Automorphism Towers

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Introduction

Let G be a group. If g is an element of G, then the map

$$\iota_g : G \longrightarrow G;\ h \longmapsto h^g = g \circ h \circ g^{-1}.$$

is an automorphism of G. We call ι_g the *inner automorphism of* G corresponding to g.

We let $\mathrm{Inn(G)}$ denote the group of all inner automorphisms of G.

The map

$$\iota_G : G \longrightarrow \mathrm{Aut(G)};\ g \longmapsto \iota_g$$

is a homomorphism of groups with $\ker(\iota_G) = \mathrm{C}(G)$.

Given $g \in G$ and $\pi \in \mathrm{Aut(G)}$, an easy computation shows that

$$\iota_{\pi(g)} \;=\; \pi \circ \iota_g \circ \pi^{-1}$$

holds and this implies that $\mathrm{Inn(G)}$ is a normal subgroup of $\mathrm{Aut(G)}$.

If G is a group with trivial centre, then ι_G is an embedding of groups and the above equality implies that

$$\mathrm{C_{Aut(G)}(Inn(G))} \;=\; \{\mathrm{id_G}\}$$

holds. In particular, $\mathrm{Aut(G)}$ is also a group with trivial center in this case.

By iterating the process of forming automorphism groups, we construct the *automorphism tower of a group G*.

Definition

Given a group G, we say that a direct system

$$\langle\langle G_n \mid n \in \mathbb{N}\rangle, \langle \iota_{m,n} : G_m \longrightarrow G_n \mid m < n\rangle\rangle$$

of groups is the *automorphism tower of* G if the following statements hold:

- $G_0 = G$.
- If $n \in \mathbb{N}$, then $G_{n+1} = \mathrm{Aut(G_n)}$ and $\iota_{n,n+1} = \iota_{G_n}$.

It is now natural to ask if the above process stops at some point.

Definition

We say that the automorphism tower $\langle\langle G_n \mid n \in \mathbb{N}\rangle, \langle \iota_{m,n} \mid m < n\rangle\rangle$ of a group G *terminates after* N*-many steps* if the map $\iota_{N,N+1}$ is an isomorphism.

The following classical result of Helmut Wielandt is the starting point of the work presented in this talk.

Theorem (Wielandt, 1939)

If G is a finite group with trivial center, then the automorphism tower of G terminates.

This result raises the following question.

Question

Does the automorphism tower of every group terminate?

In the following, we present two examples that show that both assumptions made in the theorem are necessary for the conclusion.

The infinite dihedral group D_∞

Remember that the infinite dihedral group D_∞ is the free product $\langle a \rangle * \langle b \rangle$ of cyclic subgroups generated by involutions a and b.

Let π be the automorphism of D_∞ that interchanges the elements a and b. Then $\pi \notin \mathrm{Inn}(\mathrm{D}_\infty)$. Moreover, a short proof yields the following statement.

Lemma

There is an ismorphism $f : D_\infty \longrightarrow \mathrm{Aut}(\mathrm{D}_\infty)$ of groups with $f(a) = \iota_a$ and $f(b) = \pi$.

Corollary

The automorphism tower of D_∞ does not terminate.

The dihedral group D_8

Remember that the dihedral group D_8 is given by the following presentation

$$D_8 \;=\; \langle a, b \mid a^2 = b^2 = (ab)^4 = \mathbb{1} \rangle.$$

Then $\mathrm{C}(D_8) = \langle (ab)^2 \rangle \neq \{\mathbb{1}\}$ and hence ι_{D_8} is not an isomorphism.

As above, we let π denote the automorphism of D_8 that interchanges the elements a and b.

Lemma

There is an ismorphism $f : D_8 \longrightarrow \mathrm{Aut(D_8)}$ of groups with $f(a) = \iota_a$ and $f(b) = \pi$.

Corollary

The automorphism tower of D_8 does not terminate.

The above examples show that the automorphism tower of a group might not terminate after finitely many steps.

Since we can form the direct limit of a direct system of groups, there is a natural way to continue the above construction of automorphism towers: we first form the direct limit G_∞ of the above direct system of groups, then we form $\mathrm{Aut(G_\infty)}$ and so on.

It turns out that this extended construction terminates for both of the above groups.

Let $\langle\langle G_n \mid n \in \mathbb{N}\rangle, \langle \iota_{m,n} \mid m < n\rangle\rangle$ denote the automorphism tower of D_8 and let

$$\langle G_\infty, \langle \iota_n : G_n \longrightarrow G_\infty \mid n \in \mathbb{N}\rangle\rangle$$

denote the direct limit of this system. An easy computation shows that $G_\infty = \langle \iota_0(a)\rangle$ is a cyclic group of order 2. In particular, $\mathrm{Aut(G_\infty)}$ is the trivial group and $\iota_{\mathrm{Aut(G_\infty)}}$ is an isomorphism.

Let $\langle\langle G_n \mid n \in \mathbb{N}\rangle, \langle \iota_{m,n} \mid m < n\rangle\rangle$ denote the automorphism tower of D_∞ and let

$$\langle G_\infty, \langle \iota_n : G_n \longrightarrow G_\infty \mid n \in \mathbb{N}\rangle\rangle$$

denote the direct limit of this system.

Let $\mathbb{Z}[1/2] = \{\frac{m}{2^n} \mid m \in \mathbb{Z},\ n \in \mathbb{N}\}$ denote the additive group of the dyadic rationals and let σ be the involution in $\mathrm{Aut}(\mathbb{Z}[1/2])$ with $\sigma(x) = -x$ for all $x \in \mathbb{Z}[1/2]$. Longer calculations show that

$$G_\infty \ \cong \ \mathbb{Z}[1/2] \rtimes \langle\sigma\rangle$$

and every automorphism of $\mathrm{Aut(G_\infty)}$ fixes $\mathrm{Inn(G_\infty)}$ setwise.

Since D_∞ has a trivial centre, we know that $\mathrm{Aut(G_\infty)}$ has a trivial centre and we can use the above statement to conclude that the map $\iota_{\mathrm{Aut(G_\infty)}}$ is an isomorphism.

The above examples suggests an extension of our construction of automorphism towers into the transfinite. In order to produce a rigid definition, we have to find suitable directed sets for such a construction.

The following notion generalizes the concept of counting from finite sets to larger collections.

Definition (Cantor)

A strict linear ordering $<$ of a class X is a *well-order* if every non-empty subclass of X contains a $<$-minimal element.

Example

- The canonical ordering of the natural numbers is a well-order.
- If we define

$$m \lhd n \iff n = 0 < m \ \vee \ 0 < m < n,$$

then $\langle \mathbb{N}, \lhd \rangle$ is a well-order.

Ordinals

Definition

A set x is *transitive* if every element of x is also a subset of x.

The following definition provides canonical representatives for set-sized well-orders.

Definition (von Neumann)

A set α is an *ordinal* if it is transitive and well-ordered by $\in$.

Lemma

If $(x, <)$ is a set-sized well-ordering, then there is an ordinal α such that $(x, <)$ is order-isomorphic to $(\alpha, \in)$.

Lemma

If α is an ordinal, then exactly one of the following statements holds:

- $\alpha = \emptyset$ *("α=0").*
- *There is a unique $\beta \in \alpha$ with $\alpha = \beta + 1 = \beta \cup \{\beta\}$ ("α is the direct successor of β").*
- $\alpha = \bigcup_{\beta \in \alpha} \beta$ *("α is a limit ordinal").*

Lemma

The class Ord of all ordinals is well-ordered by $\in$.

Example

We let ω denote the first limit ordinal. Then $(\mathbb{N}, <)$ is order-isomorphic to $(\omega, \in)$ and we can identify the natural numbers with the elements of ω.

The automorphism tower of a group (revised)

We can use the above notions to define transfinite automorphism towers.

Definition (revised)

Given a group G, we say that a direct system

$$\langle\langle G_\alpha \mid \alpha \in \mathrm{Ord}\rangle, \langle \iota_{\alpha,\beta} : G_\alpha \longrightarrow G_\beta \mid \alpha \in \beta\rangle\rangle$$

of groups is the *automorphism tower of* G if the following statements hold:

- $G_0 = G$.
- If $\alpha \in \mathrm{Ord}$, then $G_{\alpha+1} = \mathrm{Aut(G_\alpha)}$ and $\iota_{\alpha,\alpha+1} = \iota_{G_\alpha}$.
- If $\beta \in \mathrm{Ord}$ is a limit ordinal, then $\langle G_\beta, \langle \iota_{\alpha,\beta} \mid \alpha \in \beta\rangle\rangle$ is the direct limit of $\langle\langle G_\alpha \mid \alpha \in \beta\rangle, \langle \iota_{\alpha_0,\alpha_1} \mid \alpha_0 \in \alpha_1 \in \beta\rangle\rangle$.

We say that the automorphism tower of G *terminates after* α*-many steps* if the map $\iota_{\alpha,\alpha+1}$ is an isomorphism.

The above definition naturally leads to the following question.

Question

Does the automorphism tower of every group terminate?

In the following, we will present results of Simon Thomas and Joel Hamkins that yield an affirmative answer to this question.

We will first show that the automorphism tower of every group with trivial centre terminates. Then we show that the automorphism tower of an arbitrary group contains a group with trivial centre.

Automorphism towers of centreless groups

In this section, we will discuss the proof of the following result of Simon Thomas.

Theorem (Thomas, 1985)

The automorphism tower of every centreless group terminates.

In order to prove this result, we need to introduce more fundamental set-theoretic concepts.

Cardinals

Definition (Cantor)

An ordinal β is a *cardinal* if there is no injection $i : \beta \longrightarrow \alpha$ with $\alpha \in \beta$.

Observation

- Every finite ordinal is a cardinal.
- ω is a cardinal.
- Every infinite cardinal is a limit ordinal.

Theorem

For every set x, there is a unique cardinal $|x|$ with the property that there is a bijection $b : x \longrightarrow |x|$.

Definition

The *power set* $\mathcal{P}(x)$ *of a set* x is the set consisting of all subsets of x.

Theorem (Cantor)

Given a set x, *there is no injection* $i : \mathcal{P}(x) \longrightarrow x$.

Proof.

Assume that $i : \mathcal{P}(x) \longrightarrow x$ is such an injection and define

$$z \ = \ \{y \in x \mid \exists u \subseteq x \ [i(u) = y \ \wedge \ y \notin u]\} \in \mathcal{P}(x).$$

Then $i(z) \in z$ if and only if $i(z) \notin z$, a contradiction. $\square$

Corollary

For every cardinal κ, *there is an* $\in$*-minimal cardinal* κ^+ *with* $\kappa \in \kappa^+$.

The following notion will allow us to use the above set-theoretic concepts to show that the automorphism tower of a centreless group terminates.

Definition (Kaplan-Shelah)

Let G be a group, let A be a subset of G and let $v \notin A$.

- We let $\mathcal{F}_*(A)$ denote the free group with basis $A \cup \{v\}$.
- Given $g \in G$, we let $\pi_{A,g} : \mathcal{F}_*(A) \longrightarrow G$ denote the unique homomorphism with $\pi_{A,g}(v) = g$ and $\pi_{A,g} \restriction A = \mathrm{id}_A$.
- We define

$$\Pi_{G,A} : G \longrightarrow \mathcal{P}(\mathcal{F}_*(A)); \ g \longmapsto \ker(\pi_{A,g}).$$

- We say (G, A) is a *special pair* if the function $\Pi_{G,A}$ is an injection.

Theorem (Kaplan-Shelah, 2009)

Let $\langle\langle G_\alpha \mid \alpha \in \mathrm{Ord}\rangle, \langle \iota_{\alpha,\beta} \mid \alpha \in \beta\rangle\rangle$ *be the automorphism tower of a centreless group* G*. Then* $(G_\alpha, \iota_{0,\alpha}[G])$ *is a special pair for all* $\alpha \in \mathrm{Ord}$*.*

This result allows us to prove Thomas' theorem.

Proof of the Theorem.

Let $\langle\langle G_\alpha \mid \alpha \in \mathrm{Ord}\rangle, \langle \iota_{\alpha,\beta} \mid \alpha \in \beta\rangle\rangle$ be the automorphism tower of a centreless group G. By our assumptions, the maps $\iota_{\alpha,\beta}$ are all injective.

If $b_\alpha : \mathcal{P}(\mathcal{F}_*(\iota_{0,\alpha}[G])) \longrightarrow \mathcal{P}(\mathcal{F}_*(G))$ denotes the canonical bijection, then

$$b_\alpha \circ \Pi_{G_\alpha,\iota_{0,\alpha}[G]} = b_\beta \circ \Pi_{G_\beta,\iota_{0,\beta}[G]} \circ \iota_{\alpha,\beta}$$

for all $\alpha \in \beta$. Set $\kappa = |\mathcal{P}(\mathcal{F}_*(G))|$. Then we find a sequence

$$\langle \varphi_\alpha : G_\alpha \longrightarrow \kappa \mid \alpha \in \kappa^+\rangle$$

of injections with $\varphi_\alpha = \varphi_\beta \circ \iota_{\alpha,\beta}$ for all $\alpha \in \beta \in \kappa^+$.

Assume that for every $\alpha \in \kappa^+$ there is a $g_\alpha \in \mathrm{Aut}(\mathrm{G}_\alpha) \setminus \mathrm{Inn}(\mathrm{G}_\alpha)$. Then

$$\Phi : \kappa^+ \longrightarrow \kappa;\ \alpha \longmapsto \varphi_{\alpha+1}(g_\alpha)$$

is an injection of κ^+ into κ, a contradiction. $\square$

Automorphism towers of arbitrary groups

We discuss the proof of the following result.

Theorem (Hamkins, 1998)

If $\langle\langle G_\alpha \mid \alpha \in \mathrm{Ord}\rangle, \langle \iota_{\alpha,\beta} \mid \alpha \in \beta\rangle\rangle$ *is the automorphism tower of a group* G*, then there is an* $\alpha \in \mathrm{Ord}$ *with* $\mathrm{C}(G_\alpha) = \{\mathbb{1}_{G_\alpha}\}$*.*

Corollary

The automorphism tower of every group terminates.

The proof of the theorem relies on the following simple set-theoretic fact.

Proposition

If $F : \mathrm{Ord} \longrightarrow \mathrm{Ord}$ *is a class function, then there is a limit ordinal* β *with* $F(\alpha) \in \beta$ *for all* $\alpha \in \beta$*.*

Proof of the Theorem.

Let $\langle\langle G_\alpha \mid \alpha \in \mathrm{Ord}\rangle, \langle \iota_{\alpha,\beta} \mid \alpha \in \beta\rangle\rangle$ be the automorphism tower of a group G. Given $\alpha \in \mathrm{Ord}$, define

$$H_\alpha \ = \ \{g \in G_\alpha \mid \exists \beta \in \mathrm{Ord}\ [\alpha \in \beta \ \wedge \ \iota_{\alpha,\beta}(g) = \mathbb{1}_{G_\beta}]\}.$$

Then there is a class function $F : \mathrm{Ord} \longrightarrow \mathrm{Ord}$ with $\alpha \in F(\alpha)$ and $\iota_{\alpha,F(\alpha)}(g) = \mathbb{1}_{G_{F(\alpha)}}$ for all $\alpha \in \mathrm{Ord}$ and $g \in H_\alpha$.

Pick a limit ordinal β with $F(\alpha) \in \beta$ for all $\alpha \in \beta$ and fix $g \in \mathrm{C}(G_\beta)$.

Since G_β is a direct limit, there is $\alpha \in \beta$ and $h \in G_\alpha$ with $g = \iota_{\alpha,\beta}(h)$.

Then $g \in \mathrm{C}(G_\beta)$ implies $\iota_{\alpha,\beta+1}(h) = \iota_{\beta,\beta+1}(g) = \mathbb{1}_{G_{\beta+1}}$ and $h \in H_\alpha$.

This allows us to conclude that

$$g \ = \ \iota_{\alpha,\beta}(h) \ = \ (\iota_{F(\alpha),\beta} \circ \iota_{\alpha,F(\alpha)})(h) \ = \ \iota_{F(\alpha),\beta}(\mathbb{1}_{G_{F(\alpha)}}) \ = \ \mathbb{1}_{G_\beta}. \quad \square$$

The heights of automorphism towers

The above results allow us to make the following definitions.

Definition

Let G be a group with automorphism tower $\langle\langle G_\alpha \mid \alpha \in \mathrm{Ord}\rangle, \langle \iota_{\alpha,\beta} \mid \alpha \in \beta\rangle\rangle$. We let $\tau(G)$ denote the $\in$-minimal ordinal with the property that $\iota_{\tau(G),\tau(G)+1}$ is an isomorphism.

Definition

Given a cardinal κ, we let τ_κ denote the $\in$-minimal ordinal with the property that $\tau(G) \in \tau_\kappa$ for every group G with $|G| = \kappa$.

The following problem is completely open (even for finite cardinals):

Problem

Given a cardinal κ, find a non-trivial upper bound for τ_κ.

In contrast to Hamkins' proof, the above argument for centreless group provides an upper bound for the heights of automorphism towers of such groups.

Given a cardinal κ, we write 2^κ to denote $|\mathcal{P}(\kappa)|$.

Theorem (Thomas, 1998)

If G is an infinite centreless group and $\kappa = |G|$, then $\tau(G) \in (2^\kappa)^+$.

This result motivates the following definition.

Definition

Given a cardinal κ, we let τ^c_κ denote the $\in$-minimal ordinal with the property that $\tau(G) \in \tau^c_\kappa$ for every centreless group G with $|G| = \kappa$.

The following theorem is an easy consequence of the above theorem.

Theorem (Thomas, 1998)

If κ is an infinite cardinal, then $\tau^c_\kappa \in (2^\kappa)^+$.

Since the upper bound $(2^\kappa)^+$ is never optimal, it is natural to ask whether it is possible to give a set-theoretic characterization of the ordinal τ_κ^c.

The following result shows that this is extremely difficult.

Following Cantor, we write ω_1 to denote the first uncountable cardinal ω^+.

Theorem (Just-Shelah-Thomas, 1998)

The statement "$\tau_{\omega_1}^c \in 2^{\omega_1}$" is independent of the standard axioms of set theory.

The following problem is again completely open.

Problem

Construct a model of set theory such that it is possible to give a set theoretic characterization of the value of τ_κ^c in that model for some infinite cardinal κ in that model.

The main obstacle for a solution of the above problems is the fact that the value of $\tau(G)$ can differ in different models of set theory.

We conclude this talk by presenting better upper bounds for the value of τ^c_κ.

Theorem (Kaplan-Shelah, 2009)

Let κ be an infinite cardinal and let $\theta_{\mathcal{P}(\kappa)}$ be the least ordinal such that there is no surjection from $\mathcal{P}(\kappa)$ onto $\theta_{\mathcal{P}(\kappa)}$ in $\mathrm{L}(\mathcal{P}(\kappa))$, the smallest inner model of set theory containing all ordinals and $\mathcal{P}(\kappa)$. Then $\tau^c_\kappa < \theta_{\mathcal{P}(\kappa)}$.

Theorem (L., 2012)

Let κ be an infinite cardinal and let $\delta_{\mathcal{P}(\kappa)}$ be the ordinal height of the least transitive model of Kripke-Platek set theory that contains $\mathcal{P}(\kappa)$. Then either $\tau^c_\kappa < \delta_{\mathcal{P}(\kappa)}$ or $\tau^c_\kappa = \delta_{\mathcal{P}(\kappa)} + 1$.

In contrast to the above results, the proof of this theorem does not already show that the provided bound is not optimal.

Conjecture (L.)

It is consistent that "$\tau^c_\kappa = \delta_{\mathcal{P}(\kappa)} + 1$" holds for some infinite cardinal κ.

Thank you for listening!