Infinite Fields with free automorphism groups

Some interactions between algebra and set theory

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Abstract

We present results from [4] that provide partial answers to the question, whether it is possible to prove the existence of a field of cardinality $\aleph_1$ (the first uncountable cardinal) whose automorphism group is a free group of rank strictly greater than $\aleph_1$ from the standard axioms of set theory. These results provide general techniques to construct uncountable fields with free automorphism group of large rank. Using these methods, we construct a field of cardinality $2^{\aleph_0}$ (the cardinality of the continuum) whose automorphism group is a free group of rank strictly greater than $2^{\aleph_0}$. In particular, if the *Continuum Hypothesis* $2^{\aleph_0} = \aleph_1$ holds, then there is a field of cardinality $\aleph_1$ whose automorphism group is a free group of rank strictly greater than $\aleph_1$. Moreover, these techniques can be used to show that the non-existence of fields of cardinality $\aleph_1$ whose automorphism group is a free group of rank strictly greater than $\aleph_1$ implies the existence of a set-sized model of the axioms of ZFC (the standard axiom of set theory). By Gödels second incompleteness theorem, this shows that the non-existence of such fields cannot be derived from the axioms of ZFC. Finally, these techniques also allow us to prove that analog results hold for the free objects in various varieties of groups, like the variety of abelian groups.

This is joint work with Saharon Shelah (Hebrew University, Jerusalem, and Rutgers University, NJ).

Introduction

We consider questions of the following type: *given an abstract group G and an infinite cardinal λ, is G isomorphic to the automorphism group of a field of cardinality λ*? We start by presenting some known results related to this kind of problem.

Fix an infinite cardinal λ. With the help of a theorem of Fried and Kollár from [2], one can show that every group of cardinality λ is isomorphic to the automorphism group of a field of cardinality λ. In contrast, the automorphism group of a field of cardinality λ can be embedded into the group $\mathrm{Sym}(\lambda)$ of all permutations of λ and therefore has cardinality at most 2^λ. Since a result of De Bruijn from [1] shows that the group $\mathrm{Fin}(\lambda^+)$ consisting of all finite permutations of λ^+ cannot be embedded into $\mathrm{Sym}(\lambda)$, there always is a group of cardinality λ^+ that is not isomorphic to the automorphism group of a field of cardinality λ. Moreover, Sanerib showed in [5] that the group $\mathrm{Sym}(\lambda)$ has 2^{2^λ}-many pairwise non-isomorphic subgroups and this result implies that there also is a subgroup of $\mathrm{Sym}(\lambda)$ that is not isomorphic to the automorphism group of a field of cardinality λ, because, up to isomorphism, there are only 2^λ-many fields of cardinality λ.

In the following, we focus on free groups and the following question.

Question. Given an infinite cardinal λ, is there a field of cardinality λ whose automorphism group is a free group of rank strictly greater than λ?

The above question was first asked by David Evans for the case $\lambda = \aleph_0$. In [6], Shelah answered Evan's question in the negative by showing that a free group of uncountable rank is not isomorphic to the automorphism group of a countable first-order structure. In contrast, a theorem of Just, Shelah, and Thomas from [3] shows that a positive answer to the above question in the case $\lambda = \aleph_1$ is consistent with the axioms of ZFC. The Continuum Hypothesis holds in the model of set theory constructed in the proof of their result. The main result of our work shows that such fields exist in all models of the Continuum Hypothesis.

Theorem 1 ([4])**.** *Let λ be an infinite cardinal with $\lambda = \lambda^{\aleph_0}$ (i.e. the set of all functions from the set of natural numbers to λ has cardinality λ). Then there is a field of cardinality λ whose automorphism group is a free group of rank 2^λ.*

This theorem shows that there always is a cardinal for which the above question has a positive answer and that the Continuum Hypothesis implies that this question has a positive answer in the case $\lambda = \aleph_1$.

Corollary. *There is a field of cardinality $2^{\aleph_0}$ whose automorphism group is a free group of rank $2^{2^{\aleph_0}}$.*

Corollary. *If the Continuum Hypothesis holds, then there is a field of cardinality $\aleph_1$ whose automorphism group is a free group of rank strictly greater than $\aleph_1$.*

The question whether the axioms of set theory alone prove the existence of a field of cardinality $\aleph_1$ whose automorphism group is a free group of rank strictly greater than $\aleph_1$ remains open. As a partial answer, the techniques of [4] can be used to show that in order to construct models of set theory without such fields, one has to start with assumptions that are stronger than the consistency of the axioms of ZFC.

Theorem 2 ([4])**.** *Assume that there are no fields of cardinality $\aleph_1$ whose automorphism group is a free group of rank strictly greater than $\aleph_1$. Then there is an inner model of set theory with an inaccessible cardinal. In particular, there is a set-sized model of* ZFC*.*

The main construction

In the following, we outline the main ideas behind the proofs of the above results. The following theorem shows that it suffices to construct a small inductive system of groups whose limit is a free group of large rank.

Theorem ([4])**.** *Let $\mathbb{D}$ be a directed set and let*

$$\mathbb{G} = \langle \langle G_p \mid p \in \mathbb{D} \rangle, \langle f_{p,q} : G_q \longrightarrow G_p \mid p \leq_{\mathbb{D}} q \rangle \rangle$$

be an inverse system of groups over $\mathbb{D}$. If λ is an infinite cardinal with $|\mathbb{D}| \leq \lambda$ and $|G_p| \leq \lambda$ for all $p \in \mathbb{D}$, then there is a field of cardinality λ whose automorphism group is an inverse limit of $\mathbb{G}$.

The next result shows that it is possible to construct such an inverse system of groups from a suitable inverse system of sets. Let $F : \mathsf{Set} \longrightarrow \mathsf{Grp}$ denote the canonical functor from the category of sets into the category of groups that sends a set A to the free group $F(A)$ with basis A.

Theorem ([4])**.** *Let $\mathbb{D}$ be a directed set and let*

$$\mathbb{I} = \langle \langle A_p \mid p \in \mathbb{D} \rangle, \langle f_{p,q} : A_q \longrightarrow A_p \mid p \leq_{\mathbb{D}} q \rangle \rangle$$

be an inverse system of sets over $\mathbb{D}$ with inverse limit $A_{\mathbb{I}}$. Let

$$\mathbb{G}_{\mathbb{I}} = \langle \langle F(A_p) \mid p \in \mathbb{D} \rangle, \langle F(f_{p,q}) : F(A_q) \longrightarrow F(A_p) \mid p \leq_{\mathbb{D}} q \rangle \rangle$$

denote the induced inverse system of groups. If every countable subset of $\mathbb{D}$ has an upper bound in $\mathbb{D}$, then the group $F(A_{\mathbb{I}})$ is an inverse limit of $\mathbb{G}_{\mathbb{I}}$.

The combination of the above theorems yields the following result.

Corollary. *Let λ be an infinite cardinal, let $\mathbb{D}$ be a directed set and let*

$$\mathbb{I} = \langle \langle A_p \mid p \in \mathbb{D} \rangle, \langle f_{p,q} : A_q \longrightarrow A_p \mid p \leq_{\mathbb{D}} q \rangle \rangle$$

be an inverse system of sets over $\mathbb{D}$. Assume that the following statements hold:

1. *$|\mathbb{D}| \leq \lambda$ and $|A_p| \leq \lambda$ for all $p \in \mathbb{D}$.*
2. *Every countable subset of $\mathbb{D}$ has an upper bound in $\mathbb{D}$.*
3. *The inverse limit of $\mathbb{I}$ has cardinality strictly greater than λ.*

Then there is a field of cardinality λ whose automorphism group is a free group of rank strictly greater than λ.

We show how the main theorems of [4] can be derived from this result.

Proof of Theorem 1. Let λ be a cardinal with $\lambda = \lambda^{\aleph_0}$. Let $\mathbb{D}$ denote the directed set consisting of countable subsets of λ ordered by inclusion. Given $p \subseteq q \in \mathbb{D}$, let A_p denote the set of all functions from q to the natural numbers and let $f_{p,q} : A_q \longrightarrow A_p$ denote the restriction map. Then the resulting inverse system of sets satisfies the assumptions of the above corollary, because there is a natural bijection between the inverse limit of this system and the set of all functions from λ to the natural numbers. By the above corollary, there is a field of cardinality λ whose automorphism group is a free group of rank strictly greater than λ. □

Proof of Theorem 2. Let λ be an uncountable regular cardinal with the property that λ^+ is not inaccessible in an inner model. In this situation, classical results on the structure of models of the form $\mathrm{L}[A]$ imply that there is a set-theoretic tree $\mathbb{T}$ of height λ with the property that every level of $\mathbb{T}$ has cardinality at most λ and the set of all cofinal branches through $\mathbb{T}$ has cardinality strictly greater than λ. Define $\mathbb{D}$ to be the directed system $\langle \lambda, \leq \rangle$. Given $\alpha \leq \beta \in \mathbb{D}$, let A_α denote the α-th level of $\mathbb{T}$ and let $f_{\alpha,\beta} : A_\beta \longrightarrow A_\alpha$ denote the canonical map that sends a node in A_β to its unique predecessor in A_α. Then the resulting inverse system of sets satisfies the assumptions listed in the above corollary, because there is a natural bijection between the inverse limit of this system and the set of all cofinal branches through $\mathbb{T}$. By the above corollary, there is a field of cardinality λ whose automorphism group is a free group of rank strictly greater than λ. □

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