Lightface Σ^1_1-subsets of ${}^{\omega_1}\omega_1$

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Introduction

Assume that κ is an uncountable regular cardinal.

The *generalized Baire space of* κ is the set ${}^\kappa\kappa$ of all functions from κ to κ equipped with the topology whose basic open sets are of the form

$$N_s \;=\; \{x \in {}^\kappa\kappa \mid s \subseteq x\}$$

for some s contained in the set ${}^{<\kappa}\kappa$ of all functions $t : \alpha \longrightarrow \kappa$ with $\alpha < \kappa$.

A subset of ${}^\kappa\kappa$ is a $\mathbf{\Sigma}^1_1$*-set* if it is equal to the projection of a closed subset of ${}^\kappa\kappa \times {}^\kappa\kappa$.

A subset X of ${}^\kappa\kappa$ is a $\mathbf{\Delta}^1_1$*-set* if both X and ${}^\kappa\kappa \setminus X$ are $\mathbf{\Sigma}^1_1$-sets.

The following folklore result shows that the class of $\mathbf{\Sigma}^1_1$-sets contains many interesting objects.

Proposition

As subset of ${}^\kappa\kappa$ is a $\mathbf{\Sigma}^1_1$-set if and only if it is definable over the structure $\langle \mathrm{H}(\kappa^+), \in \rangle$ by a Σ_1-formula with parameters.

This observation can also be used to show that many basic questions about the class of $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$ are not settled by the axioms of $\mathbf{ZFC}$ together with large cardinal axioms.

In the following, we discuss two important examples of such questions.

Separating the club filter from the nonstationary ideal

Define

$$\mathsf{CLUB}_\kappa \;=\; \{x \in {}^\kappa\kappa \mid \exists C \subseteq \kappa \textit{ club } \forall \alpha \in C \; x(\alpha) > 0\}$$

and

$$\mathsf{NS}_\kappa \;=\; \{x \in {}^\kappa\kappa \mid \exists C \subseteq \kappa \textit{ club } \forall \alpha \in C \; x(\alpha) = 0\}.$$

Then the *club filter* CLUB_κ *on* κ and the *non-stationary ideal* NS_κ *on* κ are disjoint $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$.

In the light of the *Lusin Separation Theorem theorem* it is natural to ask the following question.

Question

Is there a $\mathbf{\Delta}^1_1$-subset A of ${}^\kappa\kappa$ that separates CLUB_κ from NS_κ, in the sense that $\mathsf{CLUB}_\kappa \subseteq A \subseteq {}^\kappa\kappa \setminus \mathsf{NS}_\kappa$ holds?

For a large class of cardinals it is known that the above question is not settled by the axioms of $\mathbf{ZFC}$ together with large cardinal axioms.

With the help of results of Mekler-Shelah and Hyttinen-Rautila, the following theorem produces positive answers to the above question.

Theorem (Friedman-Hyttinen-Kulikov)

Assume that the GCH *holds and* κ *is not the successor of a singular cardinal. In a cofinality preserving forcing extension of the ground model, there is a* $\mathbf{\Delta}^1_1$*-subset of* ${}^\kappa\kappa$ *that separates* CLUB_κ *from* NS_κ*.*

In contrast, it is possible to combine results of Halko-Shelah and Friedman-Hyttinen-Kulikov (or L.-Schlicht) to show that a negative answer to the above question is also consistent.

Theorem

If $\kappa = \kappa^{<\kappa}$ *and* G *is* $\mathrm{Add}(\kappa, \kappa^+)$*-generic over* V*, then there is no* $\mathbf{\Delta}^1_1$*-subset* A *of* ${}^\kappa\kappa$ *that separates* $\mathsf{CLUB}(\kappa)$ *from* $\mathsf{NS}(\kappa)$ *in* $\mathrm{V}[G]$*.*

Existence of $\mathbf{\Sigma}^1_1$-definable well-orders

Given $x \subseteq \omega$, a classical theorem of Mansfield shows that there is a well-ordering of the reals that is definable over $\langle \mathrm{H}(\omega_1), \in\rangle$ by a Σ_1-formula with parameter x if and only if all reals are contained in $\mathrm{L}[x]$.

In particular, the existence of large cardinals implies that no well-ordering of the reals is definable in this way.

Since $\mathbf{\Sigma}^1_1$-definable well-orderings of ${}^\kappa\kappa$ exist in models of the form $\mathrm{L}[x]$ with $x \subseteq \kappa$, Mansfield's result motivates the following questions.

Question

What are the provable consequences of the existence of a $\mathbf{\Sigma}^1_1$-definable well-ordering of ${}^\kappa\kappa$?

Question

Does the existence of $\mathbf{\Sigma}^1_1$-definable well-orderings of ${}^\kappa\kappa$ imply the non-existence of certain large cardinals above κ?

It turns out that the existence of $\mathbf{\Sigma}^1_1$-definable well-orderings of ${}^\kappa\kappa$ is independent from $\mathbf{ZFC}$ together with large cardinal axioms.

Theorem

If $\kappa = \kappa^{<\kappa}$ and G is $\mathrm{Add}(\kappa, \kappa^+)$-generic over V, then no well-ordering of $({}^\kappa\kappa)^{\mathrm{V}[G]}$ is definable over the structure $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in \rangle$ by a formula with parameters.

Theorem (Holy-L.)

Assume that $\kappa = \kappa^{<\kappa}$ holds and 2^κ is regular. Then there is a partial order $\mathbb{P}$ with the following properties.

- *$\mathbb{P}$ is $<\kappa$-closed and forcing with $\mathbb{P}$ preserves cofinalities less than or equal to 2^κ and the value of 2^κ.*
- *If G is $\mathbb{P}$-generic over the ground model V, then there is a well-ordering of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ that is definable over the structure $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in \rangle$ by a Σ_1-formula with parameters.*

The results presented above show that there are many interesting questions about $\mathbf{\Sigma}^1_1$-subsets that are not settled by the axioms of $\mathbf{ZFC}$ together with large cardinal axioms. In particular, these axioms do not provide a nice structure theory for the class of $\mathbf{\Sigma}^1_1$-sets.

If we do not want to restrict ourselves to consistency proofs of individual structural statements, then there are two obvious ways to proceed:

- Consider other canonical extensions of $\mathbf{ZFC}$ that provide a rich structure theory for $\mathbf{\Sigma}^1_1$-sets (Example: *Closed maximality principles*).
- Consider smaller classes of sets that still contain many interesting subsets and have the property that the axioms of $\mathbf{ZFC}$ together with large cardinal axioms prove a nice structure theory for these classes.

We will discuss examples of the second approach.

Lightface Σ^1_1-subsets of ${}^{\omega_1}\omega_1$

Definition

Let κ be an uncountable regular cardinal and A be a subset of ${}^\kappa\kappa$.

- We say that A is a Σ^1_1-*set* if A is definable over the structure $\langle \mathrm{H}(\kappa^+), \in\rangle$ by a Σ_1-formula with parameter κ.
- We say that A is a Δ^1_1-*set* if both A and ${}^\kappa\kappa \setminus A$ are Σ^1_1-sets.

This class contains many interesting and important subsets of ${}^\kappa\kappa$. For example, it contains the club filter, the nonstationary ideal and the isomorphism relations for structures of cardinality κ.

We present results that suggest that this class is suitable for the second approch outlined above in the case where either $\kappa = \omega_1$ or κ itself is a large cardinal.

Theorem

Assume that there is a Woodin cardinal with a measurable cardinal above it. Then there is no Σ^1_1-definable well-ordering of ${}^{\omega_1}\omega_1$.

Using results of Woodin on the Π_2-maximality of the $\mathbb{P}_{max}$-extension of $\mathrm{L}(\mathbb{R})$, it is easy to derive the conclusion of this theorem from the existence of infinitely many Woodin cardinals with a measurable cardinal above them.

Theorem

Assume that there is a Woodin cardinal with a measurable cardinal above it. Then there is no Δ^1_1-subset of ${}^{\omega_1}\omega_1$ that separates CLUB_{ω_1} from NS_{ω_1}.

Friedman and Wu used Woodin's $\mathbb{P}_{max}$-results to show that the existence of infinitely many Woodin cardinals with a measurable cardinal above them implies that the club filter CLUB_{ω_1} is not definable over $\langle \mathrm{H}(\omega_2), \in \rangle$ by a Π_1-formula with parameter ω_1.

We start by discussing the proof of the second theorem.

The statement of the theorem follows from the following result.

Theorem

Assume $M_1^\#(X)$ exists for every $X \subseteq \omega_1$. Then the following statements hold for every Σ^1_1-subset A of ${}^{\omega_1}\omega_1$.

- $A \setminus \mathsf{NS}_{\omega_1}$ *is contained in the closure of* $A \cap \mathsf{CLUB}_{\omega_1}$ *in* ${}^{\omega_1}\omega_1$.
- $A \setminus \mathsf{CLUB}_{\omega_1}$ *is contained in the closure of* $A \cap \mathsf{NS}_{\omega_1}$ *in* ${}^{\omega_1}\omega_1$.

We sketch how the second theorem can be derived from this theorem.

Theorem

If $M_1^\#(X)$ exists for every $X \subseteq \omega_1$, then the following statements hold for every Σ^1_1-subset A of ${}^{\omega_1}\omega_1$.

- *$A \setminus \mathsf{NS}_{\omega_1}$ is contained in the closure of $A \cap \mathsf{CLUB}_{\omega_1}$ in ${}^{\omega_1}\omega_1$.*
- *$A \setminus \mathsf{CLUB}_{\omega_1}$ is contained in the closure of $A \cap \mathsf{NS}_{\omega_1}$ in ${}^{\omega_1}\omega_1$.*

Theorem

Assume that there is a Woodin cardinal with a measurable cardinal above it. Then there is no Δ^1_1-subset of ${}^{\omega_1}\omega_1$ that separates CLUB_{ω_1} from NS_{ω_1}.

Proof.

Assume that there is a Woodin cardinal with a measurable cardinal above it. Then $M_1^\#(X)$ exists for every $X \subseteq \omega_1$. Let A and B be Σ^1_1-subsets of ${}^{\omega_1}\omega_1$ with $A \cap B = \emptyset$, $\mathsf{CLUB}_{\omega_1} \subseteq A$ and $\mathsf{NS}_{\omega_1} \subseteq B$.

By the above theorem, we know that $A \setminus \mathsf{CLUB}_{\omega_1}$ is contained in the closure of $A \cap \mathsf{NS}_{\omega_1} = \emptyset$ and hence $A = \mathsf{CLUB}_{\omega_1}$. Another application of the above theorem shows that $B \setminus \mathsf{NS}_{\omega_1}$ is contained in the closure of $B \cap \mathsf{CLUB}_{\omega_1} = \emptyset$ and hence $B = \mathsf{NS}_{\omega_1}$. This shows that $A \cup B \neq {}^{\omega_1}\omega_1$. $\square$

To motivate the proof of the last theorem, we discuss the proof of an analogous result for large cardinals.

Theorem

Let κ be a regular uncountable cardinal with the property that the set

$$E = \{\delta < \kappa \mid \delta \text{ is a measurable cardinal}\}$$

is stationary in κ. Then there is

$$x \in {}^{\kappa}2 \setminus (\mathsf{CLUB}_\kappa \cup \mathsf{NS}_\kappa)$$

with the property that x is a limit point of $A \cap \mathsf{CLUB}_\kappa$ for every Σ^1_1-subset A of ${}^{\kappa}\kappa$ with $x \in A$.

Corollary

If κ satisfies the above assumptions, then the set ${}^{\kappa}\kappa \setminus \mathsf{CLUB}_\kappa$ is not a Σ^1_1-subset of ${}^{\kappa}\kappa$. □

Note that Friedman and Wu showed that ${}^{\kappa}\kappa \setminus \mathsf{CLUB}_\kappa$ is not a Σ^1_1-subset of ${}^{\kappa}\kappa$ whenever κ is weakly compact.

Proof of the Theorem.

Set $S = \kappa \setminus E$, where E is the stationary set of measurable cardinals less than κ. Then our assumptions imply that E is a bistationary subset of κ.

Let $x \in {}^{\kappa}2 \setminus (\mathsf{CLUB}_\kappa \cup \mathsf{NS}_\kappa)$ denote the characteristic function of S.

Pick $\beta < \kappa$ and a Σ_0-formula $\varphi(v_0, v_1, v_2)$ such that $\varphi(x, y, \kappa)$ holds for some $y \in \mathrm{H}(\kappa^+)$.

Pick a strictly increasing continuous chain $\langle M_\alpha \mid \alpha < \kappa \rangle$ of elementary submodels of $\mathrm{H}(\kappa^+)$ of cardinality less than κ such that $\beta \cup \{x, y\} \subseteq M_0$ and $\kappa_\alpha = \kappa \cap M_\alpha \in \kappa$ for all $\alpha < \kappa$.

Then $C = \{\kappa_\alpha \mid \alpha \in \kappa \cap \mathrm{Lim}\}$ is a club in κ.

Let δ denote the minimal element of $\kappa \cap \mathrm{Lim}$ with $\kappa_\delta \in E$. Since κ_δ is a measurable cardinal and therefore regular, we know that $\delta = \kappa_\delta > \beta$.

Let $\pi : M_\delta \longrightarrow M$ denote the transitive collapse of M_δ. Then $\pi(\kappa) = \delta$, $\pi(x) = x \restriction \delta$ and Σ_0-absoluteness implies that $\varphi(x \restriction \delta, \pi(y), \delta)$ holds in V.

Moreover, $C \cap \delta$ is a club in δ, the minimality of δ implies that $C \cap \delta \subseteq S \cap \delta$ and therefore $x(\gamma) = 1$ for all $\gamma \in C \cap \delta$.

Proof (cont.).

In the above computations, we found $\beta < \delta < \kappa$ measurable, $D \subseteq \delta$ club and $z \in \mathrm{H}(\delta^+)$ such that $\varphi(x \restriction \delta, z, \delta)$ holds in V and D witnesses that $x \restriction \delta$ is an element of CLUB_δ.

Pick a normal ultrafilter U on δ and let

$$\langle \langle N_\alpha \mid \alpha \in \mathrm{On}\rangle, \ \langle j_{\bar{\alpha},\alpha} : N_{\bar{\alpha}} \longrightarrow N_\alpha \mid \bar{\alpha} \leq \alpha \in \mathrm{On}\rangle\rangle$$

denote the corresponding iteration of V by U. Set $j = j_{0,\kappa} : \mathrm{V} \longrightarrow N_\kappa$.

Then $j(\delta) = \kappa$ and $j(D)$ is a club in κ that witnesses that $j(x \restriction \delta)$ is an element of CLUB_κ.

In this situation, Σ_0-absoluteness and elementarity imply that $\varphi(j(x \restriction \delta), j(z), \kappa)$ holds in V and $j(z \restriction \delta) \in N_{x \restriction \beta}$. □

In order to derive the above conclusion, it suffices to assume that κ is regular and a stationary limit of ω_1-iterable cardinals.

Next, we discuss the proof of the related result for $\kappa = \omega_1$.

Theorem

If $M_1^\#(X)$ exists for every $X \subseteq \omega_1$, then the following statements hold for every Σ^1_1-subset A of ${}^{\omega_1}\omega_1$.

- *$A \setminus \mathsf{NS}_{\omega_1}$ is contained in the closure of $A \cap \mathsf{CLUB}_{\omega_1}$ in ${}^{\omega_1}\omega_1$.*
- *$A \setminus \mathsf{CLUB}_{\omega_1}$ is contained in the closure of $A \cap \mathsf{NS}_{\omega_1}$ in ${}^{\omega_1}\omega_1$.*

Proof of the Theorem.

Fix $x \in {}^\kappa\kappa \setminus \mathsf{NS}_{\omega_1}$, $\beta < \omega_1$ and a Σ_0-formula $\varphi(v_0, v_1, v_2)$ such that $\varphi(x, y, \omega_1)$ holds for some $y \in \mathrm{H}(\omega_2)$. Then the set

$$S \ = \ \{\alpha < \omega_1 \mid x(\alpha) > 0\}$$

is stationary. Fix $A \subseteq \omega_1$ such that $\omega_1 = \omega_1^{\mathrm{L}[A]}$ and $x, y \in \mathrm{L}[A]$.

Pick a countable elementary submodel N of $M_1^\#(A)$ with $x, y \in M$, $\beta \subseteq M$ and let $\pi : N \longrightarrow M$ denote the corresponding transitive collapse.

Then $\varphi(\pi(x), \pi(y), \omega_1^M)$ holds.

Let $\mathbb{P} = \mathbb{C}(S)^{M_1^\#(A)}$ denote the canonical partial order in $M_1^\#(A)$ that shoots a club through the stationary set S using countable conditions and let g be $\pi(\mathbb{P})$-generic over M.

Then there is a club $C \subseteq \omega_1^M$ with $C \in M[g]$ and $\pi(x)(\alpha) > 0$ for all $\alpha \in C$.

Let δ denote the unique Woodin cardinal of M. Since $M[g]$ is countable and transitive, δ is a Woodin cardinal in $M[g]$ and there is a measure on $M[g]$ above δ, the model $M[g]$ is iterable with respect to the countable stationary tower $\mathbb{Q}_{<\delta}^{M[g]}$ and its images. Let

$$\langle\langle M_\alpha \mid \alpha \leq \omega_1\rangle, \ \langle j_{\bar{\alpha},\alpha} : M_{\bar{\alpha}} \longrightarrow \alpha \mid \bar{\alpha} \leq \alpha \leq \omega_1\rangle\rangle$$

denote such a generic iteration of $M[g]$ and set $j = j_{0,\omega_1} : M[g] \longrightarrow M_{\omega_1}$.

Then $\omega_1 = j\left(\omega_1^{M[g]}\right)$ and $\varphi((j \circ \pi)(x), (j \circ \pi)(y), \omega_1)$ holds in V.

Moreover, we have $(j \circ \pi)(x) \in N_{x \restriction \beta}$, $j(C)$ is a club in ω_1 and $(j \circ \pi)(x)(\alpha) > 0$ for all $\alpha \in j(C)$.

The above computations show that, whenever A is a Σ^1_1-subset of ${}^{\omega_1}\omega_1$, then $A \setminus \mathsf{NS}_{\omega_1}$ is contained in the closure of $A \cap \mathsf{CLUB}_{\omega_1}$. The second statement of the theorem can be shown in the same way. □

In the following, we sketch the proof of the well-order result.

Theorem

Assume that $M_1^\#(A)$ exists for every $A \subseteq \omega_1$. Then there is no Σ^1_1-definable well-ordering of ${}^{\omega_1}\omega_1$.

Proof of the Theorem.

Given $x \in {}^\omega\omega$, let $\bar{x}$ denote the unique element of ${}^{\omega_1}\omega_1$ with $\bar{x} \restriction \omega = x$ and $\bar{x}(\alpha) = 0$ for all $\omega \leq \alpha < \omega_1$.

Assume that there is a Σ_0-formula $\varphi(v_0, \dots, v_3)$ such that

$$\lhd \;=\; \{\langle x, y\rangle \in {}^{\omega_1}\omega_1 \times {}^{\omega_1}\omega_1 \mid \exists z \in \mathrm{H}(\omega_2)\ \varphi(x, y, u, \omega_1)\}$$

is a well-ordering of ${}^{\omega_1}\omega_1$.

Define $\blacktriangleleft$ to be the set of all pairs $\langle x, y\rangle \in {}^\omega\omega \times {}^\omega\omega$ with the property that there is a countable transitive model M of $\mathbf{ZFC}^-$ and $\delta \in M$ such that $x, y \in M$, δ is a Woodin cardinal in M, $\varphi(\bar{x}, \bar{y}, z, \omega_1^M)$ for some $z \in \mathrm{H}(\omega_2)^M$ and M is ω_1-iterable with respect to the countable stationary tower $\mathbb{Q}^M_{<\delta}$ and its images.

Proof (cont.).

Claim

The relation $\blacktriangleleft$ is a Σ^1_3-subset of ${}^\omega\omega \times {}^\omega\omega$.

Claim

Given $x, y \in {}^\omega\omega$, then $x \blacktriangleleft y$ if and only if $\bar{x} \lhd \bar{y}$.

Our assumptions imply that $\boldsymbol{\Sigma}^1_2$-determinacy holds and therefore every $\boldsymbol{\Sigma}^1_3$-set of reals has the Baire property. This yields a contradiction, because the above claims show that $\blacktriangleleft$ is a Σ^1_3-well-ordering of ${}^\omega\omega$. □

These results naturally lead to the question whether the above large cardinal assumptions are necessary.

The following theorem shows that the assumptions of the well-order result are close to optimal by showing that the existence of a Σ^1_1-definable well-ordering of ${}^{\omega_1}\omega_1$ is compatible with the existence of a Woodin cardinal.

Theorem

Assume M_1 exists. In M_1, the canonical well-ordering of ${}^{\omega_1}\omega_1$ is Σ^1_1-definable.

The proof of this result can also be combined with techniques developed by Holy-L. to show that the existence of a Σ^1_1-definable well-ordering of ${}^{\omega_1}\omega_1$ is compatible with a Woodin cardinal and a failure of the GCH at ω_1.

A result of Friedman-Wu-Zdomskyy shows that the set ${}^{\omega_1}\omega_1 \setminus \mathsf{CLUB}_{\omega_1}$ is Σ^1_1-definable in a cardinal preserving forcing extension of L.

It is plausible that the same statement holds for M_1. This would show that the large cardinal assumption of our second theorem is also close to optimal.

Open questions

Question

Is it possible to derive non-trivial structural statements about Σ^1_1-definable equivalence relations on ${}^{\omega_1}\omega_1$ from the axioms of $\mathbf{ZFC}$ together with large cardinal axioms?

Question

Given a regular uncountable cardinal κ, does the existence of a Σ^1_1-definable well-order of ${}^{\kappa}\kappa$ imply that there are no supercompact cardinals above κ?

Question

Is the existence of Σ^1_1-definable well-orders of ${}^{\kappa}\kappa$ consistent for supercompact cardinals κ?

Thank you for listening!