The influence of closed maximality principles on generalized Baire space

Philipp Moritz Lücke

Mathematisches Institut
Rheinische Friedrich-Wilhelms-Universität Bonn
http://www.math.uni-bonn.de/people/pluecke/

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$\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$

Throughout this talk, we let κ denote an uncountable cardinal satisfying $\kappa = \kappa^{<\kappa}$.

The *generalized Baire space of* κ is the set ${}^\kappa\kappa$ of all functions from κ to κ equipped with the topology whose basic open sets are of the form

$$N_s \ = \ \{x \in {}^\kappa\kappa \mid s \subseteq x\}$$

for some s contained in the set ${}^{<\kappa}\kappa$ of all functions $t : \alpha \longrightarrow \kappa$ with $\alpha < \kappa$.

A subset of ${}^\kappa\kappa$ is a $\mathbf{\Sigma}^1_1$*-set* if it is equal to the projection of a closed subset of ${}^\kappa\kappa \times {}^\kappa\kappa$.

The following folklore result shows that the class of $\mathbf{\Sigma}^1_1$-sets contains many interesting objects.

Proposition

As subset of ${}^\kappa\kappa$ is a $\mathbf{\Sigma}^1_1$-set if and only if it is definable over the structure $\langle \mathrm{H}(\kappa^+), \in \rangle$ by a Σ_1-formula with parameters.

This observation can also be used to show that many basic questions about the class of $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$ are not settled by the axioms of $\mathbf{ZFC}$ together with large cardinal axioms.

In the following, we will discuss three examples of such questions.

Separating the club filter from the nonstationary ideal

Given $S \subseteq \kappa$, we define

$$\mathrm{Club}(S) \ = \ \{x \in {}^{\kappa}\kappa \mid \exists C \subseteq \kappa \textit{ club } \forall \alpha \in C \cap S \ x(\alpha) > 0\}$$

and

$$\mathrm{NStat}(S) \ = \ \{x \in {}^{\kappa}\kappa \mid \exists C \subseteq \kappa \textit{ club } \forall \alpha \in C \cap S \ x(\alpha) = 0\}.$$

Then the *club filter* $\mathrm{Club}(\kappa)$ and the *non-stationary ideal* $\mathrm{NStat}(\kappa)$ are disjoint $\mathbf{\Sigma}^1_1$-subsets of ${}^{\kappa}\kappa$.

In the light of the *Lusin Separation Theorem theorem* it is natural to ask the following question.

Question

Is there a $\mathbf{\Delta}^1_1$-subset A of ${}^{\kappa}\kappa$ that separates $\mathrm{Club}(\kappa)$ from $\mathrm{NStat}(\kappa)$, in the sense that $\mathrm{Club}(\kappa) \subseteq A \subseteq {}^{\kappa}\kappa \setminus \mathrm{NStat}(\kappa)$ holds?

If S is a stationary subset of κ, then Club(S) is a $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa$ that separates Club(κ) from NStat(κ).

The following theorem builds upon results of Mekler/Shelah and Hyttinen/Rautila and shows that sets of this form can be forced to be $\mathbf{\Delta}^1_1$-definable at many regular cardinals.

Theorem (Friedman/Hyttinen/Kulikov)

Assume that GCH *holds and* κ *is not the successor of a singular cardinal. There is a cofinality preserving forcing* $\mathbb{P}$ *such that* Club(S^κ_ω) *is a* $\mathbf{\Delta}^1_1$*-subset of* ${}^\kappa\kappa$ *in ever* $\mathbb{P}$*-generic extension of the ground model.*

This shows that a positive answer to the above question is consistent.

In contrast, it is possible to combine results of Halko/Shelah and Friedman/Hyttinen/Kulikov (or L./Schlicht) to show that a negative answer to the above question is also consistent.

Theorem

If G is $\mathrm{Add}(\kappa, \kappa^{+})$-generic over V, then there is no $\mathbf{\Delta}^1_1$-subset A of ${}^{\kappa}\kappa$ that separates $\mathsf{Club}(\kappa)$ *from* $\mathsf{NStat}(\kappa)$ *in $\mathrm{V}[G]$.*

Lengths of Σ^1_1-definable well-orders

We call well-order $\langle A, \prec\rangle$ a $\mathbf{\Sigma}^1_1$-*well-ordering of a subset of* ${}^\kappa\kappa$ if $\prec$ is a $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa \times {}^\kappa\kappa$.

It is easy to see that for every $\alpha < \kappa^+$, there is a $\mathbf{\Sigma}^1_1$-well-ordering of a subset of ${}^\kappa\kappa$ of order-type α.

Moreover, if there is an $x \subseteq \kappa$ such that κ^+ is not inaccessible in $\mathrm{L}[x]$, then there is a $\mathbf{\Sigma}^1_1$-well-ordering of a subset of ${}^\kappa\kappa$ of order-type κ^+.

The following question is motivated by the *Kunen-Martin Theorem*.

Question

What is the least upper bound for the order-types of $\mathbf{\Sigma}^1_1$-well-orderings of subsets of ${}^\kappa\kappa$?

With the help of generic coding techniques, it is possible to make arbitrary subsets of ${}^\kappa\kappa$ $\mathbf{\Sigma}^1_1$-definable in a cofinality preserving forcing extension of the ground model.

In particular, these techniques allow us to force the existence of a $\mathbf{\Sigma}^1_1$-well-ordering of a given length α.

Theorem

Given $\alpha < (2^\kappa)^+$, there is a partial order $\mathbb{P}$ with the property that forcing with $\mathbb{P}$ preserves all cofinalities and the value of 2^κ and there is a $\mathbf{\Sigma}^1_1$-well-ordering of a subset of ${}^\kappa\kappa$ of order-type α in every $\mathbb{P}$-generic extension of the ground model.

In the other direction, both κ^+ and $2^\kappa > \kappa^+$ can consistently be upper bounds for the length of these well-orders.

Theorem

Let $\nu > \kappa$ be a cardinal, G be $\mathrm{Add}(\kappa,\nu)$-generic over V and $\langle A, \prec\rangle$ be a $\mathbf{\Sigma}^1_1$-well-ordering of a subset of ${}^\kappa\kappa$ in $\mathrm{V}[G]$, Then $A \neq ({}^\kappa\kappa)^{\mathrm{V}[G]}$ and the order-type of $\langle A, \prec\rangle$ has cardinality at most $(2^\kappa)^{\mathrm{V}}$ in $\mathrm{V}[G]$.

Theorem

If $\nu > \kappa$ is inaccessible, $G \times H$ is $(\mathrm{Col}(\kappa, {<}\nu) \times \mathrm{Add}(\kappa,\nu))$-generic over V and $\langle A, \prec\rangle$ is a $\mathbf{\Sigma}^1_1$-well-ordering of a subset of ${}^\kappa\kappa$ in $\mathrm{V}[G,H]$, then A has cardinality κ in $\mathrm{V}[G,H]$.

Note that the conclusion of the last theorem implies that κ^+ is inaccessible in $\mathrm{L}[x]$ for every $x \subseteq \kappa$.

The bounding and the dominating number of $\langle \mathcal{TO}_\kappa, \leq \rangle$

Let $\mathcal{T}_\kappa$ denote the class of all trees of cardinality and height κ and $\mathcal{TO}_\kappa$ denote the class of all trees in $\mathcal{T}_\kappa$ without a branch of length κ.

Given $\mathbb{T}_0, \mathbb{T}_1 \in \mathcal{T}_\kappa$, we write $\mathbb{T}_0 \leq \mathbb{T}_1$ if there is a function $f : \mathbb{T}_0 \longrightarrow \mathbb{T}_1$ such that $f(s) <_{\mathbb{T}_0} f(t)$ holds for all $s, t \in \mathbb{T}_0$ with $s <_{\mathbb{T}_1} t$.

The elements of the resulting partial order $\langle \mathcal{TO}_\kappa, \leq \rangle$ can be viewed as generalizations of countable ordinals.

We can identify $\mathcal{TO}_\kappa$ with a $\mathbf{\Pi}^1_1$-subset of ${}^\kappa\kappa$ and the ordering $\leq$ with a $\mathbf{\Sigma}^1_1$-definable relation on this set.

We are interested in the order-theoretic properties of $\langle \mathcal{TO}_\kappa, \leq \rangle$.

More specifically, we interested in the value of the following cardinal characteristics.

- The *bounding number of* $\langle \mathcal{TO}_\kappa, \leq \rangle$ is the smallest cardinal $\mathfrak{b}_{\mathcal{TO}_\kappa}$ with the property that there is a $U \subseteq \mathcal{TO}_\kappa$ of this cardinality such that there is no tree $\mathbb{T} \in \mathcal{TO}_\kappa$ with $\mathbb{S} \leq \mathbb{T}$ for all $\mathbb{S} \in U$.

- The *dominating number of* $\langle \mathcal{TO}_\kappa, \leq \rangle$ is the smallest cardinal $\mathfrak{d}_{\mathcal{TO}_\kappa}$ with the property that there is a subset $D \subseteq \mathcal{TO}_\kappa$ of this cardinality such that for every $\mathbb{S} \in \mathcal{TO}_\kappa$ there is a $\mathbb{T} \in D$ with $\mathbb{S} \leq \mathbb{T}$.

It is easy to see that

$$\kappa^+ \leq \mathfrak{b}_{\mathcal{TO}_\kappa} \leq \mathfrak{d}_{\mathcal{TO}_\kappa} \leq 2^\kappa$$

holds. In particular, if $2^\kappa = \kappa^+$, then these cardinal characteristics are equal. We may therefore ask if this is always the case.

Question

Is $\mathfrak{b}_{\mathcal{TO}_\kappa}$ equal to $\mathfrak{d}_{\mathcal{TO}_\kappa}$?

With the help of κ-Cohen forcing it is possible to show that a negative answer to this question is also consistent.

Theorem

If G is $\mathrm{Add}(\kappa, (2^\kappa)^+)$-generic over V, then

$$\mathfrak{b}_{\mathcal{TO}_\kappa}^{\mathrm{V}[G]} \leq (2^\kappa)^{\mathrm{V}} < (2^\kappa)^{\mathrm{V}[G]} = \mathfrak{d}_{\mathcal{TO}_\kappa}^{\mathrm{V}[G]}.$$

The results presented above show that there are many interesting questions about $\mathbf{\Sigma}^1_1$-subsets that are not settled by the axioms of $\mathbf{ZFC}$ together with large cardinal axioms. In particular, these axioms do not provide a nice structure theory for the class of $\mathbf{\Sigma}^1_1$-sets.

This observation leads us to the following question.

Question

Are there canonical extensions of $\mathbf{ZFC}$ that settle these questions by providing a strong structure theory for the class of $\mathbf{\Sigma}^1_1$-sets?

In the following, we will show that forcing axioms called *closed maximality principle* are examples of such extensions of $\mathbf{ZFC}$.

Closed maximality principles

We will present axioms that are variations of the *maximality principles* introduced by Stavi/Väänänen and Hamkins.

We say that a sentence φ in the language of set theory is *forceably necessary* if there is a partial order $\mathbb{P}$ such that $\mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}} \Vdash \varphi$ holds for every $\mathbb{P}$-name $\dot{\mathbb{Q}}$ for a partial order.

Example

The sentence "$\omega_1 > \omega_1^{\mathrm{L}}$" is forceably necessary.

The *maximality principle for forcing* is the scheme of axioms stating that every forceably necessary sentence is true.

This formulation is motivated by the *maximality principle* $\Diamond \Box \varphi \longrightarrow \varphi$ of modal logic by interpreting the modal statement $\Diamond\varphi$ ("*φ is possible*") as "*φ holds in some forcing extension of the ground model*" and the statement $\Box\varphi$ ("*φ is necessary*") as "*φ holds in every forcing extension of the ground model*".

We will modify this principle in the following ways.

- By restricting the complexity of the considered formulas.
- By restricting the class of forcings that can be used to witness that a given statement is possible.
- By restricting the class of forcings that need to be considered in order to check that a given statement is necessary.
- By allowing statements containing parameters.

Note that the first two modifications weaken the principle, while the last two strengthen it.

The axioms discussed in this talk consider classes of $<\kappa$-closed forcings and allow statements with parameters of bounded hereditary cardinality.

We will refer to these principle as *boldface closed maximality principles*.

Definition

Let $\Phi(v_0, v_1)$ be a formula and z be a set.

- We say that a statement $\varphi(x_0, \ldots, x_{n-1})$ is (Φ, z)-*forceably necessary* if there is a partial order $\mathbb{P}$ with $\Phi(\mathbb{P}, z)$ and $\mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}} \Vdash \varphi(\check{x}_0, \ldots, \check{x}_{n-1})$ for every $\mathbb{P}$-name $\dot{\mathbb{Q}}$ for a partial order with $\mathbb{1}_{\mathbb{P}} \Vdash \Phi(\dot{\mathbb{Q}}, \check{z})$.
- Given an infinite cardinal ν and $0 < n < \omega$, we let $\mathrm{MP}^n_{\Phi,z}(\nu)$ denote the statement that every (Φ, z)-forceably necessary Σ_n-statement with parameters in $\mathrm{H}(\nu)$ is true.

Note that, with the help of a universal Σ_n-formula, the principle $\mathrm{MP}^n_{\Phi,z}(\nu)$ can be expressed by a single statement using the parameters ν and z.

Let $\Phi_{cl}(v_0, v_1)$ be the formula defining the class of $<\kappa$-closed partial orders using the parameter κ.

We write CMP^n_κ instead of $\mathrm{MP}^n_{\Phi_{cl},\kappa}(\kappa^+)$.

The principles CMP^n_κ were studied in depth by Gunter Fuchs.

The following remark shows that they may be viewed as strengthenings of the statement that Σ_1-statements with parameters in $\mathrm{H}(\kappa^+)$ are absolute with respect to $<\kappa$-closed forcings.

Corollary

The principle CMP^1_κ *holds.*

The following result of Fuchs gives bounds for the consistency strength of these principles.

Remember that a cardinal δ is Σ_n-*reflecting* if it is inaccessible and $\langle \mathrm{V}_\delta, \in \rangle$ is a Σ_n-elementary submodel of $\langle \mathrm{V}, \in \rangle$.

Theorem (Fuchs)

Let $0 < n < \omega$*.*

- *If* $\delta > \kappa$ *is a* Σ_{n+2}*-reflecting cardinal and* G *is* $\mathrm{Col}(\kappa, \delta)$*-generic over* V*, then* CMP^n_κ *holds in* $\mathrm{V}[G]$*.*
- *If* $\mathrm{CMP}^{n+1}_\kappa$ *holds and* $\delta = \kappa^+$*, then* δ *is* Σ_n*-reflecting in* L*.*

We consider closed maximality principles for statements of arbitrary complexities.

Let $\mathcal{L}_{\in,\dot{\nu}}$ denote the language of set theory extended by an additional constant symbol $\dot{\nu}$.

Let REFL denote the $\mathcal{L}_{\in,\dot{\nu}}$-theory consisting of $\mathbf{ZFC}$ together with the scheme of $\mathcal{L}_{\in,\dot{\nu}}$-sentences stating that $\dot{\nu}$ is Σ_n-reflecting for all $0 < n < \omega$.

Let CMP denote the $\mathcal{L}_{\in,\dot{\nu}}$-theory consisting of $\mathbf{ZFC}$ together with the scheme of $\mathcal{L}_{\in,\dot{\nu}}$-sentences stating that $\mathrm{CMP}^n_{\dot{\nu}}$ holds for all $0 < n < \omega$.

Corollary (Fuchs)

- *Assume that* $\langle \mathrm{V}, \in, \delta \rangle$ *is a model of* REFL *with* $\delta > \kappa$*. If* G *is* $\mathrm{Col}(\kappa,\delta)$*-generic over* V*, then* $\langle \mathrm{V}[G], \in, \kappa \rangle$ *is a model of* CMP*.*
- *Assume that* $\langle \mathrm{V}, \in, \kappa \rangle$ *is a model of* CMP *and* $\delta = \kappa^+$*. Then* $\langle \mathrm{L}, \in, \delta \rangle$ *is a model of* REFL*.*

The axiom CMP^2_κ induces a strong structure theory for $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$. In particular, it settles the first two questions posed above.

Theorem

If CMP^2_κ *holds, then there is no* $\mathbf{\Delta}^1_1$*-subset of* ${}^\kappa 2$ *that separates* Club(κ) *from* NStat(κ)*.*

Sketch of the proof.

- CMP^2_κ *implies* $\mathbf{\Sigma}^1_2$*-absoluteness for* $<\kappa$*-closed forcings.*
- $\mathbf{\Sigma}^1_2$*-absoluteness for* $\mathrm{Add}(\kappa, 1)$ *implies that all* $\mathbf{\Delta}^1_1$*-sets have the* κ*-Baire property.*
- *A result of Halko/Shelah shows that no set with the* κ*-Baire property separates* Club(κ) *from* NStat(κ)*.*

Theorem

If CMP^2_κ holds, then the least upper bound for the order-types of $\mathbf{\Sigma}^1_1$-well-orderings of subsets of ${}^\kappa\kappa$ is equal to κ^+.

Sketch of the proof.

- *If a $<\kappa$-closed forcing adds an element to a $\mathbf{\Sigma}^1_1$-set, then this set contains a perfect subset.*
- *This shows that CMP^2_κ implies that all $\mathbf{\Sigma}^1_1$-sets have the perfect set property.*
- *$\mathbf{\Sigma}^1_2$-absoluteness for $\mathrm{Add}(\kappa, 1)$ implies that the domains of $\mathbf{\Sigma}^1_1$-well-orderings of subsets of ${}^\kappa\kappa$ do not contain perfect subsets.*

In contrast, such principles do not answer the third question

Theorem

If the theory CMP *is consistent, then it does not decide the statement* $\mathfrak{b}_{\mathcal{TO}_{\dot{\nu}}} = \mathfrak{d}_{\mathcal{TO}_{\dot{\nu}}}$.

This result is a consequence of the above theorem on the values of the cardinal characteristics in $\mathrm{Add}(\kappa,(2^\kappa)^+)$-generic extensions and a result of Fuchs showing that $\langle \mathrm{V}[G,H],\in,\kappa\rangle$ is a model of CMP whenever $\langle \mathrm{V},\in,\delta\rangle$ is a model of REFL with $\delta > \kappa$ and $G \times H$ is $(\mathrm{Col}(\kappa,\delta) \times \mathrm{Add}(\kappa,\delta^+))$-generic over V.

Closed maximality principles with more parameters

The proof of the above negative result suggests that we consider maximality principles for statements containing parameters of higher cardinalities. To do so we have to restrict ourselves to cardinality-preserving forcings. Natural candidates are classes of all $<\kappa$-closed partial orders satisfying the κ^+-chain condition.

It turns out that such principles are connected to generalizations of classical forcing axioms to κ.

Given a partial order $\mathbb{P}$ and an infinite cardinal ν, we let $\mathsf{FA}_\nu(\mathbb{P})$ denote the statement that for every collection $\mathcal{D}$ of ν-many dense subsets of $\mathbb{P}$, there is a filter G on $\mathbb{P}$ that meets all elements of $\mathcal{D}$.

Proposition

Let $\Phi(v_0, v_1)$ be a formula, z be set and $\nu \geq \kappa$ be a cardinal.

- *If $\mathrm{MP}^1_{\Phi,z}(\nu^+)$ holds and $\mathbb{P}$ is a partial order of cardinality at most ν with $\Phi(\mathbb{P}, z)$, then $\mathsf{FA}_\nu(\mathbb{P})$ holds.*
- *Assume that every partial order $\mathbb{P}$ with $\Phi(\mathbb{P}, z)$ satisfies the ν^+-chain condition and $\mathsf{FA}_\nu(\mathbb{P})$ holds for all such $\mathbb{P}$. Then $\mathrm{MP}^1_{\Phi,z}(\nu^+)$ holds*

A result of Shelah shows that there is a $<\kappa$-closed partial order $\mathbb{P}$ satisfying the κ^+-chain condition such that $\mathsf{FA}_{\kappa^+}(\mathbb{P})$ fails.

Together with the above observation, this shows that we have to restrict the class of forcings considered even further to obtain a consistent maximality principle. In particular, $\mathsf{FA}_{\kappa^+}(\mathbb{P})$ should consistently hold for every partial orders $\mathbb{P}$ in this class.

An example of such a class can be found in Baumgartner's work on generalization of Martin's Axiom to higher cardinalities.

Baumgartner's Axiom for κ (BA_κ) is the assumption that $\mathsf{FA}_\nu(\mathbb{P})$ holds for all $\nu < 2^\kappa$ and partial orders $\mathbb{P}$ that is $<\kappa$-closed, κ-linked and well-met.

Let $\Phi_B(v_0, v_1)$ be the canonical formula that defines the class of all $<\kappa$-support products of $<\kappa$-closed, κ-linked and well-met partial orders using the parameter κ.

In this talk, we consider the maximality principles $\mathrm{MP}^n_{\Phi_B,\kappa}(2^\kappa)$ associated to this class.

We abbreviate these principles by BMP^n_κ.

These principles may be viewed as strengthenings of BA_κ.

Proposition

If $\nu^{<\kappa} < 2^\kappa$ for all $\nu < 2^\kappa$, then BMP^1_κ holds if and only if $\mathsf{FA}_\nu(\mathbb{P})$ holds for all $\nu < 2^\kappa$ and every partial order $\mathbb{P}$ with $\Phi_B(\mathbb{P}, \kappa)$.

Theorem

If BMP^2_κ holds, then 2^κ is a weakly inaccessible cardinal and $\nu^{<\kappa} < 2^\kappa$ holds for all $\nu < 2^\kappa$.

The consistency strength of this principle can be bounded in similar way as for the principles discussed in the last section.

Theorem

Let $0 < n < \omega$.

- *Given an inaccessible cardinal* $\delta > \kappa$, *there is a partial order* $\mathbb{B}(\kappa, \delta)$ *that is uniformly definable in parameters* κ *and* δ *with the property that if* δ *is* Σ_{n+2}*-reflecting, then* BMP^n_κ *holds in every* $\mathbb{B}(\kappa, \delta)$*-generic extension of the ground model* V.
- *If* $\mathrm{BMP}^{n+1}_\kappa$ *holds and* $\delta = 2^\kappa$, *then* δ *is* Σ_n*-reflecting in* L.

As above, we also consider versions of this maximality principle for statements of unbounded complexity.

We let BMP denote the $\mathcal{L}_{\in,\dot{\nu}}$-theory consisting of the axioms of $\mathbf{ZFC}$ together with the scheme of $\mathcal{L}_{\in,\dot{\nu}}$-sentences stating that $\mathrm{BMP}^n_{\dot{\nu}}$ holds for all $0 < n < \omega$.

Corollary

- *Assume that* $\langle \mathrm{V}, \in, \delta \rangle$ *is a model of* REFL *with* $\delta > \kappa$. *If* G *is* $\mathbb{B}(\kappa, \delta)$*-generic over* V, *then* $\langle \mathrm{V}[G], \in, \kappa \rangle$ *is a model of* BMP.
- *Assume that* $\langle \mathrm{V}, \in, \kappa \rangle$ *is a model of* BMP *and* $\delta = 2^\kappa$. *Then* $\langle \mathrm{L}, \in, \delta \rangle$ *is a model of* REFL.

The axiom BMP^2_κ provides a strong structure theory for $\mathbf{\Sigma}^1_1$-sets that decides all of the above questions.

Theorem

If BMP^2_κ holds, then there is no $\mathbf{\Delta}^1_1$-subset of ${}^\kappa 2$ that separates Club(κ) *from* NStat(κ).

Sketch of the proof.

- CMP^2_κ *implies $\mathbf{\Sigma}^1_2$-absoluteness for* $\mathrm{Add}(\kappa, 1)$.
- *$\mathbf{\Sigma}^1_2$-absoluteness for $\mathrm{Add}(\kappa, 1)$ implies that all $\mathbf{\Delta}^1_1$-sets have the κ-Baire property.*
- *A result of Halko/Shelah shows that no set with the κ-Baire property separates* Club(κ) *from* NStat(κ).

Theorem

If BMP^2_κ holds, then the least upper bound for the order-types of $\mathbf{\Sigma}^1_1$-well-orderings of subsets of ${}^\kappa\kappa$ is equal to 2^κ and every $\gamma < 2^\kappa$ is equal to the order-type of such a well-ordering.

Sketch of the proof.

- *If a $<\kappa$-closed forcing adds an element to a $\mathbf{\Sigma}^1_1$-set, then this set contains a perfect subset.*
- *This shows that BMP^2_κ implies that all $\mathbf{\Sigma}^1_1$-sets of cardinality 2^κ contain a perfect subset.*
- *$\mathbf{\Sigma}^1_2$-absoluteness for $\mathrm{Add}(\kappa, 1)$ implies that the domains of $\mathbf{\Sigma}^1_1$-well-orderings of subsets of ${}^\kappa\kappa$ do not contain perfect subsets.*
- *Using almost disjoint coding forcing at κ, it can be seen that BMP^2_κ implies that every subset of ${}^\kappa\kappa$ of cardinality less than 2^κ is a $\mathbf{\Sigma}^0_2$-set.*

Theorem

If BMP^2_κ holds, then $\mathfrak{b}_{\mathcal{TO}_\kappa} = \mathfrak{d}_{\mathcal{TO}_\kappa} = 2^\kappa$.

Sketch of the proof.

- BMP^2_κ *implies that every subset of ${}^\kappa\kappa$ of cardinality less than 2^κ is a $\mathbf{\Sigma}^1_1$-set.*
- *A result of Mekler/Väänänen (Boundedness Lemma for $\mathcal{TO}_\kappa$) shows that for every $\mathbf{\Sigma}^1_1$-subset A of $\mathcal{TO}_\kappa$ there is a $\mathbb{T} \in \mathcal{TO}_\kappa$ with $\mathbb{S} \leq \mathbb{T}$ for all $\mathbb{S} \in A$.*
- *Together, this shows that BMP^2_κ implies that $\mathfrak{b}_{\mathcal{TO}_\kappa} = 2^\kappa$.*

Further results and open questions

The above results show that the axioms CMP^2_κ and BMP^2_κ decide the least upper bounds for the lengths of $\mathbf{\Sigma}^1_1$-definable well-orders.

Motivated by the results of classical descriptive set theory, it is natural to ask the same question for prewell-orders.

Question

Is the least upper bound of the lengths of $\mathbf{\Delta}^1_1$-prewell-orders on subsets of ${}^{\dot{\nu}}\dot{\nu}$ determined by the axioms of BMP or CMP?

There are bigger classes of $<\kappa$-closed partial orders satisfying the κ^+-chain condition such that the corresponding maximality principle is consistent.

For example, the maximality principle for the class $<\kappa$-support products of $<\kappa$-closed and strongly κ-linked partial orders is consistent and implies that all $\mathbf{\Sigma}^1_1$-subsets of ${}^\kappa\kappa$ satisfy the *Hurewicz Dichotomy*.

In the light of classical forcing axioms, it is natural to ask the following question.

Question

Are there natural classes of $<\kappa$-closed partial orders satisfying the κ^+-chain condition such that for each class it is consistent that this class consists of all $<\kappa$-closed partial orders $\mathbb{P}$ that satisfy the κ^+-chain condition and $\mathsf{FA}_{\kappa^+}(\mathbb{P})$?

It there a unique class with this property?

We proposed the above maximality principles as candidates for extensions of $\mathbf{ZFC}$ that provide a strong structure theory for $\mathbf{\Sigma}^1_1$-sets.

Therefore it is natural to ask whether these axioms can hold globally, i.e. is it consistent that BMP^n_ν (or CMP^n_ν) holds for every uncountable cardinal ν with $\nu = \nu^{<\nu}$?

Theorem (Fuchs)

The class of all uncountable cardinals ν with $\nu = \nu^{<\nu}$ and CMP^3_ν is bounded in On.

Theorem

The class of all uncountable cardinals ν with $\nu = \nu^{<\nu}$ and BMP^2_ν is bounded in On.

Question

Is the class of all uncountable cardinals ν with $\nu = \nu^{<\nu}$ and CMP^2_ν always bounded in On?

Following Fuchs, we may consider weakenings of the above principles called *localized maximality principles*.

These principles can consistently hold at every uncountable cardinals ν with $\nu = \nu^{<\nu}$.

Moreover, they have the same influence on $\mathbf{\Sigma}^1_1$-subsets of ${}^\nu\nu$ as the full maximality principles.

Thank you for listening!