times M_\kappa$.

Some admissible set theory

The *Recursion Theorem* holds in admissible sets and this allows us to prove the following statements.

Proposition

Let $\mathbb{A}$ be an admissible set, $\alpha = \mathrm{On} \cap \mathbb{A}$ and $\mathcal{L}$ be a finite first order language. If $\mathcal{M}$ is an $\mathcal{L}$-model with $\mathcal{M} \in \mathbb{A}$ and $\varphi(v_0, \ldots, v_{n-1})$ is a $\mathcal{L}^n_{\mathcal{M}}$-formula, then $I^\beta_\varphi \in \mathbb{A}$ for all $\beta < \alpha$ and the map

$$F_\varphi : \alpha \longrightarrow \mathbb{A};\ \beta \longmapsto I^\beta_\varphi$$

is definable in $\langle \mathbb{A}, \in \rangle$ by a Σ_1-formula using parameters.

Proposition

Let $\mathbb{A}$ be an admissible set, $\alpha = \mathrm{On} \cap \mathbb{A}$ and $F : \mathbb{A} \longrightarrow \alpha$ be a function that is definable in $\langle \mathbb{A}, \in \rangle$ by a Σ_1-formula using parameters. If $x \in \mathbb{A}$, then $F"x$ is bounded in α.

We are now ready to show how the proof of the theorem works.

Theorem

Let κ be an infinite cardinal and α be $\mathcal{P}(\kappa)$-admissible. If $\tau_\kappa \neq \alpha + 1$, then $\tau_\kappa < \alpha$.

Proof of the Theorem

Fix an admissible set $\mathbb{A}$ with $\alpha = \mathrm{On} \cap \mathbb{A}$ and $\mathcal{P}(\kappa) \in \mathbb{A}$. Let G be a centreless group of cardinality κ and $\langle G_\alpha \mid \alpha \in \mathrm{On} \rangle$ be an automorphism tower of G. We may assume that the domain of G is κ. For $\beta \in \mathrm{On}$, we set

$$\mathcal{G}_\beta = \{\mathsf{qft}_G(g) \mid g \in G_\beta\} \in \mathbb{A}.$$

We sketch the idea behind the proof of the theorem.

- *Given* $\pi \in G_{\alpha+1}$ *and* $\beta < \alpha$*, we show that the function*

$$c_\beta : \mathcal{G}_\beta \longrightarrow \alpha;\ \mathsf{qft}_G(g) \longmapsto \min\{\gamma < \alpha \mid g^\pi, g^{\pi^{-1}} \in G_\gamma\}$$

is definable in $\mathbb{A}$ *by a* Σ_1*-formula with parameters. In particular, the range of* c_β *is bounded below* α*.*

- *We construct a strictly increasing function* $f : \omega \longrightarrow \alpha$ *that is definable in* $\mathbb{A}$ *by a* Σ_1*-formula with parameters and satisfies* $c_{f(n)}(\mathsf{qft}_G(g)) < f(n+1)$ *for all* $g \in G_{f(n)}$ *and* $n < \omega$*. Then* $\alpha^* = \sup_{n<\omega} f(n) \in \mathrm{Lim} \cap \alpha$ *and* $\iota_\pi \restriction G_{\alpha^*} \in \mathrm{Aut}(G_{\alpha^*})$*.*
- *There is a* $\pi^* \in G_{\alpha^*+1}$ *with* $\iota_{\pi^*} \restriction G_{\alpha^*} = \iota_\pi \restriction G_{\alpha^*}$ *and this means* $\pi = \pi^* \in G_{\alpha^*+1} \subseteq G_\alpha$*, because*

$$\pi^* \pi^{-1} \in \mathrm{C}_{G_{\alpha+1}}(G) = \{\mathbb{1}_G\}.$$

- *This shows* $\tau_\kappa \leq \alpha + 1$.
- *If* $\tau_\kappa \leq \alpha$, *then the function*

$$t : \{\mathsf{qft}_G(\mathbb{1}_G) \mid G \textit{ is centreless group with domain } \kappa\} \to \alpha \; ;$$
$$\mathsf{qft}_G(\mathbb{1}_G) \longmapsto \tau(G)$$

is definable in $\mathbb{A}$ *by a* Σ_1*-formula with parameters and therefore its range is bounded in* α.

Details?

$\square$

Thank you for listening!

We let $\mathcal{M}_\kappa \in \mathbb{A}$ *denote the structure produced by the above theorem and* φ *be the corresponding formula.*
Given a term $\mathrm{t} \equiv \mathrm{t}(v_0, v_1)$ *in* $\mathcal{T}^2_G$*, we define*

$$\Psi_{\mathrm{t}} : \mathcal{P}(\mathcal{T}^2_G) \longrightarrow \mathcal{P}(\mathcal{T}^1_G);\ X \longmapsto \{\mathrm{s}(v_0) \mid \mathrm{s}(\mathrm{t}(v_0, v_1)) \in X\}.$$

Given $\beta \in \mathrm{On}$, $g, h \in G_\beta$ *and* $k \in \langle G \cup \{g, h\}\rangle$ *with* $k = \mathrm{t}(g, h)$*, it is easy to see that* $\mathsf{qft}_G(k) = \Psi_{\mathrm{t}}(\mathsf{qft}_G(g, h))$.
We define N *to be the smallest class of terms in* $\mathcal{T}^2_G$ *that contains* v_1 *and* $\dot{g}$ *for every* $g \in G$ *and is closed under* $\circ$, ${}^{-1}$ *and* $[\mathrm{t} \mapsto v_0 \mathrm{t} v_0^{-1}]$.
Let $\pi \in G_{\alpha+1}$ *and* $g \in G_\alpha$*. The function*

$$c : \mathrm{N} \longrightarrow \alpha;\ \mathrm{t} \longmapsto \min\{\beta < \alpha \mid \Psi_{\mathrm{t}}(\mathsf{qft}_G(\pi, g)) \in \mathcal{G}_\beta\}$$

is well-defined and definable in $\langle \mathbb{A}, \in\rangle$ *by a* Σ_1*-formula with parameters.*
Since $\mathrm{N} \in \mathbb{A}$*, the above Proposition shows that there is a* $\beta < \alpha$ *such that* $\Psi_{\mathrm{t}}(\mathsf{qft}_G(\pi, g)) \in \mathcal{G}_\beta$ *for all* $\mathrm{t} \in \mathrm{N}$ *and this implies* $g^\pi, g^{\pi^{-1}} \in G_\beta$.

If $h \in G_\alpha$, then the following statements are equivalent.

- $h = g^\pi$.
- *There is $X \subseteq \mathcal{T}_G^2$ and $\beta < \alpha$ with*
  - *X is closed under $\circ$, ${}^{-1}$, $[\mathrm{t} \mapsto v_0 \mathrm{t} v_0^{-1}]$ and reductions,*
  - $\mathsf{qft}_G(\pi) = \Psi_{v_0}(X)$, $\mathsf{qft}_G(g) = \Psi_{v_1}(X)$ *and* $\mathsf{qft}_G(h) = \Psi_{v_0 v_1 v_0^{-1}}(X)$,
  - *For all $\mathrm{t}_0, \mathrm{t}_1, \mathrm{t}_2 \in \mathrm{N}$, we have $\mathrm{t}_0\mathrm{t}_1\mathrm{t}_2^{-1} \in X$ if and only if*

$$\langle \Psi_{\mathrm{t}_0}(X), \Psi_{\mathrm{t}_1}(X), \Psi_{\mathrm{t}_2}(X), \mathsf{qft}_G(\mathbb{1}_G)\rangle \in I_\varphi^\beta,$$

  - $\mathsf{qft}_G(k) = \Psi_k(X)$ *for all $k \in G$.*

This shows that for each $\beta < \alpha$ the function

$$c_\beta : \mathcal{G}_\beta \longrightarrow \alpha;\ \mathsf{qft}_G(g) \longmapsto \min\{\gamma < \alpha \mid g^\pi, g^{\pi^{-1}} \in G_\gamma\}$$

is definable in $\mathbb{A}$ by a Σ_1-formula with parameters. □

Thank you for listening!