Definable pathological sets

Philipp Moritz Lücke

Joint work in progress with Sandra Müller (Vienna)

Mathematisches Institut
Rheinische Friedrich-Wilhelms-Universität Bonn
http://www.math.uni-bonn.de/people/pluecke/

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Introduction

Set-theoretic objects whose construction requires the Axiom of Choice are frequently referred to as *pathological sets*.

We list some prominent examples of such sets:

- Non-Lebesgue measurable sets of real numbers.
- Hamel bases of the real numbers over the rational numbers.
- Well-orderings of power sets of infinite cardinals.
- Bi-stationary (i.e. stationary and co-stationary) subsets of uncountable regular cardinals.
- Colourings witnessing failures of weak compactness at accessible cardinals.

It is natural to ask whether there are set-theoretical properties that can be used to distinguish pathological sets from objects that are explicitly constructed.

Results from descriptive set theory show that pathological sets of real numbers cannot be defined by simple formulas in second-order arithmetic.

Moreover, both strong large cardinal assumptions and forcing axioms imply that this implication can be extended to arbitrary formulas.

In this talk, I want to present results dealing with the *set-theoretic definability* of pathological objects, i.e. with the question whether objects usually obtained from the Axiom of Choice can be defined in the structure $\langle \mathrm{V}, \in \rangle$ using simple formulas.

I will focus on the definability of certain subsets of uncountable cardinals.

More specifically, we will consider the following types of sets:

- Long well-orderings.
- Maximal almost disjoint families.

In order to motivate our specific questions, we have to make the notion of *simple formulas* more precise.

In the following, we measure the complexity of formulas using the *Levy hierarchy*.

Remember that a formula in the language $\mathcal{L}_\in = \{\in\}$ of set theory is a Σ_0-*formula* if it is contained in the smallest collection of $\mathcal{L}_\in$-formulas that contains all atomic formulas and is closed under negations, conjunctions and bounded quantification.

Moreover, a $\mathcal{L}_\in$-formula is a Σ_{n+1}-*formula* for some $n < \omega$ if it is of the form $\exists x \neg\varphi$ for some Σ_n-formula φ.

Note that the class of all formulas that are $\mathbf{ZFC}$-provably equivalent to a Σ_{n+1}-formula is closed under existential quantification, bounded quantification, conjunctions and disjunctions.

The following concept is central for our considerations:

Definition

Given a set z, a well-ordering $\lhd$ of a class A is a *a good $\Sigma_n(z)$-well-ordering* if the class

$$I(\lhd) \ = \ \{\{y \mid y \lhd x\} \mid x \in A\}$$

of all proper initial segments of $\lhd$ can be defined by a Σ_n-formula with parameter z.

For all types of pathological subsets of some $\mathcal{P}(\kappa)$ considered in this talk, it is easy to see that $\Sigma_n(\kappa, z)$-definable examples of such sets can be constructed from good $\Sigma_n(\kappa, z)$-well-ordering of $\mathcal{P}(\kappa)$.

Proposition

- *The canonical well-ordering of* L *is a good* Σ_1*-well-ordering.*
- *The canonical well-ordering of* HOD *is a good* Σ_2*-well-ordering.*

Theorem (Friedman–Holy, L.)

Let κ *be an uncountable cardinal with* $\kappa^{<\kappa} = \kappa$ *and* $2^\kappa = \kappa^+$.

Then, in a cofinality-preserving generic extension of V, *there is a good* $\Sigma_1(\kappa, z)$*-well-ordering of* $\mathcal{P}(\kappa)$ *for some* $z \subseteq \kappa$.

Theorem (Caicedo–Veličković)

The Bounded Proper Forcing Axiom $\mathbf{BPFA}$ *implies the existence of a* $\Sigma_1(\omega_1, z)$*-well-ordering of* $\mathcal{P}(\omega_1)$ *for some* $z \subseteq \omega_1$.

Motivated by the above results, we view subsets of $\mathcal{P}(\kappa)$ that are defined by Σ_1-formulas with parameters in $\mathrm{H}(\kappa) \cup \{\kappa\}$ as simply defined objects.

We will focus on the question whether canonical extensions of $\mathbf{ZFC}$ (e.g. by large cardinal assumptions) imply that, for certain cardinals κ, pathological subsets of $\mathcal{P}(\kappa)$ are not simply definable.

In order to illustrate our approaches, we start by discussing the existence of good Σ_1-well-orderings.

Good Σ_1-well-orderings of $\mathcal{P}(\kappa)$

It is possible to use deep results of Woodin to derive the following statement:

Corollary

If there exists a measurable cardinal above infinitely many Woodin cardinals, then no well-ordering of the reals is definable by a Σ_1-formula with parameters in $\mathrm{H}(\omega_1) \cup \{\omega_1\}$.

In particular, the above large cardinal assumption implies that for all $z \in \mathrm{H}(\omega_1)$, there is no good $\Sigma_1(\omega_1, z)$-well-ordering of $\mathcal{P}(\omega_1)$.

The proof of the above result makes use of the following theorems of Woodin:

Theorem (Woodin)

If there exists a measurable cardinal above infinitely many Woodin cardinals, then the Axiom of Determinacy $\mathbf{AD}$ *holds in* $\mathrm{L}(\mathbb{R})$.

Theorem (Woodin, Π_2-*maximality of the* $\mathbb{P}_{max}$-*extension* (simplified))

If there exists a measurable cardinal above infinitely many Woodin cardinals, then there exists a partial order $\mathbb{P}_{max} \in \mathrm{L}(\mathbb{R})$ *such that the following statements hold:*

- $\mathbb{P}_{\mathrm{max}}$ *is* σ*-closed and weakly homogeneous in* $\mathrm{L}(\mathbb{R})$.
- $\mathbb{1}_{\mathbb{P}_{\mathrm{max}}} \Vdash^{\mathrm{L}(\mathbb{R})} (\mathrm{AC})$.
- *If* $z \in \mathrm{H}(\omega_1)$ *and* $\varphi(v_0, v_1)$ *is a* Σ_2*-formula such that* $\langle \mathrm{H}(\omega_2), \in \rangle \models \neg\varphi(\omega_1, z)$ *holds, then* $\langle \mathrm{H}(\omega_2)^{\mathrm{L}(\mathbb{R})[G]}, \in \rangle \models \neg\varphi(\omega_1, z)$ *holds whenever* G *is* $\mathbb{P}_{max}$*-generic over* $\mathrm{L}(\mathbb{R})$.

Proof of the Corollary.

Assume there exists a measurable cardinal above infinitely many Woodin cardinals and there is a well-ordering of the reals that is definable by a Σ_1-formula $\varphi(v_0, \ldots, v_3)$ using the parameters ω_1 and $z \in \mathrm{H}(\omega_1)$.

Then the statement that the formula φ and the parameters ω_1 and z define such a well-ordering can be expressed over $\langle \mathrm{H}(\omega_2), \in \rangle$ by the negation of a Σ_2-formula and the parameters ω_1 and z.

Let G be $\mathbb{P}_{\max}$-generic over $\mathrm{L}(\mathbb{R})$. By the Π_2-maximality of $\mathrm{L}(\mathbb{R})[G]$, the formula φ and the parameters ω_1 and z define a well-ordering of the reals in $\mathrm{L}(\mathbb{R})[G]$.

Since $\mathbb{P}_{max}$ is weakly homogeneous in $\mathrm{L}(\mathbb{R})$, this well-ordering is an element of $\mathrm{L}(\mathbb{R})$. This contradicts the fact that $\mathbf{AD}$ holds in $\mathrm{L}(\mathbb{R})$. $\square$

In many interesting cases, it is possible to replace generic iterations of countable models by iterations of V to generalize results about the non-Σ_1-definability of pathological subsets of $\mathcal{P}(\omega_1)$ in the presence of large cardinals to certain cardinals above measurable cardinals.

In particular, this is possible for good well-orderings.

Theorem (L.–Schlicht)

Let δ be a measurable cardinal and let $\nu > \delta$ be a cardinal with $\mathrm{cof}(\nu) \neq \delta$ and $\lambda^\delta < \nu$ for all $\lambda < \nu$.

If $\kappa \in \{\nu, \nu^+\}$ and $z \in \mathrm{H}(\delta)$, then there is no good $\Sigma_1(\kappa, z)$-well-ordering of $\mathcal{P}(\kappa)$.

In addition, it is possible to show that, in $\mathrm{L}[U]$, good $\Sigma_1(\kappa)$-well-orderings of $\mathcal{P}(\kappa)$ exist for all other cardinals κ.

Finally, measurability turns out to be the weakest large cardinal property that influences the existence of good Σ_1-well-orderings, because the canonical well-ordering of the *Dodd–Jensen core model* is a good $\Sigma_1(\kappa)$-well-ordering for every uncountable cardinal κ.

Long well-orderings

The notion of a good Σ_1-well-ordering of $\mathcal{P}(\kappa)$ can be weakened in two obvious ways:

- Σ_1-definable well-orderings of $\mathcal{P}(\kappa)$.
- *Long good Σ_1-well-orderings in $\mathcal{P}(\kappa)$*, i.e. Σ_1-definable injections from κ^+ into $\mathcal{P}(\kappa)$.

Both of these notions have the following common weakening:

- *Long Σ_1-well-orderings in $\mathcal{P}(\kappa)$*, i.e. Σ_1-definable well-orderings of Σ_1-definable subsets of $\mathcal{P}(\kappa)$ of cardinality greater than κ.

The above argument about good Σ_1-well-orderings of $\mathcal{P}(\omega_1)$ can easily be generalized to long Σ_1-well-orderings in $\mathcal{P}(\kappa)$.

Theorem

Assume that there is a measurable cardinal above infinitely many Woodin cardinals. Let $\mathcal{A}$ be a subset of $\mathcal{P}(\omega_1)$ and let $<_{\mathcal{A}}$ be a well-ordering of $\mathcal{A}$.

If both $\mathcal{A}$ and $<_{\mathcal{A}}$ are definable by Σ_1-formulas with parameters in $\mathrm{H}(\omega_1) \cup \{\omega_1\}$, then $\mathcal{A}$ has cardinality at most $\aleph_1$.

We again start by showing that the relevant objects do not exist in $\mathrm{L}(\mathbb{R})$.

Lemma ($\mathbf{ZF}$)

Let κ be an infinite cardinal. If there is an injection from κ^+ into $\mathcal{P}(\kappa)$, then there is no normal ultrafilter on κ^+.

Corollary ($\mathbf{ZF} + \mathbf{DC} + \mathbf{AD}$)

If $<_{\mathcal{A}}$ is a well-ordering of a subset $\mathcal{A}$ of $\mathcal{P}(\omega_1)$, then $\mathrm{otp}(\mathcal{A}, <_{\mathcal{A}}) < \omega_2$.

Lemma

Assume that $\mathbf{AD}$ *holds in* $\mathrm{L}_(\mathbb{R})$ *and* V *is a* $\mathbb{P}_{max}$*-generic extension of* $\mathrm{L}(\mathbb{R})$*. If* $<_{\mathcal{A}} \in \mathrm{OD}(\mathbb{R})$ *is a well-ordering of a subset* $\mathcal{A}$ *of* $\mathcal{P}(\omega_1)$*, then* $\mathcal{A}$ *has cardinality* $\aleph_1$*.*

Proof.

By using the $<_{\mathcal{A}}$-ranks of elements of $\mathcal{A}$ as parameters, it is easy to see that $\mathcal{A} \subseteq \mathrm{OD}(\mathbb{R})$.

Since $\mathbb{P}_{max}$ is weakly homogeneous in $\mathrm{L}(\mathbb{R})$, this implies that $\mathcal{A} \subseteq \mathrm{L}(\mathbb{R})$ and hence $\mathcal{A}, <_{\mathcal{A}} \in \mathrm{L}(\mathbb{R})$.

By an earlier lemma, we now have $\mathrm{otp}(\mathcal{A}, <_{\mathcal{A}}) < \omega_2^{\mathrm{L}(\mathbb{R})}$ and hence $|\mathcal{A}| = \aleph_1$. $\square$

Proof of the Theorem.

Assume that there is a measurable cardinal above infinitely many Woodin cardinals.

Let $\mathcal{A}$ be a subset of $\mathcal{P}(\omega_1)$ of cardinality greater than $\aleph_1$ that is definable by a Σ_1-formula $\varphi(v_0, v_1, v_2)$ and the parameters ω_1 and $z \in \mathrm{H}(\omega_1)$, and let $<_\mathcal{A}$ be a well-ordering of $\mathcal{A}$ that is definable by a Σ_1-formula $\psi(v_0, \dots, v_3)$ and the parameters ω_1 and z.

Then, in $\langle \mathrm{H}(\omega_2), \in \rangle$, the statement that the formula φ and the parameters ω_1 and z define a subset of $\mathcal{P}(\omega_1)$ of cardinality greater than $\aleph_1$ that is well-ordered by a relation defined by the formula ψ and the parameters ω_1 and z can be expressed by a Π_2-formula with parameters ω_1 and z.

Let G be $\mathbb{P}_{max}$-generic over $\mathrm{L}(\mathbb{R})$. Then Π_2-maximality implies that the above statement also holds in $\langle \mathrm{H}(\omega_2)^{\mathrm{L}(\mathbb{R})[G]}, \in \rangle$. In particular, there is a set $\mathcal{B} \in \mathrm{OD}(\mathbb{R})^{\mathrm{L}(\mathbb{R})[G]}$ with $|\mathcal{B}|^{\mathrm{L}(\mathbb{R})[G]} > \aleph_1$ consisting of subsets of ω_1 and a relation $<_\mathcal{B} \in \mathrm{OD}(\mathbb{R})^{\mathrm{L}(\mathbb{R})[G]}$ that well-orders $\mathcal{B}$. This conclusion contradicts an earlier lemma. $\square$

With the help of the following classical result of Dehornoy, we can generalize some of the above statements to cardinals above measurable cardinals.

Theorem (Dehornoy)

Let U be a normal ultrafilter on a measurable cardinal δ and let

$$\langle\langle N_\alpha \mid \alpha \in \mathrm{On}\rangle, \ \langle j_{\alpha,\beta} : N_\alpha \longrightarrow N_\beta \mid \alpha \leq \beta \in \mathrm{On}\rangle\rangle$$

denote the system of iterated ultrapowers of $\langle \mathrm{V}, \in, U\rangle$.

Given $\alpha \in \mathrm{On}$, define $M_\alpha = \bigcap\{N_{\bar{\alpha}} \mid \bar{\alpha} < \alpha\}$.

Then the following statements hold for all $\alpha \in \mathrm{Lim}$:

- *M_α is a model of $\mathbf{ZF}$.*
- *If there is an $\bar{\alpha} < \alpha$ with $\mathrm{cof}(\alpha)^{N_{\bar{\alpha}}} > \omega$, then $M_\alpha = N_\alpha$.*
- *If $\alpha \neq \bar{\alpha} + \omega$ and $\mathrm{cof}(\alpha)^{N_{\bar{\alpha}}} = \omega$ for all $\bar{\alpha} < \alpha$, then there is a subset $\mathcal{G}_\alpha \in M_\alpha$ of $\mathcal{P}(j_{0,\alpha}(\delta))$ that is not well-orderable in M_α.*

Theorem

If δ is a measurable cardinal, $\nu > \delta$ is a cardinal with $\mathrm{cof}(\nu) \neq \delta$ and $\lambda^\delta < \nu$ for all $\lambda < \nu$, and $\kappa \in \{\nu, \nu^+\}$, then the following statements hold:

- *No injection from κ^+ into $\mathcal{P}(\kappa)$ is definable by a Σ_1-formula with parameters in $\mathrm{H}(\delta) \cup \{\kappa\}$.*
- *No well-ordering of $\mathcal{P}(\kappa)$ is definable by a Σ_1-formula with parameters in $\mathrm{H}(\delta) \cup \{\kappa\}$.*

Lemma

In the situation of the above theorem, if U is a normal ultrafilter on δ and

$$\langle \langle N_\alpha \mid \alpha \in \mathrm{On} \rangle, \ \langle j_{\alpha,\beta} : N_\alpha \longrightarrow N_\beta \mid \alpha \leq \beta \in \mathrm{On} \rangle \rangle$$

denotes the system of iterated ultrapowers of $\langle \mathrm{V}, \in, U \rangle$, then $j_{0,\alpha}(\kappa) = \kappa$ holds for all $\alpha < \kappa$.

Proof of the first statement.

Assume that there is an injection $\iota : \kappa^+ \longrightarrow \mathcal{P}(\kappa)$ that is definable by a Σ_1-formula and parameters in $\mathrm{H}(\delta) \cup \{\kappa\}$. Let U be a normal ultrafilter on δ and let

$$\langle \langle N_\alpha \mid \alpha \in \mathrm{On} \rangle, \ \langle j_{\alpha,\beta} : N_\alpha \longrightarrow N_\beta \mid \alpha \leq \beta \in \mathrm{On} \rangle \rangle$$

denote the system of iterated ultrapowers of $\langle \mathrm{V}, \in, U \rangle$.

Given $\alpha < \kappa$, the map $j_{0,\alpha}$ is the identity on $\mathrm{H}(\delta) \cup \{\kappa, \kappa^+\}$.

By elementarity and Σ_1-upwards absoluteness, this implies that $j_{0,\alpha}(\iota) = \iota$ holds for all $\alpha < \kappa$.

In particular, we have $\iota \in \bigcap \{N_\alpha \mid \alpha < \kappa\} = N_\kappa$ and $|\mathcal{P}(\kappa)^{N_\kappa}| \geq \kappa^+$.

Since $j_{0,\kappa}(\delta) = \kappa$ and $j_{0,\alpha}(2^\delta) < \kappa$ for all $\alpha < \kappa$, we also know that

$$\mathcal{P}(\kappa)^{N_\kappa} \ = \ \{ j_{\alpha,\kappa}(x) \mid \alpha < \kappa, \ x \in \mathcal{P}(j_{0,\alpha}(\delta))^{N_\alpha} \}$$

and hence $|\mathcal{P}(\kappa)^{N_\kappa}| \leq \kappa$, a contradiction. $\square$

Proof of the second statement.

Assume that there is a well-ordering $\lhd$ of $\mathcal{P}(\kappa)$ that is definable by a Σ_1-formula and parameters in $\mathrm{H}(\delta) \cup \{\kappa\}$.

Let U be a normal ultrafilter on δ and let

$$\langle\langle N_\alpha \mid \alpha \in \mathrm{On}\rangle, \ \langle j_{\alpha,\beta} : N_\alpha \longrightarrow N_\beta \mid \alpha \leq \beta \in \mathrm{On}\rangle\rangle$$

denote the system of iterated ultrapowers of $\langle \mathrm{V}, \in, U\rangle$.

Given $\alpha < \kappa$, the map $j_{0,\alpha}$ is the identity on $\mathrm{H}(\delta) \cup \{\kappa\}$ and therefore Σ_1-upwards absoluteness implies that

$$j_{0,\alpha}(\lhd) \ = \ \lhd \restriction (\mathcal{P}(\kappa)^{N_\alpha} \times \mathcal{P}(\kappa)^{N_\alpha}).$$

Set $M = \bigcap\{N_\alpha \mid \alpha < \omega^2\}$ and $\blacktriangleleft = \bigcap\{j_{0,\alpha}(\lhd) \mid \alpha < \omega^2\}$.

Then $\blacktriangleleft \in M$ is a well-ordering of $\mathcal{P}(\kappa)^M$, contradicting the fact that that M contains a subset $\mathcal{G}$ of $\mathcal{P}(j_{0,\omega^2}(\delta))$ that is not well-orderable in M. $\square$

The next observation shows that, for much stronger large cardinal properties, the above result can be strengthened to *long well-orders*.

Remember that an $\mathrm{I0}$-embedding is a non-trivial elementary embedding $j : \mathrm{L}(\mathrm{V}_{\lambda+1}) \longrightarrow \mathrm{L}(\mathrm{V}_{\lambda+1})$ with $\mathrm{crit}(j) < \lambda$ and $\lambda = \sup_{n<\omega} j^n(\mathrm{crit}(j))$.

Proposition

Let $j : \mathrm{L}(\mathrm{V}_{\lambda+1}) \longrightarrow \mathrm{L}(\mathrm{V}_{\lambda+1})$ *be an* $\mathrm{I0}$*-embedding, let* $\mathcal{A} \subseteq \mathcal{P}(\lambda)$ *and let* $<_{\mathcal{A}}$ *be a well-ordering of* $\mathcal{A}$*. If both* $\mathcal{A}$ *and* $<_{\mathcal{A}}$ *are definable by* Σ_1*-formulas with parameters in* $\mathrm{H}(\lambda) \cup \{\lambda\}$*, then* $|A| \leq \lambda$*.*

Proof.

By the Σ_1-Reflection Principle, both $\mathcal{A}$ and $<_{\mathcal{A}}$ are contained in $\mathrm{L}(\mathrm{V}_{\lambda+1})$.

Since results of Woodin show that there is a normal ultrafilter on λ^+ in $\mathrm{L}(\mathrm{V}_{\lambda+1})$, we may repeat an earlier argument to show that $\langle \mathcal{A}, <_{\mathcal{A}} \rangle$ has order-type less than λ^+. $\square$

In contrast, the next result shows that, in general, we cannot replace *good long well-orders* by *long well-orders* in the statement of the above theorem.

Theorem

Let U be a normal ultrafilter on a measurable cardinal δ with $\mathrm{V} = \mathrm{L}[U]$ and let $\kappa > \delta$ either be a weakly compact cardinal or a singular cardinal with $\mathrm{cof}(\kappa) < \delta$.

Then there is a well-ordering $<_{\mathcal{A}}$ of a subset $\mathcal{A}$ of $\mathcal{P}(\kappa)$ of order-type $\kappa^+ \cdot \kappa$ with the property that both $\mathcal{A}$ and $<_{\mathcal{A}}$ are definable by Σ_1-formulas with parameter κ.

The proof of this result relies on the following lemma:

Lemma

Let U be a normal ultrafilter on a measurable cardinal δ with $\mathrm{V} = \mathrm{L}[U]$ and let $\kappa > \delta$ be a cardinal that is either weakly compact or singular. If $\lambda > \kappa$ is an ordinal with

$$\mathrm{L}_\lambda[U] \models \mathbf{ZFC}^- + \text{“}\kappa \textit{ is either weakly compact or singular with } \mathrm{cof}(\kappa) < \delta\text{”},$$

then

$$\mathcal{P}(\kappa)^{\mathrm{Ult}(\mathrm{L}_\lambda[U],U)} = \mathcal{P}(\kappa)^{\mathrm{L}_\lambda[U]} \cap \mathrm{Ult}(\mathrm{V}, U).$$

Proof of the Theorem.

We let

$$\langle\langle N_\alpha \mid \alpha \in \mathrm{On}\rangle, \ \langle j_{\alpha,\beta} : N_\alpha \longrightarrow N_\beta \mid \alpha \leq \beta \in \mathrm{On}\rangle\rangle$$

denote the system of iterated ultrapowers of $\langle \mathrm{V}, \in, U\rangle$.

Define $\mathcal{A}$ *to consist of all subsets* A *of* κ *such that* $A \in N_\alpha \setminus N_{\alpha+1}$ *holds for some* $\alpha < \kappa$. *Then* $|\{A \in \mathcal{A} \mid o(A) = \alpha\}| = \kappa^+$ *for all* $\alpha < \kappa$.

Given $A \in \mathcal{A}$, *let* $o(A)$ *denote the unique* $\alpha < \kappa$ *with* $A \in N_\alpha \setminus N_{\alpha+1}$.

Given $\alpha < \kappa$, *set* $U_\alpha = j_{0,\alpha}(U)$. *Define a binary relation* $<_{\mathcal{A}}$ *on* $\mathcal{A}$ *by setting*

$$A <_{\mathcal{A}} B \iff o(A) < o(B) \ \vee \ (o(A) = o(B) \ \wedge \ A <_{\mathrm{L}[U_{o(A)}]} B)$$

for all $A, B \in \mathcal{A}$. *Then* $\mathrm{otp}(\mathcal{A}, <_{\mathcal{A}}) = \kappa^+ \cdot \kappa$.

Proof (cont.).

In the following, we call a triple $\langle M, \varepsilon, F\rangle$ suitable for $A \subseteq \kappa$ if the following statements hold:

- *M is a transitive model of $\mathbf{ZFC}^-$ with $\kappa, A, F \in M$ and*

$$M \models \text{“}\kappa \textit{ is either weakly compact or singular with } \mathrm{cof}(\kappa) < \varepsilon\text{”},$$

- *$\varepsilon < \kappa$ is an uncountable regular cardinal in M.*
- *F is a normal ultrafilter on ε in M with $M = \mathrm{L}_\lambda[F]$ for some $\kappa < \lambda < \kappa^+$.*
- *$A \notin \mathrm{Ult}(M, F)$.*

With the help of the above lemma and classical results of Kunen, it is now possible to prove the following statement:

Claim

Given $A \subseteq \kappa$, if the tuple $\langle M, \varepsilon, F\rangle$ is suitable for A, then $A \in \mathcal{A}$, $\varepsilon = j_{0,o(A)}(\delta)$ and $F = U_{o(A)}$.

Proof (cont.).

With the help of the above claim, it is now possible to prove the following statement whose conjunction proves the theorem.

Claim

A subset A of κ is an element of $\mathcal{A}$ if and only if there is a triple $\langle M, \varepsilon, F\rangle$ that is suitable for A.

Claim

Given $A, B \in \mathcal{A}$, we have $A <_{\mathcal{A}} B$ if and only if there is a triple $\langle M, \varepsilon, F\rangle$ that is suitable for A and either $B \in \mathrm{Ult}(M, F)$ or the triple $\langle M, \varepsilon, F\rangle$ is also suitable for B and $A <_{\mathrm{L}[F]} B$ holds.

Claim

The set $\mathcal{A}$ and the relation $<_{\mathcal{A}}$ are definable by Σ_1-formulas with parameter κ. □

M.a.d. families

The next definition generalizes a classical concept to larger cardinalities:

Definition

Given an infinite cardinal κ, a subset $\mathcal{A}$ of $\mathcal{P}(\kappa)$ is a *κ-m.a.d. family* if the following statements hold:

- If $A \in \mathcal{A}$, then A is unbounded in κ.
- If $A, B \in \mathcal{A}$ with $A \neq B$, then $A \cap B$ is bounded in κ.
- If $X \subseteq \kappa$ is unbounded in κ, then there is $A \in \mathcal{A}$ with $A \cap X$ unbounded in κ.
- The set $\mathcal{A}$ does not have cardinality less than $\mathrm{cof}(\kappa)$.

Theorem

Assume that there is a measurable cardinal above infinitely many Woodin cardinals. Then no ω_1-m.a.d. family is definable by a Σ_1-formula with parameters in $\mathrm{H}(\omega_1) \cup \{\omega_1\}$.

Theorem (Chan–Jackson–Tran, $\mathbf{ZF} + \mathbf{AD}^+ + \mathrm{V} = \mathrm{L}(\mathcal{P}(\mathbb{R}))$)

There are no ω_1-m.a.d. families.

The proof of this result relies on the following two observations:

Lemma ($\mathbf{ZF}$)

If κ is an infinite cardinal and $\mathcal{A}$ is a κ-m.a.d. family, then there is no bijection between $\mathrm{cof}(\kappa)$ and $\mathcal{A}$.

Proposition ($\mathbf{ZF} + \mathbf{DC}$)

If X is a Polish space and $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ is a sequence of pairwise disjoint non-meager subsets of X, then there is an $\alpha < \omega_1$ such that the subset A_α does not have the property of Baire.

Proof of the Theorem.

Assume that $\mathbf{ZF} + \mathbf{AD}^+ + \mathrm{V} = \mathrm{L}(\mathcal{P}(\mathbb{R}))$ *holds and* $\mathcal{A}$ *is an* ω_1*-m.a.d. family.*

Then $\mathcal{A}$ *is not well-orderable, because otherwise the above lemma would imply that* $\mathcal{A}$ *has cardinality greater than* $\aleph_1$ *and hence there would be an injection of* ω_2 *into* $\mathcal{P}(\omega_1)$*.*

By a theorem of Woodin, the above observation implies that there is an injection $\iota : \mathbb{R} \longrightarrow \mathcal{A}$*. Define*

$$c : [\mathbb{R}]^2 \longrightarrow \omega_1;\ \{x, y\} \longmapsto \min\{\alpha < \omega_1 \mid \iota(x) \cap \iota(y) \subseteq \alpha\}$$

and set $E_\alpha = c^{-1}\{\alpha\} \subseteq \mathbb{R} \times \mathbb{R}$ *for all* $\alpha < \omega_1$*.*

Then $\bigcup\{E_\alpha \mid \alpha < \omega_1\} = [\mathbb{R}]^2$ *is comeager in* $\mathbb{R} \times \mathbb{R}$*.*

Proof (cont.).

Using the above proposition about disjoint non-meager sets, we can find $\lambda < \omega_1$ with the property that the set $\bigcup\{E_\alpha \mid \alpha < \lambda\}$ is comeager.

By a classical result of Mycielski, we can now find an injection $e : \mathbb{R} \longrightarrow \mathbb{R}$ with the property that for all $x, y \in \mathbb{R}$ with $x \neq y$, there is an $\alpha < \lambda$ with $\langle e(x), e(y)\rangle \in E_\alpha$.

In this situation, we know that

$$(\iota \circ e)(x) \ \cap \ (\iota \circ e)(y) \ \subseteq \ \lambda$$

holds for all $x, y \in \mathbb{R}$ with $x \neq y$. In particular, the map

$$i : \mathbb{R} \longrightarrow \kappa; \ x \longmapsto \min((\iota \circ e)(x) \setminus \lambda)$$

is an injection, a contradiction. $\square$

Theorem

Assume that $\mathbf{AD}$ *holds in* $\mathrm{L}(\mathbb{R})$ *and* V *is* $\mathbb{P}_{max}$*-generic extension of* $\mathrm{L}(\mathbb{R})$*. Then no* ω_1*-m.a.d. family is contained in* $\mathrm{OD}(\mathbb{R})$*.*

The proof of this theorem again makes use of results of Woodin.

Theorem (Woodin, *Perfect set theorem for* ω_1)

Assume that $\mathbf{AD}$ *holds in* $\mathrm{L}(\mathbb{R})$ *and* V *is* $\mathbb{P}_{max}$*-generic extension of* $\mathrm{L}(\mathbb{R})$*. If* $\mathcal{B} \in \mathrm{OD}(\mathbb{R})$ *with* $\mathcal{B} \nsubseteq \mathrm{L}(\mathbb{R})$*, then an unbounded subset* U *of* ω_1 *and a function* $\pi : {}^{<\omega_1}2 \longrightarrow [\omega_1]^\omega$ *such that the following statements hold:*

- *If* $s, t \in {}^{<\omega_1}2$ *with* $s \subseteq t$*, then* $\pi(s) \subseteq \pi(t)$ *and* $\pi(s) \cap \alpha = \pi(t) \cap \alpha$ *for all* $\alpha \in \pi(s)$*.*
- *Given* $s \in {}^{<\omega_1}2$ *and* $\alpha \in \mathrm{dom}(s) \cap U$*, we have* $\alpha \in \pi(s)$ *if and only if* $s(\alpha) = 1$*.*
- *If* $x \in {}^{\omega_1}2$*, then* $\bigcup\{\pi(x \restriction \alpha) \mid \alpha < \omega_1\} \in \mathcal{A}$*.*

Proof of the Theorem.

Assume that $\mathbf{AD}$ holds in $\mathrm{L}(\mathbb{R})$, V is $\mathbb{P}_{max}$-generic extension of $\mathrm{L}(\mathbb{R})$ and $\mathcal{A} \in \mathrm{OD}(\mathbb{R})$ is an ω_1-m.a.d.-family.

Then Woodin's theorem shows that $\mathcal{A} \subseteq \mathrm{L}(\mathbb{R})$.

Since $\mathbb{P}_{max}$ is weakly homogeneous in $\mathrm{L}(\mathbb{R})$, this implies that $\mathcal{A} \in \mathrm{L}(\mathbb{R})$.

Then $\mathcal{A}$ is an ω_1-m.a.d.-family in $\mathrm{L}(\mathbb{R})$, contradicting an earlier theorem. $\square$

Another application of the Π_2-maximality of the $\mathbb{P}_{max}$-extension of $\mathrm{L}(\mathbb{R})$ now shows that, in the presence of large cardinals, no ω_1-m.a.d.-family is definable by a Σ_1-formula with parameters in $\mathrm{H}(\omega_1) \cup \{\omega_1\}$.

Concluding remarks and open questions

The above results motivate the question whether the non-existence of Σ_1-definable m.a.d.-families can be generalized to certain cardinals above measurable cardinals.

Question

Let δ be a measurable cardinal and let $\kappa > \delta$ be a cardinal with $\mathrm{cof}(\kappa) \neq \delta$ and $\lambda^\delta < \kappa$ for all $\lambda < \kappa$.

Is it possible that a κ-m.a.d. family can be defined by a Σ_1-formula with parameter κ?

Question

Let $j : \mathrm{L}(\mathrm{V}_{\lambda+1}) \longrightarrow \mathrm{L}(\mathrm{V}_{\lambda+1})$ be an $\mathrm{I0}$-embedding.

Is it possible that a λ-m.a.d. family can be defined by a Σ_1-formula with parameters in $\mathrm{H}(\lambda) \cup \{\lambda\}$?

In order to find more differences between the three settings studied above, one may consider the Σ_1-definability of ultrafilters.

Question

Let δ be a measurable cardinal and let $\kappa > \delta$ be a cardinal with $\mathrm{cof}(\kappa) \neq \delta$ and $\lambda^\delta < \kappa$ for all $\lambda < \kappa$.

Is it possible that a non-principal ultrafilter on κ can be defined by a Σ_1-formula with parameter κ?

Question

Do strong large cardinal assumptions imply that no non-principal ultrafilter on ω_1 is definable by a Σ_1-formula with parameters in $\mathrm{H}(\omega_1) \cup \{\omega_1\}$?

By combining coding results of Caicedo–Veličković, Hoffelner and Holy–L., it is possible to show that the influence of large cardinal on Σ_1-definablity cannot be extended from ω_1 to ω_2.

Theorem

Assume that the GCH *holds. If* δ *is an inaccessible cardinal above a Mahlo cardinal and* $\vec{C}$ *is a ladder system, then there is a semi-proper partial order of cardinality less than* δ *that forces the existence of a good* $\Sigma_1(\omega_2, \vec{C})$*-well-ordering of* $\mathcal{P}(\omega_2)$*.*

Question

Do very strong large cardinal assumptions imply that no well-ordering of $\mathcal{P}(\omega_2)$ is definable by a Σ_1-formula with parameter ω_2?

Other questions about the Σ_1-definability of certain objects also reveal much information about the underlying model of set theory.

Question

Is it consistent that the set $\{\omega_1\}$ is not definable by a Σ_1-formula with parameter ω_ω?

Note that if ω_ω is Rowbottom, then the set $\{\omega_1\}$ is not definable by a Σ_1-formula with parameter ω_ω.

Thank you for listening!