Forcings that characterize large cardinals

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(joint work in progress with Peter Holy)

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Introduction

Many important results in set theory show that the relative consistency of set-theoretic principles can be established by collapsing a large cardinal to be the successor of a smaller cardinal.

The following classical result is an important example of such an argument.

Theorem (Solovay)

If λ is an uncountable regular cardinal, $\kappa > \lambda$ is a Mahlo cardinal and G is $\mathrm{Col}(\lambda, {<}\kappa)$-generic over the ground model V, then $\square_\lambda$ fails in $\mathrm{V}[G]$.

In many important cases, it is also possible to show that the considered principle implies that the relevant successor cardinal is a large cardinal in some canonical inner model. By combining both types of arguments, it is possible to prove equiconsistency results for such principles.

Theorem (Jensen)

If $\square_\lambda$ fails for an uncountable regular cardinal λ, then λ^+ is a Mahlo cardinal in Gödel's constructible universe L.

In contrast, it is not always possible to recover the corresponding large cardinal in the ground model of the collapse, i.e. it is sometimes also possible to obtain the considered principle by collapsing a cardinal without the given large cardinal property.

Theorem (Larson)

Assume that the Proper Forcing Axiom PFA *holds. If* $\mathbb{P}$ *is a* $<\aleph_2$*-directed closed partial order and* G *is* $\mathbb{P}$*-generic over* V*, then* PFA *holds in* $\mathrm{V}[G]$*.*

Theorem (Todorčević)

If PFA *holds, then* $\square_\lambda$ *fails for every uncountable cardinal* λ*.*

Corollary

It is consistent (from large cardinals) that there is an inaccessible, non-Mahlo cardinal κ *with the property that* $\square_{\aleph_2}$ *fails in* $\mathrm{V}[G]$*, whenever* G *is* $\mathrm{Col}(\aleph_2, <\kappa)$*-generic over* V*.*

In this talk, we will consider the question whether certain collapse forcings characterize large cardinal properties through the validity of set-theoretic principles in their forcing extensions, in the sense that the axioms of $\mathbf{ZFC}$ prove that the collapse forces the principle to hold if and only if the collapsed cardinal possess the corresponding large cardinal property in the ground model.

More specifically, we look at a sequence $\langle \mathbb{P}_\alpha \mid \alpha \in \mathrm{On} \rangle$ of partial orders and a class Γ of cardinals that are both definable without parameters. Then we ask whether there is a formula $\varphi(v)$ with the property that the statement

$$\forall \kappa \in \mathrm{On} \; [\kappa \in \Gamma \;\longleftrightarrow\; \mathbb{1}_{\mathbb{P}_\kappa} \Vdash \varphi(\check{\kappa})]$$

is provable from the axioms of $\mathbf{ZFC}$.

The following trivial observation shows that such a characterization does not exist for the Levy collapse and the class of inaccessible cardinals.

Proposition

Fix $n < \omega$. Assume that there is a formula $\varphi(v)$ with the property that

$$\mathbf{ZFC} \vdash \forall\kappa \in \mathrm{On}\ [\kappa \text{ is an inaccessible cardinal} \longleftrightarrow \mathbb{1}_{\mathrm{Col}(\aleph_n, <\kappa)} \Vdash \varphi(\check{\kappa})].$$

Then the axioms of $\mathbf{ZFC}$ *prove that there are no inaccessible cardinals.*

Proof.

Assume that V is a model of $\mathbf{ZFC}$ + "*there is an inaccessible cardinal*".

Let κ be an inaccessible cardinal and let G be $\mathrm{Col}(\aleph_n, <\kappa)$-generic over V.

Since the partial orders $\mathrm{Col}(\aleph_n, <\kappa)$ and $\mathrm{Col}(\aleph_n, <\kappa) \times \mathrm{Col}(\aleph_n, <\kappa)$ are forcing-equivalent in V, we can find G_0 $\mathrm{Col}(\aleph_n, <\kappa)$-generic over V and G_1 $\mathrm{Col}(\aleph_n, <\kappa)$-generic over $\mathrm{V}[G_0]$ with $\mathrm{V}[G] = \mathrm{V}[G_0, G_1]$.

By our assumptions, we know that $\varphi(\kappa)$ holds in $\mathrm{V}[G]$ and, by the homogeneity of $\mathrm{Col}(\aleph_n, <\kappa)$ in $\mathrm{V}[G_0]$, this implies that $\mathbb{1}_{\mathrm{Col}(\aleph_n, <\kappa)} \Vdash \varphi(\check{\kappa})$ holds in $\mathrm{V}[G_0]$.

In this situation, our assumption implies that κ is inaccessible in $\mathrm{V}[G_0]$, a contradiction. $\square$

In the remainder of this talk, I will present results that show that a small modification of the Levy collapse produces a partial order that can be used to characterize many important types of large cardinals in the above way.

More specifically, we will show that the partial order

$$\mathrm{Add}(\aleph_0, 1) * \mathrm{Col}(\aleph_1, {<}\kappa)$$

that adds a Cohen real and then collapses κ to become $\aleph_2$ can characterize inaccessible, Mahlo, weakly compact, indescribable, measurable, supercompact and huge cardinals in the above sense.

Similar results also hold for other classes of partial orders. In joint work with Ana Njegomir, we are working on characterizations of large cardinals through *Neeman's pure side condition forcing*.

Characterizing larger large cardinals

The following basic observation is the starting point of our analysis.

It will allow us to restrict our characterizations to regular, $\aleph_0$-inaccessible cardinals greater than $\aleph_1$.

Proposition

The following statements are equivalent for every ordinal λ:

- λ *is a regular, $\aleph_0$-inaccessible cardinal greater than $\aleph_1$.*
- $\mathbb{1}_{\mathrm{Add}(\aleph_0,1)*\mathrm{Col}(\aleph_1,<\lambda)} \Vdash$ "$2^{\aleph_0} = \aleph_1 \ \wedge \ \check{\lambda} = \aleph_2$".

With the help of results of Apter and Hamkins on the *approximation* and *cover property*, it is now possible to derive the desired characterization of large cardinal properties defined through the existence of elementary embeddings of V into some inner model.

The forward implication of the following result is a classical result of Jech, Magidor, Mitchell and Prikry. The backward implication is proven using results of Hamkins on the pullback of elementary embeddings in forcing extensions to the ground model.

Theorem

Let κ be a regular, $\aleph_0$-inaccessible cardinal greater than $\aleph_1$. Then κ is a measurable cardinal if and only if

$$\mathbb{1}_{\mathrm{Add}(\aleph_0,1)*\mathrm{Col}(\aleph_1,<\kappa)} \Vdash \text{"There is a precipitous ideal } I \text{ on } \aleph_2 \text{ such that } \mathcal{P}(\aleph_2)/I \text{ contains a dense } \sigma\text{-closed suborder."}$$

Small variations of the proof of this result allow us to derive similar characterizations for supercompact and huge cardinals.

Theorem

Let κ be a regular, $\aleph_0$-inaccessible cardinal greater than $\aleph_1$.

Then κ is a supercompact cardinal if and only if

$\mathbb{1}_{\mathrm{Add}(\aleph_0,1)*\mathrm{Col}(\aleph_1,<\kappa)} \Vdash$ "*For every $\aleph_1$-inaccessible cardinal $\lambda > \aleph_2$, there is a precipitous ideal I on $\mathcal{P}_{\aleph_2}(\lambda)$ such that $\mathcal{P}(\mathcal{P}_\kappa(\lambda))/I$ contains a dense σ-closed suborder.*"

Theorem

Let κ be a regular, $\aleph_0$-inaccessible cardinal greater than $\aleph_1$.

Then κ is a huge cardinal if and only if

$\mathbb{1}_{\mathrm{Add}(\aleph_0,1)*\mathrm{Col}(\aleph_1,<\kappa)} \Vdash$ "*There is an inaccessible cardinal λ and a precipitous ideal I on $[\lambda]^{\aleph_2}$ with the property that $\mathcal{P}([\lambda]^{\aleph_2})/I$ contains a dense σ-closed suborder.*"

Characterizing smaller large cardinals

Next, we consider characterizations of smaller large cardinals through partial orders of the form $\mathrm{Add}(\aleph_0, 1) * \mathrm{Col}(\aleph_1, {<}\kappa)$.

These characterization rely on combinatoral characterization of these cardinals through the non-existence of certain trees and a result of L.-Schlicht on the non-existence of certain subtrees in forcing extensions with the σ-cover property.

Theorem

Let κ be a regular, $\aleph_0$-inaccessible cardinal greater than $\aleph_1$.

Then κ is an inaccessible cardinal if and only if

$$\mathbb{1}_{\mathrm{Add}(\aleph_0,1)*\mathrm{Col}(\aleph_1,<\kappa)} \Vdash \text{"}\textit{Every tree } \mathbb{T} \textit{ of height } \omega_1 \textit{ with more than } \aleph_1\textit{-many branches contains a subtree isomorphic to } {}^{<\omega_1}2.\text{"}$$

Theorem

Let κ be an inaccessible cardinal.

Then κ is a Mahlo cardinal if and only if

$$\mathbb{1}_{\mathrm{Add}(\aleph_0,1)*\mathrm{Col}(\aleph_1,<\kappa)} \Vdash \text{"Every special } \aleph_2\text{-Aronszajn tree contains a subtree isomorphic to } {}^{<\omega_1}2.\text{"}$$

Theorem

Let κ be an inaccessible cardinal.

Then κ is a weakly compact cardinal if and only if

$$\mathbb{1}_{\mathrm{Add}(\aleph_0,1)*\mathrm{Col}(\aleph_1,<\kappa)} \Vdash \text{"Every } \aleph_2\text{-Aronszajn tree contains a subtree isomorphic to } {}^{<\omega_1}2.\text{"}$$

Characterizing indescribable cardinals

The above results leave open the questions whether there are similar characterizations of large cardinal properties in between weak compactness and measurability. In the remainder of this talk, we consider characterizations of Π^1_n-indescribable cardinals for $n > 1$.

Since these cardinals cannot be characterized by the non-existence of certain trees or the existence of precipitous ideals in forcing extensions, a different approach is needed for such results.

Our approach is to characterize various types of large cardinals as the images of certain elementary embeddings and then consider liftings of these embeddings in generic extensions of the ground model.

It turns out that this approach leads to a characterization of Π^1_n-indescribability through the partial order $\mathrm{Add}(\aleph_0, 1) * \mathrm{Col}(\aleph_1, {<}\kappa)$.

Moreover, most of the above characterizations can also be obtained through this approach.

The following result provides a new characterization of Π^1_n-indescribability.

Its proof relies on results of Hauser that characterize these cardinals through the existence of certain elementary embeddings of κ-models.

Theorem

The following statements are equivalent for every $0 < n < \omega$ and every uncountable regular cardinal κ:

- *κ is Π^1_n-indescribable.*
- *For every cardinal $\theta > \kappa$, there is an elementary embedding $j : M \longrightarrow \mathrm{H}(\theta)$ such that $j(\mathrm{crit}(j)) = \kappa$, $\mathrm{crit}(j)$ is an inaccessible cardinal and $\mathrm{H}(\mathrm{crit}(j)^+)^M \prec_{\Sigma_n} \mathrm{H}(\mathrm{crit}(j)^+)$.*

By considering liftings of elementary embeddings of the above type, we can now phrase our new characterization of Π^1_n-indescribability.

Definition

Given $0 < n < \omega$ and a regular cardinal λ, we say that λ^+ is *internally closed Π^1_n-indescribable* if for every regular cardinal $\theta > \lambda^+$ and every $x \in \mathrm{H}(\theta)$, there is an elementary embedding $j : M \longrightarrow \mathrm{H}(\theta)$ and a transitive model $\mathbf{ZFC}^-$-model N such that the following statements hold:

- There is a $<\lambda$-closed partial order $\mathbb{P}$ in N with the property that $\mathrm{H}(\theta)$ is a $\mathbb{P}$-generic extension of N.
- We have $j(\mathrm{crit}(j)) = \lambda^+$, $x \in \mathrm{ran}(j)$, $\mathrm{crit}(j)$ is a regular cardinal in N and $\mathrm{H}(\mathrm{crit}(j)^+)^M \prec_{\Sigma_n} \mathrm{H}(\mathrm{crit}(j)^+)^N$.

Theorem

The following statements are equivalent for every $0 < n < \omega$ and every inaccessible cardinal κ:

- *κ is Π^1_n-indescribable.*
- $\mathbb{1}_{\mathrm{Add}(\aleph_0,1)*\mathrm{Col}(\aleph_1,<\kappa)} \Vdash$ "*$\aleph_2$ is internally closed Π^1_n-indescribable*".

Thank you for listening!