I

Representing sets of cofinal branches as continuous images (joint with P. Schlicht).

We review some standard definitions.

- A partial order $\mathbb{T}$ is a <u>tree</u> if it has a unique minimal element and the set $\mathrm{pred}_{\mathbb{T}}(t) = \{ s \in \mathbb{T} \mid s <_{\mathbb{T}} t \}$ is well-ordered by $<_{\mathbb{T}}$ for every $t \in \mathbb{T}$.
- Given $t \in \mathbb{T}$, we define $\mathrm{rnk}_{\mathbb{T}}(t)$ to be the order-type of $\langle \mathrm{pred}_{\mathbb{T}}(t), <_{\mathbb{T}} \rangle$.
- Given $\alpha \in \mathrm{On}$, we define $\mathbb{T}(\alpha) = \{ t \in \mathbb{T} \mid \mathrm{rnk}_{\mathbb{T}}(t) = \alpha \}$ and $\mathbb{T}_{<\alpha} = \bigcup_{\bar{\alpha} < \alpha} \mathbb{T}(\bar{\alpha})$.
- Set $\mathrm{ht}(\mathbb{T}) = \min \{ \alpha \in \mathrm{On} \mid \mathbb{T}(\alpha) = \emptyset \}$.
- We say that $b \subseteq \mathbb{T}$ is a <u>cofinal branch through $\mathbb{T}$</u> if it is a maximal linearly ordered subset of $\mathbb{T}$.

Let κ be an infinite cardinal and $\mathbb{T}$ be a tree of height κ. We let $[\mathbb{T}]$ denote the set of all cofinal branches of order-type κ through $\mathbb{T}$. We equip $[\mathbb{T}]$ with the topology whose basic open sets are of the form

$$N_t = \{ b \in [\mathbb{T}] \mid t \in b \}$$

for some node $t \in \mathbb{T}$.

II.

Given a cardinal ν, we let ${}^{<\kappa}\nu$ denote the tree consisting of all functions $t: \alpha \to \nu$ with $\alpha < \kappa$ ordered by inclusion. We identify the set $[{}^{<\kappa}\nu]$ with the set ${}^{\kappa}\nu$ of all functions from κ to ν.

The following observation is the starting point of our work.

<u>Lemma:</u> If κ is a cardinal of countable cofinality and $\mathbb{T}$ is a tree of cardinality ν and height κ with $[\mathbb{T}] \neq \emptyset$, then $[\mathbb{T}]$ is a continuous image of ${}^{\kappa}\nu$.

We call a subset T of ${}^{<\kappa}\nu$ a <u>subtree</u> if it is closed under initial segments. Given such a subtree T, we define
$$[T] = \{ x \in {}^{\kappa}\nu \mid \forall \alpha < \kappa \ \ x \restriction \alpha \in T \}.$$
If we turn T into a tree by ordering its elements by inclusion, then the set $[T]$ corresponds to the set of cofinal branches of length κ through this tree

<u>Proposition:</u> If κ is an infinite cardinal and $\mathbb{T}$ is a tree of cardinality ν and height κ, then there is a subtree T of ${}^{<\kappa}\nu$ such that the spaces $[\mathbb{T}]$ and $[T]$ are homeomorphic.

III.

<u>Proof</u>: Fix an enumeration $\langle t_\gamma \mid \gamma < \nu \rangle$ of $\mathbb{T}$ and define

$$T = \{ s \in {}^{<\kappa}\nu \mid \exists t \in \mathbb{T} \, [\mathrm{dom}(s) \leq \mathrm{ht}_{\mathbb{T}}(t) \wedge \forall \alpha \in \mathrm{dom}(s) \; \{ t_{s(\alpha)} \} = \mathrm{pred}_{\mathbb{T}}(t) \cap \mathbb{T}(\alpha) \}$$

Then T is a subtree of ${}^{<\kappa}\nu$ and the unique function $i : [\mathbb{T}] \to [T]$ with

$$\{ t_{i(b)(\alpha)} \} = b \cap \mathbb{T}(\alpha)$$

for all $b \in [\mathbb{T}]$ and $\alpha < \kappa$ is an homeomorphism. $\boxtimes$

<u>Proof of the Lemma</u>: By the above Proposition, it suffices to show that for every subtree T of ${}^{<\kappa}\nu$ with ~~$[T] \neq \emptyset$ there is a continuous surjection $r : {}^{\kappa}\nu \to [T]$. Let $\langle \alpha_n \mid n < \omega \rangle$ be strictly increasing and cofinal in κ with $\alpha_0 = 0$.~~

the property that for every $t \in T$ there is an $x_t \in [T]$ with $t \subseteq x_t$ there is a continuous surjection $r : {}^{<\kappa}\nu \to [T]$. Let $\langle \alpha_n \mid n < \omega \rangle$ be strictly increasing and cofinal in κ with $\alpha_0 = 0$. Then there is a unique function $r : {}^{\kappa}\nu \to [T]$ such that $r(x) = x$ for all $x \in [T]$ and $r(x) = x_{x \restriction \alpha_m}$ with $m = \max\{ m < \omega \mid x \restriction \alpha_m \in T \}$ for all $x \in {}^{\kappa}\nu \setminus [T]$. It is easy to see that r is continuous and surjective. $\boxtimes$

IV

The following result shows that the above statement does not generalize to cardinalities of higher cofinalities

<u>Theorem (L.-Schlicht):</u> If κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$, then there is a tree $\mathbb{T}$ of cardinality and height κ such that $[\mathbb{T}]$ is not a continuous image of ${}^{\kappa}\kappa$.

<u>Proof:</u> Let C denote the club of all elements of κ closed under Gödel pairing $\langle \cdot, \cdot \rangle$. Given $t \in {}^{\lambda}2$ with $\lambda \in C \cup \{\kappa\}$, we define a relation $\in_t$ on λ by setting

$$\beta \in_t \gamma \iff t(\langle \beta, \gamma \rangle) = 1.$$

Define

$$T = \{ t \in {}^{<\kappa}2 \mid \forall \lambda \in C \cap \mathrm{dom}(t) \ \langle \lambda, \in_t \rangle \text{ is a well-order} \}.$$

Then T is a subtree of ${}^{\kappa}2$ and

$$[T] = \{ x \in {}^{\kappa}2 \mid \langle \kappa, \in_x \rangle \text{ is a well-order} \}.$$

Assume, towards a contradiction, that $[T]$ is a continuous image of ${}^{\kappa}\kappa$. Then there is a $<\kappa$-closed subtree $\mathcal{U}$ of ${}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$ without end-nodes such that $[T] = p[\mathcal{U}]$.

V

Given $\langle s, t\rangle \in \mathcal{U}$ and $\alpha < \kappa$, define
$$\mathrm{mr}_{\mathcal{U}}(s, t, \alpha) = \sup\{\mathrm{mr}_{\langle \kappa, \in_x\rangle}(\alpha) \mid \langle x, y\rangle \in [\mathcal{U}_{\langle s,t\rangle}]\}$$
Then $\mathrm{mr}_{\mathcal{U}}(\emptyset, \emptyset, \alpha) = \kappa^+$ for all $\alpha < \kappa$.

<u>Claim</u>: Let $\langle s, t\rangle \in \mathcal{U}$ and $\alpha < \kappa$ with $\mathrm{mr}_{\mathcal{U}}(s, t, \alpha) = \kappa^+$. If $\gamma < \kappa^+$, then there is an extension $\langle u, v\rangle$ of $\langle s, t\rangle$ in $\mathcal{U}$ and $\beta < \kappa$ such that $\mathrm{dom}(u) \in C$, $\alpha < \beta < \mathrm{dom}(u)$, $\beta \in_u \alpha$ and $\mathrm{mr}_{\mathcal{U}}(u, v, \beta) \geq \gamma$.

<u>Proof of the Claim</u>: Since $\mathrm{mr}_{\mathcal{U}}(s, t, \alpha) = \kappa^+$, we can find $\langle x, y\rangle \in [\mathcal{U}_{\langle s,t\rangle}]$ with $\mathrm{mr}_{\langle \kappa, \in_x\rangle}(\alpha) \geq \gamma + \kappa$. Then there is $\beta > \alpha$ with $\beta \in_x \alpha$.(*) Pick $\beta < \delta \in C$ and define $u = x \restriction \delta$ and $v = y \restriction \delta$. Then $\langle x, y\rangle$ witnesses that $\mathrm{mr}_{\mathcal{U}}(u, v, \beta) \geq \gamma$. $\boxtimes$

<u>Claim</u>: If $\langle s, t\rangle \in \mathcal{U}$ and $\alpha < \kappa$ with $\mathrm{mr}_{\mathcal{U}}(s, t, \alpha) = \kappa^+$, then there is $\langle u, v\rangle \in \mathcal{U}_{\langle s,t\rangle}$ and $\alpha < \beta < \kappa$ with $\mathrm{lh}(u) \in C$, $\beta \in_u \alpha$ and $\mathrm{mr}_{\mathcal{U}}(u, v, \beta) = \kappa^+$.

<u>Proof of the Claim</u>: By the above claim, for every $\gamma < \kappa^+$, there is $\langle u_\gamma, v_\gamma\rangle \in \mathcal{U}_{\langle s,t\rangle}$ and $\alpha < \beta_\gamma < \kappa$ with $\mathrm{lh}(u_\gamma) \in C$, $\beta_\gamma \in_{u_\gamma} \alpha$ and $\mathrm{mr}_{\mathcal{U}}(u_\gamma, v_\gamma, \beta_\gamma) \geq \gamma$. Since we assumed that $\kappa = \kappa^{<\kappa}$, there is a triple

(*) and $\mathrm{mr}_{\langle \kappa, \in_x\rangle}(\beta) \geq \gamma$.

VI

$\langle u, v, \beta \rangle$ such that $\langle u, v, \beta \rangle = \langle u_\gamma, v_\gamma, \beta_\gamma \rangle$ for κ^+-many $\gamma < \kappa^+$. This shows that $\langle u, v, \beta \rangle$ satisfies the above properties. $\boxtimes$

Using this claim, we build a sequence $\langle\langle u_n, v_n, \beta_n \rangle \mid n < \omega \rangle$ such that $u_n \subsetneq u_{n+1}$, $v_n \subseteq v_{n+1}$, $\mathrm{lh}(u_n) \in C$ and $\beta_{n+1} \in_{u_{n+1}} \beta_n$ for all $n < \omega$. Set $u = \bigcup_{n<\omega} u_n$ and $v = \bigcup_{n<\omega} v_n$. Then our assumptions imply that $\langle u, v \rangle \in U$ and there is $\langle x, y \rangle \in [U_{\langle u, v \rangle}]$. Hence $x \in p[U] = [T]$ and $\langle \kappa, \in_x \rangle$ is a well-order. But $\beta_{n+1} \in_x \beta_n$ for all $n < \omega$, a contradiction. $\boxtimes$

VII

Given an uncountable cardinal κ with $\kappa = \kappa^{<\kappa}$, we define $c(\kappa)$ to be the minimal cardinality ν such that $[T]$ is a continuous image of ${}^{\kappa}\nu$ whenever T is a tree of cardinality and height κ. Since every such set $[T]$ is clearly a continuous image of ${}^{\kappa}(2^{\kappa})$, the above theorem implies that

$$\kappa < c(\kappa) \leq 2^{\kappa}.$$

The following result shows that we can manipulate the value of the cardinal characteristic $c(\kappa)$ by forcing.

<u>Theorem (L.-Schlicht):</u> Assume that

- κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$,
- $\mu \geq 2^{\kappa}$ is a cardinal with $\mu = \mu^{\kappa}$,
- $\theta \geq \mu$ is a cardinal with $\theta = \theta^{\kappa}$.

Then the following statements hold in a cofinality preserving forcing extension of V:

(i) $2^{\kappa} = \theta$

(ii) If T is a tree of cardinality μ and height κ, then $[T]$ is a continuous image of ${}^{\kappa}\mu$.

(iii) There is a tree T of cardinality and height κ such that $[T]$ is not a continuous image of ${}^{\kappa}\bar{\mu}$ for some $\bar{\mu} < \mu$ with $\bar{\mu}^{<\kappa} < \mu$.

VIII

The above forcing extension is of the form $V[G,H,K]$, where

- G is $\mathrm{Add}(\kappa,\mu)$-generic over V,
- H is $\mathrm{Add}(\omega,1)$-generic over $V[G]$,
- K is $\mathrm{Add}(\kappa,\Theta)$-generic over $V[G,H]$.

The second statement of the theorem follows from the following result.

<u>Lemma:</u> Let $\omega < \kappa = \kappa^{<\kappa} < \mu = 2^\kappa \leq \Theta$. Then $\mathrm{Add}(\kappa,\Theta)$ forces that every tree T of cardinality μ and height κ $[T]$ is a continuous image of ${}^\kappa\mu$.

<u>Sketch of the Proof:</u> It suffices to prove the claim for subtrees of ${}^{<\kappa}\mu$ contained in V.
Let $\dot{x}$ be an $\mathrm{Add}(\kappa,\Theta)$-name for an element of $[T]$ with support $X \in [\Theta]^{\leq\kappa}$.
Then there is a $<\kappa$-closed subtree $U_{\dot{x}}$ of ${}^{<\kappa}V \times {}^{<\kappa}\mathrm{Add}(\kappa,X)$ without end-nodes such that $\dot{x}^G \in p[U_{\dot{x}}]^{V[G]} \subseteq [T]^{V[G]}$ holds whenever G is $\mathrm{Add}(\kappa,\Theta)$-generic over V.
Let $\pi \in \mathrm{Aut}(\mathrm{Add}(\kappa,\Theta))$ with $\pi''\mathrm{Add}(\kappa,X) \subseteq \mathrm{Add}(\kappa,\kappa)$. Then $p[U_{\dot{x}}]^{V[G]} = p[U_{\pi(\dot{x})}]^{V[G]}$ holds whenever G is $\mathrm{Add}(\kappa,\Theta)$-generic over V.
This shows that $[T]^{V[G]}$ is the union of μ-many continuous images of ${}^\kappa\kappa$ in every $\mathrm{Add}(\kappa,\Theta)$-generic extension $V[G]$ and therefore

itself equal to such a continuous image. $\square$

The following results can be combined to prove the third statement of the theorem.

<u>Proposition:</u> Let $\omega < \kappa = \kappa^{<\kappa} \leq \mu$ and T be a subtree of ${}^{<\kappa}\kappa$ with $|[T]| > \mu^{<\kappa}$. If $c : {}^{\kappa}\mu \longrightarrow [T]$ is a continuous surjection, then there is an order-preserving embedding of the tree ${}^{\leq\omega}\omega$ into T.

<u>Sketch of the Proof:</u> We start by proving the following claim.

<u>Claim:</u> If $U \subseteq {}^{\kappa}\mu$ is open with $|c[U]| > \mu^{<\kappa}$, then there is an $x \in U$ with $|c[N_{x \restriction \alpha}]| > \mu^{<\kappa}$ for all $\alpha < \kappa$.

<u>Proof of the Claim:</u> Otherwise, for every $x \in U$ there is $\alpha_x < \kappa$ with $|c[N_{x \restriction \alpha_x}]| \leq \mu^{<\kappa}$. This implies

$$|c[U]| \leq \Big| \bigcup_{x \in U} c[N_{x \restriction \alpha_x}] \Big| \leq \mu^{<\kappa},$$

a contradiction. $\square$

<u>Claim:</u> If $U \subseteq {}^{\kappa}\mu$ is open with $|c[U]| > \mu^{<\kappa}$, then there are $\langle x_n \mid n < \omega \rangle$ with $c(x_m) \neq c(x_n)$ and $|c[N_{x_m \restriction \alpha}]| > \mu^{<\kappa}$ for all $m < n < \omega$ and $\alpha < \kappa$. $\square$

This claim allows us to build order-embeddings $\ell : {}^{<\omega}\omega \to {}^{<\kappa}\mu$ and $i : {}^{<\omega}\omega \to T$ such that the following statements hold for all $s \in {}^{<\omega}\omega$:

(i) $|C[N_{\ell(s)}]| > \mu^{<\kappa}$.

(ii) $C[N_{\ell(s)}] \subseteq N_{i(s)}$.

(iii) For all $m, n < \omega$, $N_{i(s^\frown\langle m\rangle)} \cap N_{i(s^\frown\langle n\rangle)} \neq \emptyset$ if and only if $m = n$.

Given $x \in {}^{\omega}\omega$, we have

$$C[N_{\bigcup_{n<\omega} \ell(x\restriction n)}] \subseteq N_{\bigcup_{n<\omega} i(x\restriction n)}$$

and this allows us to extend i to an order-embedding of ${}^{\leq\omega}\omega$ into T. $\blacksquare$

Remember that a tree is called <u>superclosed</u> if above every node there is a node with infinitely-many direct successors.

<u>Theorem (Veličković-Woodin)</u>: Let T be a superperfect subtree of ${}^{<\omega}\omega$. If M is an inner model with $[T]^V \subseteq M$, then $\mathbb{R}^V \subseteq M$.

(*)

<u>Theorem (L.-Schlicht)</u>: Assume that M is an inner model with $\mathbb{R}^V \nsubseteq M$ that satisfies the O-cover property. If κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$ and $T = ({}^{<\kappa}\kappa)^M$, then $[T]$ is not a continuous image of ${}^{\kappa}\mu$ in V for all cardinals μ with $\mu^{<\kappa} < |(2^\kappa)^M|^V$.

* $\hookrightarrow$

XI

<u>Sketch of the Proof:</u> Assume that there is a cardinal μ with $\mu < |(2^\kappa)^M|^V$ and a continuous surjection $c : {}^\kappa\mu \to [T]$. By the above proposition, there is an order-embedding $i : {}^{\leq\omega}\omega \to T$ and a covering argument yields an order-embedding $i_* : {}^{\leq\omega}\omega \to ({}^{\leq\omega}\omega)^M$. Then the downwards closure of $i_*[{}^{<\omega}\omega]$ is a superperfect subtree of ${}^{<\omega}\omega$ whose set of cofinal branches is equal to $i_*[{}^\omega\omega]$. By the Velickovic-Woodin result, this implies $\mathbb{R} \subseteq M$, a contradiction. $\boxtimes$

The third statement of the above theorem follows directly from this result, because ~~[a theorem of Hamkins shows that~~ $(\mathrm{Add}(\omega,1) * \mathrm{Add}(\kappa,\theta))$-generic extensions have the θ-cover property in the above setting.