The infinite productivity of Knaster properties

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(partially joint work with Sean Cox)

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Introduction

The work presented in this talk studies the following well-known property of partial orders.

Definition

Given an uncountable regular cardinal κ, we say that a partial order $\mathbb{P}$ is κ-*Knaster* if every set of κ-many conditions in $\mathbb{P}$ contains a subset of cardinality κ consisting of pairwise compatible conditions.

This property clearly strengthens the κ-chain condition. It is typically used because of its nice product behavior.

For example, the product of a κ-Knaster partial order and a partial order satisfying the κ-chain condition satisfies the κ-chain condition.

We present another well-known example of such a productivity property that motivates the work presented in this talk.

Lemma

If κ is an uncountable regular cardinal, then the class of κ-Knaster partial orders is closed under products.

Proof.

Let $\mathbb{P}$ and $\mathbb{Q}$ be κ-Knaster partial orders and let A be a subset of $\mathbb{P} \times \mathbb{Q}$ of cardinality κ.

We may assume that for all $p \in \mathbb{P}$ and all $q \in \mathbb{Q}$, the sets

$$A_p = \{s \in \mathbb{Q} \mid \langle p, s \rangle \in A\} \text{ and } A^q = \{r \in \mathbb{P} \mid \langle r, q \rangle \in A\}$$

have cardinality less than κ.

In this situation, there is an injective partial function $F : \mathbb{P} \xrightarrow{part} \mathbb{Q}$ of cardinality κ with $F \subseteq A$. Pick $D \in [\mathrm{dom}(F)]^\kappa$ consisting of pairwise compatible conditions in $\mathbb{P}$ and $R \in [\mathrm{ran}(F \restriction D)]^\kappa$ consisting of pairwise compatible conditions in $\mathbb{Q}$. Then the set $F \restriction (F^{-1}[R])$ consists of pairwise compatible conditions in the product $\mathbb{P} \times \mathbb{Q}$. $\square$

With the help of the Δ-System Lemma, the above lemma yields the following result.

Corollary

If κ is an uncountable regular cardinal, then the class of κ-Knaster partial orders is closed under finite support products. $\square$

The above observation leads to the question whether there are uncountable regular cardinals κ with the property that the class of κ-Knaster partial orders is closed under products with larger supports.

It turns out that this productivity holds for certain large cardinals.

Proposition

Let κ be a weakly compact cardinal and let $\lambda < \kappa$ be a cardinal. If $\langle \mathbb{P}_\alpha \mid \alpha < \lambda \rangle$ is a sequence of partial orders satisfying the κ-chain condition, then the corresponding full support product $\vec{\mathbb{P}} = \prod_{\alpha<\lambda} \mathbb{P}_\alpha$ is κ-Knaster.

Proof.

Fix a sequence $\langle \vec{p}_\gamma \in \vec{\mathbb{P}} \mid \gamma < \kappa \rangle$ and let $c : [\kappa]^2 \longrightarrow \lambda + 1$ denote the unique function such that the following statements hold for all $\gamma < \delta < \kappa$:

- If $c(\gamma, \delta) = \lambda$, then the conditions $\vec{p}_\gamma$ and $\vec{p}_\delta$ are compatible in $\vec{\mathbb{P}}$.
- Otherwise, $c(\gamma, \delta)$ is the minimal $\alpha < \lambda$ with the property that the conditions $\vec{p}_\gamma(\alpha)$ and $\vec{p}_\delta(\alpha)$ are incompatible in $\mathbb{P}_\alpha$.

Since κ is weakly compact, there is $H \in [\kappa]^\kappa$ and $\beta \leq \lambda$ with $c[H]^2 = \{\beta\}$.

Then $\beta = \lambda$, because otherwise $\mathbb{P}_\beta$ would contain an antichain of size κ.

This shows that the sequence $\langle \vec{p}_\gamma \mid \gamma \in H \rangle$ consists of pairwise compatible conditions. $\square$

Corollary

If κ is a weakly compact cardinal, then the class of κ-Knaster partial orders is closed under λ-support products for every $\lambda < \kappa$. $\square$

It is now natural to ask whether this productivity characterizes weak compactness.

Question

Are the following statements equivalent for every uncountable regular cardinal κ?

- κ is weakly compact.
- The class of κ-Knaster partial orders is closed under λ-support products for every $\lambda < \kappa$.

In the following, we present results showing that the axioms of $\mathbf{ZFC}$ do not answer this question.

Ascending paths

The following result shows that the above statement about the productivity of the Knaster property characterizes weak compactness in canonical inner models.

Theorem (L.)

Let $\mathrm{L}[E]$ be a Jensen-style extender model. In $\mathrm{L}[E]$, the following statements are equivalent for every uncountable regular cardinal κ:

- *κ is weakly compact.*
- *The class of κ-Knaster partial orders is closed under λ-support products for all $\lambda < \kappa$.*

Moreover, if κ is not the successor of a subcompact cardinal in $\mathrm{L}[E]$, then the above statements are also equivalent to the following statement:

- *The class of κ-Knaster partial orders is closed under countable support products.*

We briefly outline the concepts used in the proof of this result.

The partial orders constructed in the proof of the above result will be of the following form.

Definition

Given a tree $\mathbb{T}$, we let $\mathbb{P}(\mathbb{T})$ denote the partial order whose conditions are finite partial functions $p : \mathbb{T} \xrightarrow{part} \omega$ that are injective on chains in $\mathbb{T}$ and whose ordering is given by reversed inclusion.

In order to show that partial orders of the form $\mathbb{P}(\mathbb{T})$ can be κ-Knaster for certain trees of height κ, we need to introduce the *nonstationary ideal* of a tree $\mathbb{T}$.

Definition (Todorčević)

Let κ be an uncountable regular cardinal, let S be a subset of κ and let $\mathbb{T}$ be a tree of height κ.

- A map $r : \mathbb{T} \restriction S \longrightarrow \mathbb{T}$ is *regressive* if $r(t) <_{\mathbb{T}} t$ holds for $t \in \mathbb{T} \restriction S$ that is not minimal in $\mathbb{T}$.
- We say that S *is nonstationary with respect to* $\mathbb{T}$ if there is a regressive map $r : \mathbb{T} \restriction S \longrightarrow \mathbb{T}$ with the property that for every $t \in \mathbb{T}$ there is a function $c_t : r^{-1}\{t\} \longrightarrow \kappa_t$ such that κ_t is a cardinal smaller than κ and c_t is injective on chains in $\mathbb{T}$.

Lemma (Cox-L.)

Let κ be an uncountable regular cardinal and let $\mathbb{T}$ be a normal κ-Aronszajn tree. If there is a stationary subset S of κ such that S is nonstationary with respect to $\mathbb{T}$, then the partial order $\mathbb{P}(\mathbb{T})$ is κ-Knaster.

In order to see that the productivity of the Knaster property may fail for partial orders of the form $\mathbb{P}(\mathbb{T})$, we use the following definition that directly generalizes the notion of a cofinal branch through a tree.

Definition

Given an uncountable regular cardinal κ, a tree $\mathbb{T}$ of height κ and a cardinal $\lambda > 0$, a sequence $\langle b_\gamma : \lambda \longrightarrow \mathbb{T}(\gamma) \mid \gamma < \kappa \rangle$ of functions is an *ascending path of width λ through* $\mathbb{T}$ if for all $\bar{\gamma} < \gamma < \kappa$, there are $\alpha, \bar{\alpha} < \lambda$ such that $b_{\bar{\gamma}}(\bar{\alpha}) <_{\mathbb{T}} b_\gamma(\alpha)$.

Proposition

Let κ be an uncountable regular cardinal, let $\mathbb{T}$ be a tree of height κ and let λ be an infinite cardinal. If there is an ascending path of width λ through $\mathbb{T}$, then the full support product $\prod_\lambda \mathbb{P}(\mathbb{T})$ does not satisfy the κ-chain condition.

The following result yields the above characterizations of weakly compact cardinals in canonical inner models.

Its proof relies on results of Todorčević on walks on ordinals and results of Schimmerling and Zeman on the existence of square sequences in canonical inner models that extend seminal results of Jensen.

Theorem

Assume that V *is a Jensen-style extender model. Let* κ *be an uncountable regular cardinal that is not weakly compact and not the successor of a subcompact cardinal. Then there is a normal* κ*-Aronszajn tree* $\mathbb{T}$ *and a stationary subset* S *of* S^κ_ω *such that* S *is nonstationary with respect to* $\mathbb{T}$ *and there is an ascending path of width* ω *through* $\mathbb{T}$*.*

Layered partial orders

Next, we present a result showing that the above statement about the productivity of the Knaster property can consistently hold at non-weakly compact cardinals.

This result uses the following strengthening of the Knaster property.

Definition

Let κ be an uncountable regular cardinal, let $\lambda \geq \kappa$ be a cardinal, let $\mathcal{F}$ be a normal filter on $\mathcal{P}_\kappa(\lambda)$ and let $\mathbb{P}$ be a partial order.

- We say $\mathbb{P}$ is $\mathcal{F}$-layered, if it has cardinality at most λ and

  $$\{a \in \mathcal{P}_\kappa(\lambda) \mid s[a] \textit{ is a regular suborder of } \mathbb{P}\} \in \mathcal{F}$$

  holds for every surjection $s : \lambda \longrightarrow \mathbb{P}$.
- We say that $\mathbb{P}$ is *completely $\mathcal{F}$-layered* if every subset of $\mathbb{P}$ of cardinality at most λ is contained in a regular suborder of $\mathbb{P}$ of cardinality at most λ and every regular suborder of $\mathbb{P}$ of size at most λ is $\mathcal{F}$-layered.

Lemma (Cox-L.)

In the situation of the above definition, if $\lambda = \lambda^{<\kappa}$ holds, then every completely $\mathcal{F}$-layered partial order is κ-Knaster.

Theorem (Cox-L.)

Let κ be a weakly compact cardinal and let $\mathcal{F}_{wc}$ denote the weakly compact filter on $\mathcal{P}_\kappa(\kappa)$. Then a partial order $\mathbb{P}$ satisfies the κ-chain condition if and only if $\mathbb{P}$ is completely $\mathcal{F}_{wc}$-layered.

Lemma (Cox-L.)

In the situation of the above definition, assume that $\kappa = \lambda$ is inaccessible and $\nu < \kappa$ is a cardinal with

$$\{a \in \mathcal{P}_\kappa(\kappa) \mid \varphi[{}^\nu a] \subseteq a\} \in \mathcal{F}$$

for every function $\varphi : {}^\nu\kappa \longrightarrow \kappa$. Then the class of completely $\mathcal{F}$-layered partial orders is closed under ν-support products.

Theorem (Cox-L.)

Let κ be an inaccessible cardinal with the property that there is a κ-Souslin tree $\mathbb{T}$ with $\mathbb{1}_{\mathbb{T}} \Vdash$ "$\check{\kappa}$ is weakly compact". Set

$$\mathcal{F} \ = \ \{A \subseteq \mathcal{P}_\kappa(\kappa) \mid \mathbb{1}_{\mathbb{T}} \Vdash \text{“}\check{A} \in \mathcal{F}_{wc}\text{''}\}.$$

Then:

- *$\mathcal{F}$ is a normal filter on $\mathcal{P}_\kappa(\kappa)$ with $\{a \in \mathcal{P}_\kappa(\kappa) \mid \varphi[{}^\nu a] \subseteq a\} \in \mathcal{F}$ for all $\nu < \kappa$ and every function $\varphi : {}^\nu\kappa \longrightarrow \kappa$.*
- *Every κ-Knaster partial order is completely $\mathcal{F}$-layered.*

In combination with a classical result of Kunen, this result yields the following theorem that shows that the above potential characterization of weak compactness can consistently fail.

Theorem (Cox-L.)

If κ is a weakly compact cardinal, then there is a partial order $\mathbb{P}$ such that the following statements hold in $\mathrm{V}[G]$ whenever G is $\mathbb{P}$-generic over V.

- *κ is inaccessible and not weakly compact.*
- *For every $\nu < \kappa$, the class of κ-Knaster partial orders is closed under ν-support products.*

Thank you for listening!