Simple formulas defining complicated sets

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Joint DMV and GAMM Annual Meeting
Section: Algebra, logic and set theory
Braunschweig, 08.03.2016

Introduction

Remember that a formula $\varphi(v_0, \ldots, v_n)$ in the language $\mathcal{L}_\in = \{\epsilon\}$ of set theory and parameters $p_0, \ldots, p_{n-1}$ define a set X if

$$X \;=\; \{x \mid \varphi(x, p_0, \ldots, p_{n-1})\}.$$

The work presented in this talk is motivated by the following vague question about definable sets.

Question

Is it *possible* that *simple formulas* using *simple parameters* define very *complicated sets*?

In the following, we will state the above question more precisely.

- An $\mathcal{L}_\in$-formula is a Δ_0-*formula* if it is contained in the smallest collection of $\mathcal{L}_\in$-formulas that contains all atomic $\mathcal{L}_\in$-formulas and is closed under $\neg$, $\wedge$ and $\exists x \in v$.
- An $\mathcal{L}_\in$-formula is a Σ_1-*formula* if it is of the form $\exists x \ \varphi(x)$ for some Δ_0-formula φ.

In the following, we will focus on Σ_1-formulas that use only a single ordinal number as a parameters and the question whether such a formula can define a *well-orderings of the reals*, i.e. a linear ordering of the real numbers with the property that every non-empty subset has a least element.

It is well-known that such well-orders exist in Gödel's constructible universe.

Proposition

If all reals are constructible, then there is a well-ordering of the reals that is definable by a Σ_1-formula without parameters.

In contrast, classical results of Mansfield show that the converse of the above implication is also true.

Theorem (Mansfield)

Let α be a countable ordinal. Assume that there is a well-ordering of the reals that is definable by a Σ_1-formula with parameter α. If $x \in \mathbb{R}$ with $\alpha < \omega_1^{\mathrm{L}[x]}$, then every real is constructible from x.

The above result shows that the existence of such a definable well-ordering has many restrictive implications.

Corollary

Assume that there is a well-ordering of the reals that is definable by a Σ_1-formula that only uses a countable ordinal as a parameter.

- *The Continuum Hypothesis holds.*
- *There are no measurable cardinals.*

In particular, the existence of such a definable well-ordering of the reals is independent from the axioms of $\mathbf{ZFC}$.

Moreover, many canonical extensions of $\mathbf{ZFC}$ (like larger *large cardinal axioms* or *forcing axioms*) imply the non-existence of such well-orderings.

These results motivate the following reformulations of our initial question.

Question

Is the existence of a well-ordering of reals that is definable by a Σ_1-formula with ordinal parameters compatible with strong forcing axioms or the existence of (very large) large cardinals?

Our work provides negative answers to this question for the case where the Σ_1-formula uses the first uncountable cardinal ω_1 or a large cardinal as a parameter.

$\Sigma_1(\omega_1)$-subsets of $H(\omega_2)$

The next result answers the above question for Σ_1-formulas using ω_1 as a parameter.

Theorem (L.-Schindler-Schlicht)

Assume that either there is a measurable cardinal above a Woodin cardinal or Martin's Maximum holds. Then no well-ordering of the reals is definable by a Σ_1-formula with parameter ω_1.

The next result shows that the first assumption of the theorem is close to optimal.

Theorem (L.-Schindler-Schlicht)

The existence of a well-ordering of the reals that is definable by a Σ_1-formula with parameter ω_1 is compatible with the existence of a Woodin cardinal.

The following lemma lies at the heart of the proof of the first theorem.

Lemma

Assume that either there is a measurable cardinal above a Woodin cardinal or Martin's Maximum holds. Then the following statements are equivalent for every subset X of $\mathbb{R}$.

- *The set X is Σ^1_3-definable.*
- *The set X is definable by a Σ_1-formula with parameter ω_1.*

The proof of this lemma uses iterated generic ultrapowers and Woodin's countable stationary tower forcing. The lemma directly yields the statement of the theorem, because both assumptions imply $\mathbf{\Sigma}^1_2$-determinacy holds and hence that no Σ^1_3-well-ordering of the reals exists.

Note that the forward implication is a theorem of $\mathbf{ZFC}$. In contrast, the reverse implication fails both in L and in M_1, because these models contain a projective truth predicate that is definable by a Σ_1-formula with parameter ω_1.

Large cardinals as parameters

The techniques developed in the proof of the above result can be generalized to study sets definable by Σ_1-formulas that use certain large cardinals as parameters.

This approach leads to the following result.

Theorem (L.-Schindler-Schlicht)

Let κ either be a measurable cardinal above a Woodin cardinal or a Woodin cardinal below a measurable cardinal. Then no well-ordering of the reals is definable by a Σ_1-formula with parameter κ.

The proof of the above result relies on the following analog of the above lemma and the fact that our assumptions imply $\mathbf{\Sigma}^1_2$-determinacy.

Lemma

Assume that κ is either a measurable cardinal or a regular cardinal that is a stationary limit of measurable cardinals. Then the following statements are equivalent for every subset X of $\mathbb{R}$.

- *The set X is Σ^1_3-definable.*
- *The set X is definable by a Σ_1-formula with parameter κ.*

Open questions

We close with two questions raised by the above results.

Question

Is the existence of a well-ordering of the reals that is definable by a Σ_1-formula with parameter ω_2 compatible with the existence of a supercompact cardinal?

Question

Is the existence of a well-ordering of $\mathcal{P}(\omega_1)$ that is definable by a Σ_1-formula without parameters compatible with the existence of a non-constructible subset of ω_1?

Thank you for listening!